Repeated Trade and News in Markets with Asymmetric Information*

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Abstract

This paper explores the role of news in financial markets with asymmetrically-informed traders. We study a continuous-time setting in which stochastic information about a privately-informed seller's asset is revealed gradually to a market of traders. Traders' time preference for money is subject to random liquidity shocks generating future incentive to trade. In equilibrium, the price is determined not only by traders' beliefs about the fundamental value of the asset, but also by expectations of future liquidity in the market. The equilibrium involves periods of no trade in which *liquidity dries up*: assets remain in the hands of a liquidity constrained traders despite efficient gains from trade. The no-trade period ends in one of two ways: either enough good news arrives restoring confidence and re-opening markets, or bad news arrives making buyers more pessimistic and forcing market *capitulation* i.e., a partial sell-off of low-value assets. The model helps to explain a number of frequently observed trading patterns. Evidence from the mortgage-backed securities market is discussed.

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1 Introduction

The objective of this paper is to explore the role of news (i.e., stochastic information) in a dynamic market where assets are trading repeatedly among asymmetrically-informed traders. Repeated trade can occur only if there must are potential gains from doing so (Milgrom and Stokey, 1982). Liquidity reasons are both a plausible and convenient way to generate the potential for such gains. We introduce liquidity shocks to the framework developed in Daley and Green (2010) and study the effects of news on trade behavior, asset prices and liquidity.

The model is as follows. There is a single indivisible asset in the economy that delivers cashflows to the owner of the asset. The cashflows depend on the asset's type, which is privately known by the current owner. As time passes, (i) potential buyers arrive and make offers to the asset owner, (ii) stochastic information about the asset's type is gradually revealed to the market by a Brownian diffusion process and (iii) the asset owner is subject to an observable liquidity shock.

Liquidity shocks arrive randomly according to a Poisson process and increase the rate at which an owner of the asset discounts future payoffs generating a potential gain from trade. The owner of an asset is *not* forced to sell upon the arrival of a liquidity shock, but she is more eager to do so. In this setting, a trader's value for the asset depends not only on her beliefs about the asset type but also on her expectations about future liquidity in the market. Thus, a buyer's value for the asset arises endogenously through the structure of the equilibrium.

We characterize the equilibrium through a system of differential equations and boundary conditions. In equilibrium, an owner who is not liquidity constrained never sells. When the owner is constrained, trading behavior can be characterized by three distinct regions: (1) the market is liquid when the market's beliefs about the quality of the asset are favorable, constrained sellers are able to trade quickly at "fair" prices; (2) a sell-off region when the market is very pessimistic, an owner hit by a shock in this region is forced to either sell at rock-bottom prices or hold out; (3) a no-trade region where both sides of the market wait for news until either good news restores confidence to (1) or bad news forces (2).

The no-trade region leads to an inefficient allocation of the asset. This makes liquidation costly for a seller because it requires inefficient delay before the asset is transferred. Buyers correctly anticipate these liquidation costs and therefore the asset trades at prices below it's fundamental value. Asset prices decrease and the inefficiency increases with the arrival rate of shocks because traders liquidate (and incur the cost from doing so) more frequently. The occurs despite the fact that the fundamental value of the asset remains constant. As the arrival rate of liquidity shocks goes to zero, asset prices converge to fundamental values.

We present an algorithm for solving the system. The iterative process has a convenient economic interpretation: in iteration k, the algorithm computes the unique equilibrium of a game with k liquidity shocks remaining. Thus, equilibrium asset values (in the infinite-shock model) are determined by the fixed point of the algorithm.

Our results help to explain several phenomena commonly observed in financial markets. For example, the model predicts that a sell-off of assets at low prices can help stabilize a shaky market. Wall Street traders and analysts refer to this as "market capitulation" Zweig (2008); Cox (2008). In addition, we find that a small amount of bad news can lead to a drastic decrease in volume, which explains another phenomenon that traders refer to as "when liquidity dries up" Smith (2008); Reuters (2008).

The recent collapse of the mortgage-backed securities (MBS) market is particularly relevant. Until 2007, trade and issuance of mortgage-backed securities occurred in a liquid and well-functioning market. This occured despite the fact that banks issuing these securities had a significant amount of data about the underlying collateral that was inaccessible to most investors. In mid 2007, economic indicators of a decline in the real-estate market created more uncertainty in the value of the collateral and led to a catastrophic drop in both liquidity and prices.[1] Investors were unwilling to buy these securities or lend against them (even at a substantial discount/haircut) for fear of being stuck with the most "toxic" assets. Rightly so. A bank that was willing to issue or sell MBS for cents on the dollar was likely holding collateral which was least likely to perform. As a result, mortgages-backed securities remained on the balance sheet of numerous large banks despite their need for capital. Figure 1 illustrates the dramatic decrease in liquidity of MBS experienced during 2008. There was a significant tightening of credit starting in 2008 that certainly played a role in the decline of MBS issuance. However, MBS issuance as a percentage of the total US bond market experienced a similarly severe decline indicating that industry specific factors (e.g., information frictions) were also responsible.

In the next section, we present the model and establish some preliminaries. In Section 3, we describe equilibrium behavior and characterize asset values through a set of necessary conditions. An algorithm to solve for the equilibrium is presented in Section 4

[1]See Krishnamurthy (2010) or Brunnermeier (2009) for a descriptive analysis of how debt markets malfunctioned in the recent crisis.

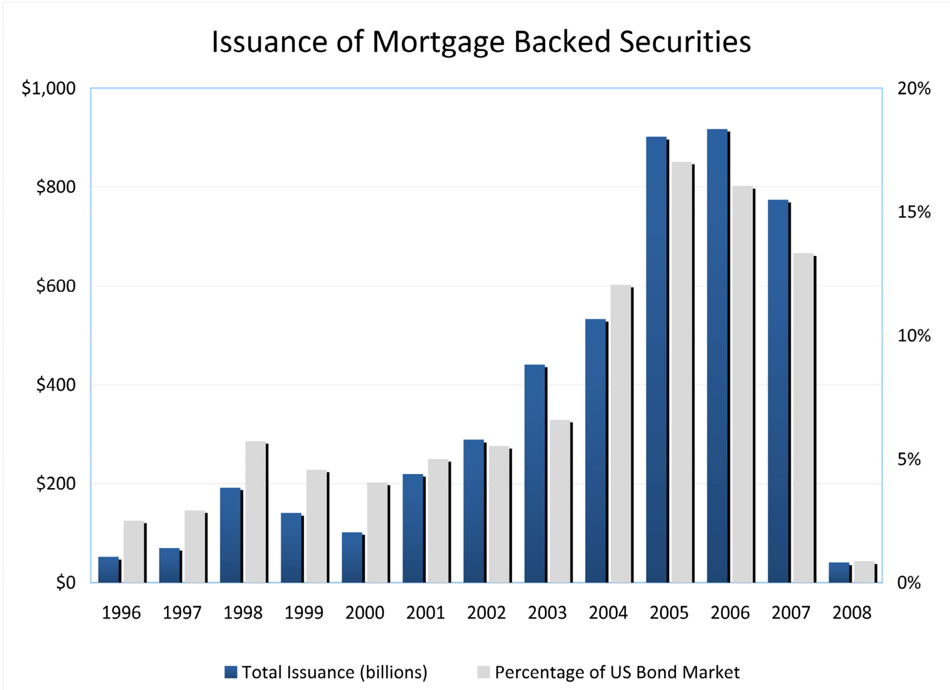


Figure 1: Both MBS issuance and percentage of MBS contributing to the US bond market fell drastically in 2008. Hence, the drop in MBS issuance cannot be attributed solely to macroeconomic shocks to credit markets (Source: SIFMO)

and is then used to prove equilibrium existence. Section 5 solves a numerical example using the algorithm. Section 6 concludes. Proofs are located in the Appendix.

2 The Model

The game begins at $t = 0$ with an indivisible asset owned by a liquidity-constrained agent denoted by A_0. The asset may be one of two types: $\theta \in \{L, H\}$. A_0 knows the asset's type, potential buyers do not. An asset of type θ generates a cash flow v_θ ($v_H > v_L$), which accrues to the current owner as a flow payoff.[2] At every $t > 0$, multiple buyers arrive and make private offers to the current owner. If a buyer's offer is accepted, he becomes the new owner and immediately learns the asset's type. All rights to future cash flows are transferred to him. If the seller rejects all offers, she retains the asset, receives the flow payoff and can entertain offers from future buyers. A buyer whose offer is rejected exits the game permanently.

The initial owner, A_0, discounts future payoffs at a rate $\bar{r}$. Initially, all other agents

[2]Alternatively, an asset of type θ generates a stochastic flow payoff with mean v_θ.

have a discount rate of $r < \bar{r}$. Therefore, it is efficient for A_0 to sell the asset immediately. We use A_t to denote the owner of the asset at time t. Liquidity shocks arrive according to a homogeneous Poisson process, $N = \{N_t : 0 \leq t \leq \infty\}$, with arrival rate λ. For all $A_t \neq A_0$, upon arrival of the first shock, the rate at which A_t discounts future payoffs increases to $\bar{r}$ where it remains ad infinitum. For simplicity, only A_t is affected by a shock arriving at time t and subsequent arrivals have no effect on the preferences of the agent. We will refer to A_t as a *seller* or a liquidity-constrained owner if she has been hit by a shock and as a *holder* or an unconstrained owner if she has not yet been hit by a shock. All players are risk neutral. Let $V_\theta = v_\theta/r$ denote the *fundamental value* of a type θ asset. For simplicity, we assume that $v_H/\bar{r} > V_L$.

A shock that arrives at time s is observable to all buyers arriving at times $t \geq s$. In addition, news about the asset is continually revealed via a Brownian diffusion process. Both type assets start with the same initial score X_0. The score process then evolves according to

$$dX_t^\theta = \mu_\theta dt + \sigma dB_t \tag{1}$$

where B is standard Brownian motion independent of N. Without loss of generality, $\mu_H \geq \mu_L$. The parameters (μ_H, μ_L, σ) are common knowledge to all agents. Define the signal-to-noise ratio $\phi \equiv (\mu_H - \mu_L)/\sigma$. When $\phi = 0$, the news is completely uninformative. Larger values of ϕ imply higher quality news.[3,4] In what follows, we assume that $\phi > 0$, unless otherwise stated.

Formally, the game takes place on a probability space $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t\}$. The state space Ω contains all possible paths of B, N, choices by nature θ, and allows for randomization by agents. The public history at time t, which also corresponds to the information set of a buyer arriving at time t contains:

- The history of news: $\{X_s : 0 \leq s \leq t\}$
- The arrival times of liquidity shocks: $\{N_s : 0 \leq s \leq t\}$
- All times (if any) at which the asset has been traded before time t: $T_t = (t_1, t_2, ...)$

Let $\{\mathcal{F}_t^B\}$ be the filtration generated by the public history. The market beliefs about the asset type are conditioned on all of the above. At time $t = 0$, the market begins

[3] The signal-to-noise ratio can be thought of as measuring the *quality* of news (how much do we expect to learn in a given amount of time), or the *rate* of news (how much time do we expect it to take to learn a given amount). For consistency, we will refer to the *quality* of news throughout.

[4] We demonstrate that ϕ is a sufficient statistic for the quality of the news. That is, if two triples $(\mu_H, \mu_L, \sigma) \neq (\mu'_H, \mu'_L, \sigma')$ but $\frac{\mu_H - \mu_L}{\sigma} = \frac{\mu'_H - \mu'_L}{\sigma'}$, then the equilibria of the two settings are payoff equivalent.

with a common prior $P_0 = \Pr_0(\theta = H)$. Let f_t^θ denote the density of type θ's score at time t, which is normally distributed with mean $\mu_\theta t$ and variance $\sigma^2 t$. Define $\hat{P}$ to be the belief process for a Bayesian who updates *only based on news* starting from the prior ($\hat{P}_0 = P_0$).

$$\hat{P}_t = g(t, X_t) \equiv \frac{P_0 f_t^H(X_t)}{P_0 f_t^H(X_t) + (1 - P_0) f_t^L(X_t)} \tag{2}$$

$\hat{P}$ is a homogenous Markov martingale; given $\hat{P}_t$, the distribution of $\hat{P}_{t'}$, $t' > t$, is independent of both t and $\hat{P}_s$ for all $s < t$, and $E[\hat{P}_{t'}|\mathcal{F}_t, P_0] = \hat{P}_t$. For reasons that will soon become apparent, it is useful to define a new process $\hat{Z} \equiv \ln(\hat{P}/(1 - \hat{P}))$, which represents the belief in terms of its log-likelihood ratio. Because the mapping from $\hat{P}$ to $\hat{Z}$ is injective, there is no loss in making this transformation. By definition,

$$\hat{Z}_t = \ln\left(\frac{\hat{P}_t}{1 - \hat{P}_t}\right) = \ln\left(\frac{\hat{P}_0 f_t^H(X_t)}{(1 - \hat{P}_0) f_t^L(X_t)}\right) = \underbrace{\ln\left(\frac{\hat{P}_0}{1 - \hat{P}_0}\right)}_{\hat{Z}_0} + \underbrace{\ln\left(\frac{f_t^H(X_t)}{f_t^L(X_t)}\right)}_{\frac{\phi}{\sigma}\left(X_t - \frac{t(\mu_H + \mu_L)}{2}\right)}$$

$$= \hat{Z}_0 + \frac{\phi}{\sigma} X_t - \frac{\phi}{2\sigma}(\mu_H + \mu_L)t$$

Applying Ito's lemma gives

$$d\hat{Z}_t = -\frac{\phi}{2\sigma}(\mu_H + \mu_L)dt + \frac{\phi}{\sigma} dX_t \tag{3}$$

Inserting the law of motion from equation (1) gives a probabilistic representation of how beliefs based solely on news evolve from the perspective of the privately-informed seller, which we denote by $\hat{Z}^\theta$. The high type expects to receive good news, hence $\hat{Z}^H$ is a submartingale (Eq. 4). The low type is expectant of bad news: $\hat{Z}^L$ is a supermartingale (Eq. 5).

$$d\hat{Z}_t^H = -\frac{\phi}{2\sigma}(\mu_H + \mu_L)dt + \frac{\phi}{\sigma} dX_t^H = \frac{\phi^2}{2} dt + \phi dB_t \tag{4}$$

$$d\hat{Z}_t^L = -\frac{\phi}{2\sigma}(\mu_H + \mu_L)dt + \frac{\phi}{\sigma} dX_t^L = -\frac{\phi^2}{2} dt + \phi dB_t \tag{5}$$

Because $\hat{Z}^\theta$ is a linear transformation of X^θ, it is also a Brownian diffusion process, retaining desirable properties, such as stationary independent increments, and making analysis more tractable than working with the corresponding non-linear processes derivable from $\hat{P}$ in probability space. Most importantly, working in log-likelihood space allows us to represent Bayesian updating as a linear process (see Eq. 7).

2.1 Strategies and Equilibrium Concept

A strategy for a buyer arriving at time t is a $\mathcal{F}_t^B$-measurable function to offers in $\mathbb{R}$.[5] Aggregating buyers' strategies over time yields a process $\overrightarrow{W} = \{\overrightarrow{W}_t : 0 \leq t \leq \infty\}$ adapted to the filtration $\{\mathcal{F}_t^B\}$, where $\overrightarrow{W}_t(\omega)$ is the collection of offers at time t given ω. Since buyer's are competitive and offers are private, the identity of the buyer making each offer as well as the level of non-maximal offers is irrelevant. Hence, our equilibrium analysis will focus on identifying the process of maximal offers $W = \max\{\overrightarrow{W}\}$ that is consistent with buyer's playing optimally.

The asset owners information set contains the public history, as well as the asset type and the collection of offers made since the owner acquired the asset. A pure strategy for an owner is a choice of which offer to accept, if any, at each time given her information set. Formally, a pure strategy for a type θ owner who acquires the asset at time t (hereafter a (θ, t)-owner) is a stopping time $\tau_{\theta,t}$ adapted to the filtration generated by the information set of the agent, which we denote by $\{\mathcal{F}_{t,h}^S\}_{h\geq 0}$. Given W, the problem facing a (θ, t)-owner is to choose a stopping time to solve:

$$\sup_{\tau} E_t^{\theta} \left[\int_t^{\tau} v_{\theta} e^{-r_s s} ds + e^{-\int_t^{\tau} r_s ds} W_{\tau} \right] \tag{6}$$

Where r_s denotes the discount rate of the owner at time s. We allow an owner to mix by choosing a distribution over stopping times. Thus a general mixed strategy for a (θ, t)-owner is a collection of stopping times and a distribution over them. At times, it will be more convenient to represent a (θ, t)-owner's strategy as a stochastic process $S_t^{\theta} = \{S_{t,h}^{\theta}, 0 \leq h \leq \infty\}$ adapted to the filtration $\{\mathcal{F}_{t,h}^S\}$, where $S_{t,h}^{\theta}(\omega) = \Pr\left(\tau_{\theta,t}(\omega) \leq t + h | \mathcal{F}_{t,h}^S\right)$. We say that S_t^{θ} solves (6), if all $\tau_{\theta,t}$ in the support of the distribution solve (6).

To be consistent with a distribution over stopping times, the process must satisfy:

(i) $\lim_{h\to 0^-} S_{t,h}^{\theta} = 0$

(ii) $S_{t,h}^{\theta}$ is weakly increasing and right-continuous in h

(iii) $S_{t,h}^{\theta} \leq 1$ for all t, h

From the time t perspective, $S_{t,h}^{\theta}$ keeps track of how much probability mass the seller will have "used up" at time $t + h$ by assigning positive probability to accepting offers at times $s \in (t, t + h)$. An upward jump in S_t^{θ} corresponds to the type θ seller accepting

[5] We restrict buyers to play pure strategies only to simplify exposition. Trade dynamics remain unchanged if we allow buyers to play mixed strategies.

with an atom of mass. S_t^θ increasing continuously corresponds to the seller accepting at a flow rate. For any given sample path, S_t^θ is a CDF over the seller's acceptance time.

At every instant in time, buyers assign a probability to the asset being of high value. Define $P = \{P_t, 0 \le t < \infty\}$ to be this process and denote a realization of this process at time t by $P_t(\omega)$, where ω is a realization of a state in Ω. P_t differs from $\hat{P}_t$ because it accounts for the possibility and realizations of trade before time t. We have implicitly assumed that there is one P process common to all buyers. Along the equilibrium path, this feature is an implication of the common prior and Bayesian updating: P must be consistent with the players' strategies as well as the news arrival. In the spirit of sequential equilibrium, we maintain this assumption off the equilibrium path as well.

Define $Z \equiv \ln(P/(1-P))$. Just as in the previous section, there is no loss in making this transformation. Because Bayes rule is linear in log-likelihood space we can decompose Z as $Z = \hat{Z} + Q$, where Q is the stochastic process that keeps track of the information conveyed by the history of past acceptances and rejections. That is, along the equilibrium path and for all $h \in (0, t_{i+1} - t_i)$ (recall that t_i denotes the time of the ith trade.)

$$Z_{t_i+h} = Z_{t_i} + \underbrace{\ln\left(\frac{f_t^H(X_{t_i+h} - X_{t_i})}{f_t^L(X_{t_i+h} - X_{t_i})}\right)}_{\hat{Z}_{t_i+h} - \hat{Z}_{t_i}} + \underbrace{\ln\left(\frac{1 - S_{t_i,h}^H}{1 - S_{t_i,h}^L}\right)}_{Q_{t_i,h}} \tag{7}$$

where $Z_0 = \ln(P_0/(1-P_0))$. Because $\hat{Z}$ follows directly from X and P_0, identifying $Q_{t,h}$ for all $t, h \ge 0$ is sufficient to uniquely identify any equilibrium belief process.

We now integrate all of the above concepts to define our equilibrium notion.

Definition 2.1. *An equilibrium of the game is a quadruple (S^L, S^H, W, Z), such that*

1. *Given W and for all histories, S_t^θ solves (6).*
2. *Given S^L, S^H and Z, W is consistent with buyers playing best responses.*
3. *Market beliefs satisfies Bayes rule whenever possible (i.e., Z satisfies (7)).*

3 Equilibrium Construction

We construct an equilibria in which buyers use stationary strategies and Z is a Markov Process. Note that the market belief is not the only payoff relevant variable. The state of the game is also contingent on whether the current owner has been hit by a shock. We use I_t to denote the (Markov) process that indicates the current owner's status: $I_t = 1$ if A_t is a seller, and $I_t = 0$ if A_t is a holder. We use (z, i) when referring to the state variable as opposed to the stochastic processes, Z, I. The reader should interpret (z, i) as any (t, ω) such that $(Z_t(\omega), I_t(\omega)) = (z, i)$. Markov strategies require that there exists a function $w : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$ such that $w(z, i)$ denotes the maximum of all offers made in the state (z, i). Given w and Z, the problem facing an asset owner is to find an optimal policy (stopping rule) to maximize her expected payoff given any initial state (z, i).

$$\sup_{\tau} E^{\theta}_{z,i} \left[\int_0^{\tau} v_\theta e^{-r_s s} ds + e^{-\int_0^\tau r_s ds} w(Z_\tau, I_\tau) \right] \tag{8}$$

where $r_s = r + I_s(\bar{r} - r)$. Given the imposed stationary structure, we can now write the problem recursively. We use F_θ to denote the value function for a *seller* of type θ. The Bellman equation for the seller's problem is

$$F_\theta(z) = \max \left\{ w(z, 1), v_\theta dt + e^{-\bar{r}dt} E^\theta \left[F_\theta(z + dZ_t) \right] \right\} \tag{SP$_\theta$}$$

A seller chooses between accepting the current offer or taking her flow payoff and waiting in hopes of a higher offer in the future. Because offers are private, the levels of the rejected offers will have no effect on future payoffs. This observation implies that the seller will follow a reservation strategy: for a given z, a type θ seller will accept any offer above a certain threshold and reject any offer below.

When $i = 0$, a holder faces a similar problem. The only difference is that by rejecting the current offer there is a positive probability (λdt) that she will be hit with a liquidity shock and become a seller . We use G_θ to denote the value function for a *holder* of type θ. The Bellman equation for the holder's problem is

$$G_\theta(z) = \max \left\{ w(z, 0), v_\theta dt + e^{-rdt} E^\theta \left[(1 - \lambda dt) G_\theta(z + dZ_t) + \lambda dt F_\theta(z + dZ_t) \right] \right\} \tag{HP$_\theta$}$$

We now provide a brief overview and explanation of the equilibrium that will be constructed in the sections that follow. In equilibrium, a holder never sells: when $i = 0$ buyers make non-serious offers which are rejected with probability one by both types of holders. Hence, beliefs evolve strictly according to news over any interval of time in which the owner is not liquidity constrained. When A_t is a seller, equilibrium play can be described by a triple (α, β, B) such that when buyers are *pessimistic* ($z < \alpha$), buyers offer $w(z,1) = V_L$ and the low-type sell accepts with positive probability causing the equilibrium beliefs to jump immediately to α. When buyers are *optimistic* ($z > \beta$), the market is efficient—$w(z,1) = B(z)$ is offered and accepted w.p.1. In this region, an owner with a credible reason to liquidate does so immediately. The asset is never traded for $z \in (\alpha, \beta)$, buyers make non-serious offers and both sides of the market wait for more information to be revealed.

One of our primary interests is in determining the function B, which corresponds to a buyer's expected value for the asset as a function of his belief about the assets type and conditional on both types trading with probability one. B also pins down asset prices for the region in which the market is optimistic. As will be made clear in the next section, the value functions of agents are intertwined. A buyer who purchases the asset immediately becomes a holder and hence a buyer's value depends on a holder's value. A holder eventually becomes a seller and hence the holders value depends on the seller's value. Of course, a seller's value depends on the price at which the asset can be sold (i.e., the buyer's value). To characterize the equilibrium, we derive a system of interdependent differential equations and specify the boundary conditions using equilibrium arguments.

3.1 Asset Values in Equilibrium

Fix $B : \mathbb{R} \to [V_L, V_H]$ (it will be derived shortly) and assume that B is continuous and weakly increasing. In the no-trade region, the seller rejects w and takes her continuation value. Applying Ito's lemma to F_θ, using the law of motion of Z^θ and taking the expectation, (SP_θ) implies a differential equation that F_θ must satisfy for all $z \in (\alpha, \beta)$. Namely, for a high-type seller

$$\frac{\phi^2}{2}(F_H''(z) + F_H'(z)) - \bar{r}F_H(z) + v_H = 0 \tag{9}$$

and for a low-type seller

$$\frac{\phi^2}{2}(F_L''(z) - F_L'(z)) - \bar{r}F_L(z) + v_L = 0 \tag{10}$$

Outside of the no-trade region (i.e., $z \notin (\alpha, \beta)$), the equilibrium specifies the following. For all $z > \beta$, both type sellers trade immediately at $w = B(z)$, therefore

$$F_H(z) = F_L(z) = B(z) \qquad \forall\, z > \beta \tag{11}$$

For all $z < \alpha$, the equilibrium beliefs jump instantaneously to α (as long as the current owner is liquidity constrained), therefore

$$F_H(z) = F_H(\alpha), \qquad F_L(z) = F_L(\alpha) = V_L, \qquad \forall\, z < \alpha \tag{12}$$

There are six boundary conditions that help pin down the seller's value function in the interior of the no-trade region. As z approaches α from above, a low-type's value approaches V_L, and she must be indifferent between accepting $w = V_L$ or taking her continuation payoff at that point.

$$F_L(\alpha^+) = V_L \tag{13}$$

$$F_L'(\alpha^+) = 0 \tag{14}$$

where $g(x^+)$ ($g(x^-)$)is used to denote the right (left) limit of the function g at x. As z approaches β from below, both types will accept an offer of $w = B(\beta)$ with probability one.

$$F_L(\beta^-) = B(\beta) \tag{15}$$

$$F_H(\beta^-) = B(\beta) \tag{16}$$

The high type is indifferent between accepting or taking her continuation payoff at $z = \beta$,

$$F_H'(\beta^-) = B'(\beta) \tag{17}$$

And finally, the belief process is purely reflecting for a high type at $z = \alpha$.

$$F_H'(\alpha^+) = 0 \tag{18}$$

The slope conditions (14) and (17) are known as *smooth pasting* conditions and are required for seller indifference in a continuous-time setting (Dixit, 1993).[6] In equilibrium,

[6] Although (14) and (18) imply the same slope condition on F_L and F_H at α, the former is a smooth pasting condition, while the latter follows immediately from the reflective behavior of Z^H (see Harrison (1985) for a discussion of necessary boundary conditions for a function of a reflected process).

the low type is mixing between accepting at $z \le \alpha$ and waiting. In order for this mixing to be optimal, she must be indifferent between the two pure strategies implying (13) is necessary.

To see that (17) is necessary to make the high type indifferent, suppose that $F'_H(\beta) < B'(\beta)$ and consider the following deviation: reject at $z = \beta$ and continue to reject until $z = \beta + \epsilon$ for some arbitrarily small $\epsilon > 0$. Instead of accepting $B(\beta)$, the high type attains a convex combination of $B(\beta + \epsilon)$ and $F_H(\beta - \epsilon)$ which lies strictly above $B(\beta)$ implying the deviation is profitable. On the other hand, if $F'_H(\beta^-) > B'(\beta)$ then the high type would prefer to accept sooner.[7]

Given B, (9)-(18) pin down (α, β) and $F_\theta(z)$ for all z, θ. Of course, the buyer's value function is also endogenous. To determine B, we must first find the asset value to each type of holder. The equilibrium prescribes that a holder never trades. A holder simply consumes her flow payoff until a shock transforms her into a seller. The value of the asset comes from the instantaneous flow payoff and the discounted expected value of the asset an instant later. With probability $1 - \lambda dt$, A_t is not hit by a shock and has a value of $G_\theta(z + dZ_t)$. With probability λdt, the shock comes and A_t becomes a liquidity-constrained seller with value $F_\theta(z + dZ_t)$. Thus, the holder's value for the asset must satisfy the following recursive equation for all z

$$G_\theta(z) = v_\theta dt + e^{-rdt} E^\theta \big[(1 - \lambda dt) G_\theta(z + dZ_t) + \lambda dt F_\theta(z + dZ_t) \big] \tag{19}$$

Using similar methods as above, (19) implies the following differential equation for the value of each type holder

$$G''_H(z) + G'_H(z) - \frac{2(r+\lambda)}{\phi^2} G_H(z) = -\frac{2}{\phi^2}(\lambda F_H(z) + v_H) \tag{20}$$

$$G''_L(z) - G'_L(z) - \frac{2(r+\lambda)}{\phi^2} G_L(z) = -\frac{2}{\phi^2}(\lambda F_H(z) + v_L) \tag{21}$$

The next step in pinning down equilibrium asset values is to determine the boundary conditions for G_L and G_H. To do so, we make use of the fact that as $z \to \infty$, the belief

[7] Technically, (17) is not a necessary condition but rather a feature of the equilibrium we construct with the following two caveats. First, the weaker condition $F'_H(\beta^-) \le B'(\beta)$ is necessary. To see this, notice that if $F'_H(\beta^-) > B'(\beta)$ then there exist a $z' < \beta$ such that $F_H(z') < B(z')$ implying that a buyer could profitably deviate by making and offer at z' of $w \in (F_H(z'), B(z'))$. Second, equilibria in which $F'_H(\beta^-) < B'(\beta)$ can be sustained only by imposing threat beliefs for off-equilibrium path rejections (i.e., the probability assigned to a high type *decreases* following an unexpected rejection). If we impose a mild refinement on off-equilibrium path beliefs, namely that beliefs cannot decrease following an unexpected rejection, then (17) becomes a necessary condition.

becomes degenerate. To see this, let $p(z) \equiv \frac{e^z}{1+e^z}$, denote the probability assigned to $\theta = H$ in state z. Then $\lim_{z\to\infty} p(z) = 1$ and the effect of news on equilibrium beliefs goes to zero. A holder is simply waiting for the shock to come, at which point she has a seller's value for the asset. The same is true as $z \to -\infty$. In the limit, a holder's value for the asset is a weighted average of the fundamental value and a seller's value. The following boundary conditions complete the characterization of the holder's value function.

$$\lim_{z\to\infty} G_\theta(z) = \frac{rV_\theta + \lambda \lim_{z\to\infty} F_\theta(z)}{r+\lambda} \qquad \theta \in \{L, H\} \tag{22}$$

$$\lim_{z\to-\infty} G_\theta(z) = \frac{rV_\theta + \lambda \lim_{z\to-\infty} F_\theta(z)}{r+\lambda} \qquad \theta \in \{L, H\} \tag{23}$$

Only the holder is directly affected by the arrival rate of the shocks. Buyer and seller values are affected indirectly through G_θ.[8]

Finally, we turn to characterizing a buyer's value function. To do so, first note that if $\lambda = 0$, a buyer does not face future liquidity concerns. Upon purchasing the asset, the buyer will hold it ad infinitum. In this case, a buyer's expected value for the asset is simply the expected fundamental value, denoted by $\Psi(z) \equiv \frac{1}{r} E_z [v_\theta]$. When $\lambda > 0$, a buyer's value depends not only on his current beliefs but also on his ability to sell the asset in the future when he is hit by a shock. In this case, the buyer immediately becomes a holder and therefore

$$B(z) = E[G_\theta(z)|z] = p(z)G_H + (1 - p(z))G_L \tag{24}$$

Note that B as defined by (24) is continuously differentiable (since G_L, G_H are) and for any finite (α, β)

$$\lim_{z\to\infty} B(z) = \lim_{z\to\infty} G_H(z) = \frac{rV_H + \lambda \lim_{z\to\infty} F_H(z)}{r+\lambda} = \frac{rV_H + \lambda \lim_{z\to\infty} B(z)}{r+\lambda} \tag{25}$$

And therefore if $\lim_{z\to\infty} B(z) < \infty$ (which it must be in equilibrium), then $\lim_{z\to\infty} B(z) = V_H$. Similarly,

$$\lim_{z\to-\infty} B(z) = \lim_{z\to-\infty} G_L(z) = \frac{rV_L + \lambda \lim_{z\to-\infty} F_L(z)}{r+\lambda} = V_L \tag{26}$$

In fact, we can say much more about the structure of the buyer's value function. In particular, we can simplify the system by deriving a direct relationship between B and

[8]Setting $\lambda = 0$ implies that $G_H(z) = V_H$ and $G_L(z) = V_L$ for all z.

F_L, F_H by utilizing (20)-(21).

Lemma 3.1. *Suppose that there exists a solution to the system (9)-(24). Then B satisfies the differential equation:*

$$B''(z) + (2p(z) - 1)B'(z) - \frac{2(r+\lambda)}{\phi^2}B(z) = \\ \frac{-2}{\phi^2}\Big(p(z)(\lambda F_H(z) + v_H) + (1-p(z))(\lambda F_L(z) + v_L)\Big) \quad (27)$$

where $p(z) = \frac{e^z}{1+e^z}$.

Lemma 3.1 allows us to reduce the complexity of the system by removing the holder value functions after we impose the following two relevant boundary conditions.

$$\lim_{z\to\infty} B(z) = V_H \quad (28)$$

$$\lim_{z\to\infty} B(z) = V_L \quad (29)$$

Lemma 3.2. *If $(\alpha, \beta) \in \mathbb{R}^2$, and F_H, F_L, B satisfy (9)-(18) and (27)-(29). Then there exists a unique G_H, G_L satisfying (20)-(23).*

Hence to find a solution to the entire system, we can focus our attention on seller and buyer value functions.

Claim 3.3. *There exist a unique $\{F_L, F_H, B\}$ and (α, β) which solves the system (9)-(18), (27)-(29).*

4 Equilibrium Computation using an Algorithmic Approach

In the previous section, we characterized equilibrium asset values through a system of differential equations and boundary conditions and argued that a solution exists. In this section, we focus on solving that system. If $\lambda = 0$, closed form solutions for the sellers' value functions can be derived and used to verify the existence of an equilibrium candidate and solve for it. This approach is less tractable when $\lambda > 0$ because the buyer's value is endogenous to the system. While it is still possible to derive closed form solutions for the system, an algorithmic approach is more practical and enables a comparison to a model with a finite number of possible liquidity shocks. In what follows, we present an algorithm for solving the equilibrium that enables us to prove the existence of an equilibrium and numerically compute the equilibrium asset values and boundaries.

The algorithm follows an iterative process.[9] In each iteration, asset values and boundaries are computed. Let B_k denote the buyer's value function in iteration k, and let $(F_{\theta,k}, G_{\theta,k})_{\theta\in\{L,H\}}$ denote the seller and holder values. Let (α_k, β_k) denote the no-trade boundaries in iteration k.

Step 0: Initialize $B_0 = \Psi$ and let $k = 0$.

Step 1: Using B_k, solve for (α_k, β_k) and $F_{L,k}(z), F_{H,k}(z)$ for $z \in (\alpha_k, \beta_k)$, using the differential equations (9) and (10) and boundary conditions (13)-(18). Define $F_{L,k}(z), F_{H,k}(z)$ for $z \notin (\alpha_k, \beta_k)$ as in (11)-(12).

Step 2: Using $F_{\theta,k}$, solve for $G_{\theta,k}$ using the differential equations (20) and (21), along with the boundary conditions (22)-(23).

Step 3: Define $B_{k+1}(z) = E[G_{\theta,k}|z]$ for all z. Increment $k = k + 1$.

Step 4: Repeat Steps 1-3 until convergence is obtained.

Lemma 4.1 shows that the algorithm has a unique, well-defined solution at each iteration. Using this result, we then show in Lemma 4.2 that the algorithm converges to a unique fixed point. To illustrate these results formally, the following notation will be useful. Let $\xi_k \equiv \{\alpha_k, \beta_k, B_k, (F_{\theta,k}, G_{\theta,k})_{\theta\in\{L,H\}}\} \in M$ denote the boundaries and value functions computed by iteration k of the algorithm and let $\Xi_k \equiv (\xi_0, \xi_1, ..., \xi_k) \in M^k$, where $M \equiv \mathbb{R}^2 \times (C(\mathbb{R}, [V_L, V_H]))^5$, where $C(U, V)$ denotes the space of absolutely continuous functions from U to V. Let T denote the operator mapping ξ_k to ξ_{k+1}.

Lemma 4.1 (Unique Solution)**.** *For each iteration $k \geq 0$, there exists a unique ξ_{k+1} such that $T(\xi_k) = \xi_{k+1}$. Moreover, $\xi_{k+1} \in M$.*

Lemma 4.2 (Convergence)**.** *The algorithm converges to a [unique] fixed point, $\xi \equiv \{\alpha, \beta, B, (F_\theta, G_\theta)_{\theta\in\{L,H\}}\}$, in the following sense. For any $\epsilon > 0$, there exists a subsequence, $(\xi_{n_1}, \xi_{n_2}, ...)$, and a K such that $d(\xi_{n_k}, \xi) < \epsilon$ for all $k > K$, where*

$$d(\xi_k, \xi) = |\alpha - \alpha_k| + |\beta - \beta_k| + \sup_z |B - B_k| + \sum_\theta \sup_z |F_\theta - F_{\theta,k}| + \sup_z |G - G_{\theta,k}|$$

[9] We are grateful to Yuliy Sannikov for suggesting this approach.

The algorithm has intuitive appeal in that iteration k of the algorithm solves for the equilibrium asset values in a world where there can be at most k shocks. To formalize this idea, let Γ_K denote the game in which at most K shocks arrive and use Γ_∞ to denote the game as specified in Section 2. Γ_K is the same as Γ_∞ except that it is common knowledge that the arrival rate of shocks goes from λ to zero after the Kth shock has arrived. When $K < \infty$, the state of the game is now characterized by the triple $(z, i, k) = \{(t, \omega) : z = Z_t(\omega), i = I_{A_t}(\omega), k = K - N_t(\omega)\}$, where k is used to denote the number of shocks left to come.

Proposition 4.3. *For all non-negative integers K, there exists an equilibrium of Γ_K, which is characterized by Ξ_K in the following sense. When there are $k \leq K$ shocks left to come, the equilibrium asset values and boundaries are given by ξ_k and strategies are given as follows:*

- *For any (z, i, k) such that $i = 0$, buyers make non-serious offers, which both type holder's reject.*

When $i = 1$ and for all k:

- *If $z > \beta_k$ and $i = 1$, $w(z, i) = B_k(z)$ and both type seller's accept with probability one.*
- *If $z < \alpha_k$ and $i = 1$, $w(z, i) = V_L$, the high type seller rejects with probability one and the low type accepts with probability $\rho_L(z, 1, k, V_L) = 1 - e^{z-\alpha_k}$.*
- *If $z \in (\alpha_k, \beta_k)$ and $i = 1$, buyers make non-serious offers $w(z, i) \leq V_L$, which both types reject with probability one.*

The equilibrium can be sustained by off-path beliefs that remain unchanged following an unexpected rejection and assign probability one to $\theta = L$ following an unexpected acceptance.[10]

This proposition gives an economic interpretation of the algorithm. It can also be used to show that the taking limit of the equilibrium of Γ_K as $K \to \infty$ results in an equilibrium of the limit game.

Theorem 4.4. *There exists an equilibrium of Γ_∞, which is characterized by the fixed point of the algorithm ξ. Furthermore, ξ solves (9)-(24).*

[10] There is flexibility in assigning off-equilibrium path beliefs that sustain this equilibrium.

5 Numerical Results and Comparative Statics

Using the algorithm described above, we compute equilibrium asset values and strategies for various sets of parameters. We vary both λ and ϕ to illustrate how equilibrium asset values depend on the arrival rate of shocks and the quality of news. The following parameters remain fixed: $v_L = 1$, $v_H = 2$, $r = 10\%$, $\bar{r} = 15\%$. Figure 2 shows the asset values for each type of seller and holder as well as a buyer's value.

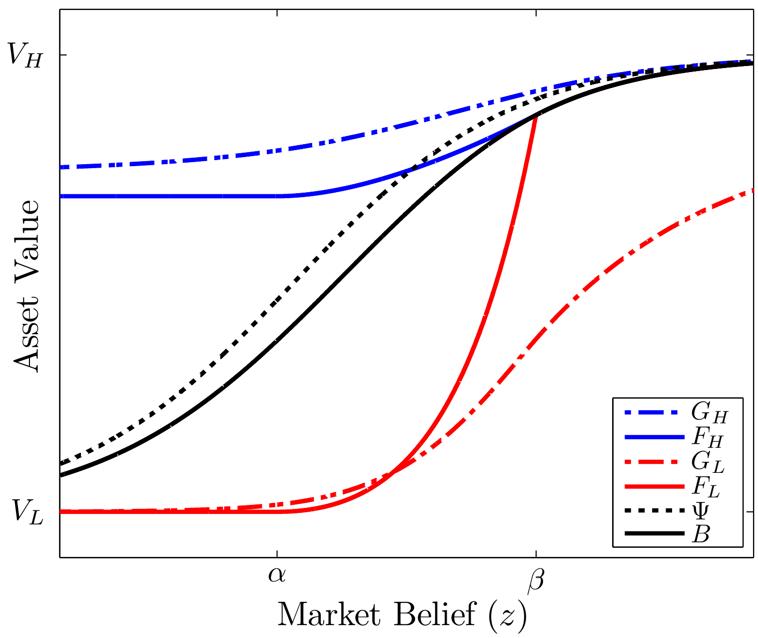
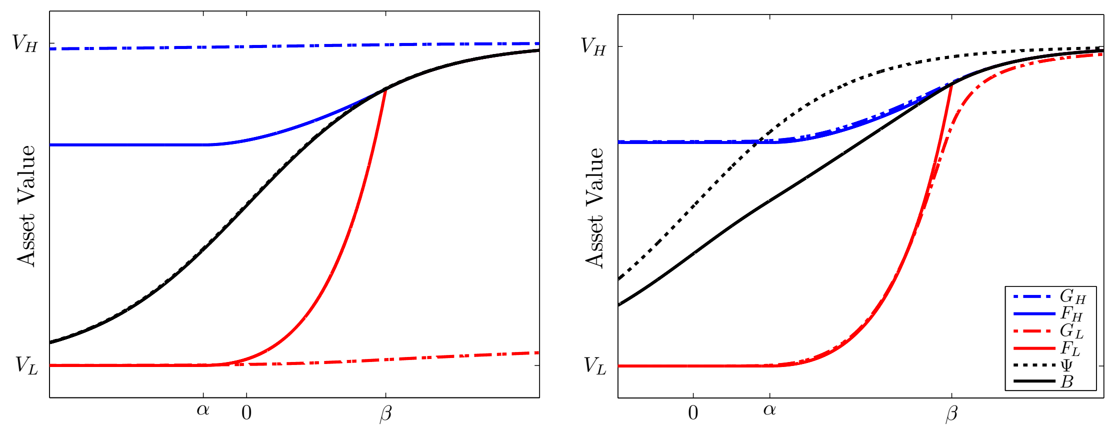


Figure 2: Equilibrium asset values for $\lambda = 0.5$, $\phi = 1$

Notice that a buyer's value lies everywhere below Ψ. Buyers realize that after purchasing the asset, they may be hit with a liquidity shock when beliefs lie in the no-trade region. In this case, rather then sell immediately, they hold the asset in hopes of a higher offer in the future despite the fact that there is common knowledge of gains from trade. Inefficient allocation of the asset in this region depresses prices below their fundamental value. Not surprisingly, the value to the owner of a high-value asset is strictly higher before she is hit by a liquidity shock. However, the same is not true for the low type asset. When beliefs are favorable, a low-type holder would prefer to become a seller because the low-type seller can trade at a price above V_L. This highlights the importance of the assumption that liquidity shocks are observable, which we discuss in more detail in the next section.

Figure 3 illustrates how the asset values depend on the arrival rate of the shocks.

The right panel illustrates that higher λ correspond to lower asset prices. The left panel shows that as $\lambda \to 0$, the equilibrium asset prices convergence to their fundamental levels ($B \to \Psi$). Furthermore, the no-trade region shifts to the right as λ increases implying that the region over which the allocation is inefficient also increases with λ. Although equilibrium asset prices decrease with λ, the same is not true of a low-type holder's value for the asset. In fact, a low-type holder is anxious for the shock to arrive because it allows her to "pool" with higher quality assets. Thus, despite the fact that higher λ leads to a lower price upon arrival of the shock, G_L increases with λ as the holder is offered the lower price more quickly.



(a) $\lambda = 0.1$, $\phi = 1$

(b) $\lambda = 10$, $\phi = 1$

Figure 3: Equilibrium asset prices decrease with λ (right panel) and converge to fundamentals as $\lambda \to 0$ (left panel)

We now turn to exploring how the asset values depend on the quality of news (ϕ). As the news quality increases, the high type seller has more incentive to wait and the size of the no-trade region increases. However, as the Figure 4 illustrates, this does not necessarily imply lower asset values. The reason is that increasing ϕ "speeds things up." Market beliefs move more quickly through the no-trade region and the asset spends less time in the hands of an inefficient owner.

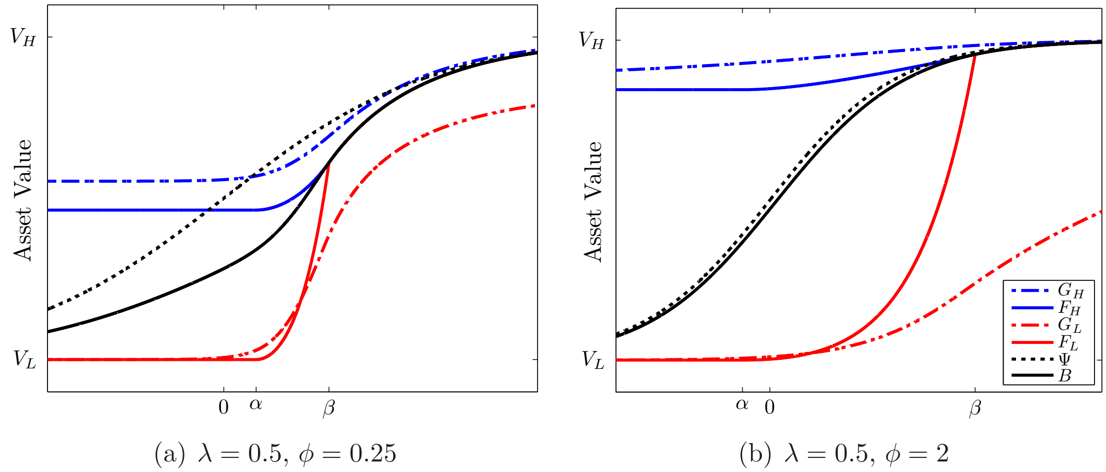


Figure 4: The effect of news quality on asset values

6 Conclusion

We have presented a model that studies the effect of gradual information arrival in a setting with both adverse selection and repeated trade.[11] Consider the insight gained from the mere description of the equilibrium. The market value of an asset varies with news and current market conditions even though fundamentals never change. Dramatic price volatility is a feature of equilibrium behavior for a subset of sample paths. For example, the price path of an asset that starts with favorable beliefs and then receives a steady stream of bad news will drop from above $B(\beta)$ to V_L with no trades (at intermediate prices) in between.

In equilibrium, a very small amount of bad news can cause a market to move from a fully liquid state to one in which liquidity completely *dries up*: when $z \in (\alpha, \beta)$ any price that a buyer is willing to offer is unacceptable to the seller.

A pessimistic market stabilizes through a partial sell-off at the lowest fundamental value. This creates a lower bound on market beliefs provided the seller has a credible reason to trade. Gradual information arrival and frequent trading opportunities are essential for this result.

Liquidity shocks decrease the asset value to both sellers and buyers. This result is fairly intuitive. The buyer's value depends on the price at which he can sell it in the future when he is hit with a shock. If beliefs lie below β when this occurs, then he may not trade immediately upon arrival of a shock. This distorts the buyer's value downward (below

[11] Korajczyk et al. (1992) and Lucas and McDonald (1990) the effect of information releases on equity issues in a setting with adverse selection.

Ψ) for all beliefs. More frequent shocks decrease the price at which buyers are willing to pay and depress asset values, despite fundamental values that remain constant—a less stable world is bad for asset trading. However, in a competitive market, the price at which assets are traded never drops below the fundamental value of a low-value asset.

With an infinite horizon, a low-value asset will eventually be "found out"—with probability one, it will trade at a price of V_L at $Z_t = \alpha$ for some finite t. Afterward buyers will only be willing to pay the fundamental value. This result is analogous to Bar-Isaac (2003) and Hendel and Lizzeri (1999).

A key feature of our model as well as those in the papers mentioned above is that agents possess information which is "long lived." Gârleanu and Pedersen (2003) study a model with private liquidity shocks and adverse selection. The private information about the asset in their model is short lived: it pertains only to cashflows arriving next period. They show that allocation costs arise and affect an asset's required return due to a the combination of a trader's private information about his liquidity preferences and private information about the asset's cashflows next period.

The observability of the shocks is an important feature of both the model and its application. The model with observable liquidity shocks corresponds to a marketplace dominated by large public companies whose balance sheets are publicly available. Traders and investors must be able to discern firms with liquidity needs and a credible reason for trading from the pure speculators. Though this may seem a daunting task, in the recent financial crisis it was not difficult to identify firms with liquidity needs (e.g., Bear Stearns, Lehman Brothers, AIG). In the model, when beliefs are favorable, the low-type holder is anxiously waiting for the shock to come allowing her to pool with a high-type seller at a price higher than the fundamental value of the asset. If shocks were unobservable, then a holder of a low-value asset would prefer to sell when in favorable market conditions before being hit by a shock, breaking the equilibrium. The moral is that an observable shock provides the owner with a credible reason to liquidate and provides evidence of this reason to the market. Without this, buyers face more severe exposure to the lemons problem.

On the other hand, a model with unobservable liquidity shocks corresponds to a marketplace either dominated by private firms (e.g., hedge funds) or one in which the identify of trading partners remains anonymous (e.g. dark pools), where the motivation for trading is often unclear. Formal analysis of a model with unobservable shocks and a comparison to the results in this paper seems a promising for future research and is one we have begun to investigate.

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A Appendix

A few comments about notation and terminology. The statement "for all z" means for all $z \in (-\infty, \infty)$. An asterisk in the superscript is used to denote a value function in equilibrium. Any value function without an asterisk refers to one computed by the algorithm unless otherwise noted. When we say Ξ_k is an/the unique equilibrium of Γ_k, we mean it characterizes an/the unique equilibrium of Γ_k in the sense of Proposition 4.3.

Proof of Lemma 3.1. Let $p(z)$ be as defined in the statement of the lemma and let $\eta_2 \equiv \frac{2(r+\lambda)}{\phi^2}$. From (24) and omitting the function arguments, we have that

$$B' = pG'_H + (1-p)G'_L + p'(G_H - G_L) \tag{30}$$
$$B'' = pG''_H + (1-p)G''_L + 2p'(G'_H - G'_L) + p''(G_H - G_L) \tag{31}$$

And therefore

$$\begin{aligned} B'' + B' - \eta_2 B &= p\left(G''_H + G'_H - \eta_2 G_H\right) + (1-p)\left(G''_L + G'_L - \eta_2 G_L\right) \\ &\quad + (p'' + p')(G_H - G_L) + 2p'(G'_H - G'_L) \\ &= p\left(G''_H + G'_H - \eta_2 G_H\right) + (1-p)\left(G''_L - G'_L - \eta_2 G_L\right) \\ &\quad + (p'' + p')(G_H - G_L) + 2p'(G'_H - G'_L) + 2(1-p)G'_L \end{aligned} \tag{32}$$

Using the definition of p, the last line of the above can be simplified:

$$\begin{aligned} (p'' + p')(G_H - G_L) + 2p'(G'_H - G'_L) + &2(1-p)G'_L \\ &= \frac{2}{1+e^z}\left(p'(G_H - G_L) + pG_H + (1-p)G_L\right) \\ &= \frac{2}{1+e^z} B' \\ &= 2(1-p)B' \end{aligned}$$

Substituting the above into (32) and rearranging gives

$$\begin{aligned} B'' + (2p-1)B' - \eta_2 B &= p\left(G''_H + G'_H - \eta_2 G_H\right) + (1-p)\left(G''_L + G'_L - \eta_2 G_L\right) \\ &= \frac{2}{\phi^2}\Big(p(\lambda F_H + v_H) + (1-p)(\lambda F_L + v_L)\Big) \end{aligned}$$

where the second inequality follows from substituting the right-hand side of (20) and (21) for the left and completes the proof. ◻

Proof of Lemma 4.1. By induction on k. For $k = 0$: that Step 1 of the algorithm results in a unique solution for $\{F_{L,0}, F_{H,0}, \alpha_0, \beta_0\}$ given $B_0 = \Psi$ is shown in Daley and Green (2010). It is also shown there that $F_{\theta,0}$ is weakly increasing, differentiable almost everywhere and

$$\begin{aligned}
\lim_{z\to\infty} F_{\theta,0}(z) &= V_H,\, \theta \in \{L,H\} \\
\lim_{z\to-\infty} F_{L,0}(z) &= V_L \\
\lim_{z\to-\infty} F_{H,0}(z) &\in (V_L, V_H)
\end{aligned}$$

Given $F_{\theta,0}$, $G_{H,0}$ is uniquely determined by (20) and (22)-(23) and $G_{L,0}$ is uniquely determined by (21) and (22)-(23). Furthermore, since $F_{\theta,0}$ is weakly increasing, $G_{\theta,0}$ is also weakly increasing.

Letting $B_1(z) = E_z[G_{\theta,0}(z)]$ for all z, we immediately have that (i) $B_0 \in C^2$, (ii) $B_1(z) \in [V_L, V_H]$, and (iii) $\lim_{z\to-\infty} B_1(z) = V_L$ and $\lim_{z\to\infty} B_1(z) = V_H$. Futhermore,

$$B_1'(z) = \frac{e^z}{1+e^z} G_{H,0}'(z) + \frac{1}{1+e^z} G_{L,0}'(z) + \frac{e^z}{1+e^z}\left(G_{H,0}(z) - G_{L,0}(z)\right) > 0$$

With the base case verified, assume the above holds for iteration $k-1$ at which point we are left with B_k, which satisfies (i)-(iii) from above and is weakly increasing.

To prove that there is a unique solution to the algorithm in iteration k, it will be useful to show a series of lemmas. First we will show that there exist two C^1 functions: $J_{L,k}$, which maps a lower boundary into an upper boundary that satisfies the low-type seller differential equations and boundary conditions and $J_{H,k}$, which maps a lower boundary into an upper boundary satisfying the high-type seller differential equation boundary conditions. Then we express the intersection of these two curves as the root of a continuous function and show that a unique real-valued root exists.

Lemma A.1. *$J_{L,k}$ is a well-defined, continuous and differentiable function. Further, for any α, $J_{L,k}(\alpha) > \alpha$, and $\lim_{\alpha\to-\infty} J_{L,k}(\alpha) = -\infty$.*

Proof. The general solution for the differential equation given in (10) is

$$F_{L,k}(z) = c_1 e^{u_1 z} + c_2 e^{u_2 z} + v_L/\bar{r} \tag{33}$$

(Polyanin and Zaitsev, 2003), where $(u_1, u_2) = \frac{1}{2}(1 \pm \sqrt{1+8\eta})$, $\eta = \frac{\bar{r}}{\phi^2}$ and c_1, c_2 are coefficients yet to be determined.

Given a lower boundary α and (33), solve (13) and (14) to get c_1, c_2.

$$c_1(\alpha) = \frac{-u_2(V_L - K_L)}{(u_1 - u_2)} e^{-u_1\alpha} \tag{34}$$

$$c_2(\alpha) = \frac{u_1(V_L - K_L)}{(u_1 - u_2)} e^{-u_2\alpha} \tag{35}$$

Notice that c_1 and c_2 are continuous and differentiable. Recall that $u_1 > 0 > u_2$, hence both c_1 and c_2 are positive, c_1 is decreasing in α, and c_2 is increasing in α.

Using boundary condition (15) gives an implicit expression for $J_{L,k}$

$$c_1(\alpha)e^{u_1 J_{L,k}(\alpha)} + c_2(\alpha)e^{u_2 J_{L,k}(\alpha)} + K_L - B_k(J_{L,k}(\alpha)) = 0 \tag{36}$$

First, notice that the function $c_1(\alpha)e^{u_1\beta} + c_2(\alpha)e^{u_2\beta} + vL/\bar{r} - B_k(\beta)$ is continuously differentiable in α and β. Starting from any pair (α, β) satisfying (36) with $J_{L,k}(\alpha) = \beta$, the implicit function theorem implies that $J_{L,k}$ is unique and continuously differentiable. That $J_{L,k}(\alpha) > \alpha$ is by definition. Finally, observe that as α goes to $-\infty$, $c_1 \to \infty$ and $c_2 \to 0$, implying $\lim_{\alpha\to-\infty} J_{L,k}(\alpha) = -\infty$. □

Lemma A.2. *$J_{H,k}$ is a well-defined, continuous and differentiable function. Further, for any α, $B_H(\alpha) > \alpha$, and $\lim_{\alpha\to-\infty} J_{H,k}(\alpha) \equiv \underline{\beta}_{H,k} > -\infty$.*

Proof. The general solution for the differential equation given in (9) is

$$F_{H,k}(z) = d_1 e^{q_1 z} + d_2 e^{q_2 z} + v_H/\bar{r} \tag{37}$$

(Polyanin and Zaitsev, 2003), where $(q_1, q_2) = \frac{1}{2}(-1 \pm \sqrt{1 + 8\eta})$, $\eta = \frac{\bar{r}}{\phi^2}$ and d_1, d_2 are coefficients yet to be determined.

First, define $\underline{\beta}_k$ implicitly by $B_k(\underline{\beta}_k) = v_H/\bar{r}$. The existence of $\underline{\beta}_k \in \mathbb{R}$ is guaranteed by $v_H/\bar{r} > V_L$. For the purpose of contradiction, suppose that for some α, $J_{H,k}(\alpha) < \underline{\beta}_k$. From equation (16), $F_{H,k}(J_{H,k}(\alpha)) = B_k(\alpha) < v_H/\bar{r}$. Calculating $F_{H,k}$ given this condition, a reflecting boundary at α, and the flow payoff of v_H, implies that $F_{H,k}$ is strictly decreasing on $[\alpha, J_{H,k}(\alpha)]$ in violation of equations (17) and (18). Hence, for all α, $J_{H,k}(\alpha) > \underline{\beta}_k$.

For a given a β, solve (16) and (17) to get

$$d_1(\beta) = \frac{B_k'(\beta) + q_2(v_H/\bar{r} - B_k(\beta))}{(q_1 - q_2)} e^{-q_1\beta} \tag{38}$$

$$d_2(\beta) = \frac{q_1(B_k(\beta) - v_H/\bar{r}) - B_k'(\beta)}{(q_1 - q_2)} e^{-q_2\beta} \tag{39}$$

Note that $d_1 > 0$ for all $\beta \in [\underline{\beta}_k, \infty)$. Using (18) we get an implicit expression for $J_{H,k}^{-1}$

$$d_1(\beta) q_1 e^{q_1 J_{H,k}^{-1}(\beta)} + d_2(\beta) q_2 e^{q_2 J_{H,k}^{-1}(\beta)} = 0 \tag{40}$$

In order for (40) to hold, a new lower bound on β arises: the first term on the LHS is always positive and thus the second must be negative. Since $q_2 < 0$, (40) requires that $d_2 > 0$, which from (39) then requires that $\Upsilon_k(\beta) \equiv q_1(B_k(\beta) - v_H/\bar{r}) - B_k'(\beta) > 0$. This expression is negative for β small (note $\Upsilon_k(\underline{\beta}_k) < 0$), eventually increasing and tends to $q_1(V_H - v_H/\bar{r}) > 0$ as $\beta \to \infty$. It has a unique real root which we denote by $\underline{\beta}_{H,k}$.

To see that $\lim_{\alpha \to -\infty} J_{H,k}(\alpha) \equiv \underline{\beta}_{H,k}$. Take $\alpha \to -\infty$, which implies that $q_1 e^{q_1\alpha} \to 0$ and $q_2 e^{q_2\alpha} \to -\infty$. Since $d_1(\beta) < \infty$ for all $\beta \in [\underline{\beta}_k, \infty]$, (40) requires that $d_2(\beta) \to 0$ implying that β must converge to $\underline{\beta}_{H,k}$.

The function $d_1(\beta) q_1 e^{q_1\alpha} + d_2(\beta) q_2 e^{q_2\alpha}$ is continuously differentiable in α and β. Starting from any pair (α, β) satisfying (40) with $J_{H,k}^{-1}(\beta) = \alpha$, the implicit function theorem implies that $J_{H,k}^{-1}$ is a continuously differentiable function. By the inverse function theorem, so too is $J_{H,k}$. □

Lemma A.3. *The two curves B_H, B_L intersect exactly once: specifically $J_{L,k}$ crosses $J_{H,k}$ from below.*

Proof. To be completed. Follows a similar argument as given in the proof of Lemma A.3 in Daley and Green (2010). □

Lemmas (A.1)–(A.3) imply a unique solution for the algorithm in iteration k completing the induction step and the proof of Lemma 4.1 □

Proof of Lemma 4.2. To be completed. □

The following properties of the value functions computed by the algorithm will be useful and are straightforward to check.

Fact A.4. *For all $k < \infty$,*

1. $F_{L,k}(z) \leq B_k(z) \leq F_{H,k}(z)$ *for all* z.

2. *$F_{\theta,k}(z)$ is weakly increasing in z.*

3. *$G_{L,k}(z) < B_k(z) < G_{H,k}(z)$ for all z.*

4. *$G_{\theta,k}(z)$ is increasing in z.*

Proof of Proposition 4.3. The proof is by induction on k. The base case, that Ξ_0 is an equilibrium for $(k, i) = (0, 1)$, was shown in Daley and Green (2010) Theorem 3.1. When $(k, i) = (0, 0)$, because when there are no shocks left to come, a high-type holder has the maximal value for an asset and will settle for nothing less than V_H. A buyer would expect to lose money on any such offer and therefore a high-type holder will never trade. Any trade that occurs between a low-type holder and a buyer must occur at V_L and is payoff irrelevant. For convenience and without affecting any results that follow, we will assume that such a trade does not occur. Therefore, Ξ_0 is an equilibrium of Γ_0.

For the inductive step, assume the proposition holds when the number of shocks left is $k-1$. The equilibrium prescribes that strategies in any subgame of Γ_K in which there are k shocks are identical to Γ_k. Thus, it suffices to prove there are no profitable deviations when there are k shocks left to come in Γ_k. Note that by following the equilibrium strategies, an asset owner's value is given by the value computed in the algorithm.

Suppose first that $i = 0$ in which case the equilibrium prescribes that buyers make non-serious offers $w(z, 0, k) < G_{L,k}(z)$ and no trade occurs. Since it is common knowledge that there are no gains from trade. The no-trade theorem can be invoked after we specify off-path beliefs which remain unchanged following a deviation. That is, any trade which occurs when $i = 0$ must be payoff equivalent to following the equilibrium strategies and no profitable deviations exist.

Now suppose that $i = 1$. We specify that beliefs remain unchanged following an unexpected rejection and place probability one on a low type following an unexpected acceptance.

Claim A.5. *No buyer can profitably deviate.*

Proof. First, consider any $z \in (\alpha_k, \beta_k)$ and recall that $F_{H,k}(z) > B_k(z) > F_{L,k}(z) > V_L$. If a buyer deviates to an offer that will be accepted by the high type it will also be accepted by the low type and hence earn negative expected profits. Likewise, an offer which attracts only the low type also loses money. For any belief $z \notin (\alpha_k, \beta_k)$, if a buyer deviates to $w < w(z, 1, k)$ (where $w(z, 1, k)$ is as specified by ξ_k), then his offer is ignored, and he earns zero profit. For $z \geq \beta_k$, any offer $w_i > B_k(z) = F_{L,k}(z) = F_{H,k}(z)$ is accepted by both types and earns negative expected profit. For $z \leq \alpha_k$, an offer $w > V_L$

either attracts only the low type or attracts both types (if $w > F_{H,k}(z) > B_k(z)$). In either case, it earns negative expected profit. $\square$

That the seller's strategy is optimal follows by construction: given w and Z, $F_{\theta,k}(z)$ solves (SP_θ) if $\mathcal{A}^\theta F_{\theta,k}(z) = 0$ for all z such that θ rejects and $\mathcal{A}^\theta F_{\theta,k}(z) < 0$ for all z such that θ accepts, where $\mathcal{A}^\theta$ is type θ differential operator. Noting that from (27), $\mathcal{A}^\theta B_k < 0$ for all $z > \beta_k$ completes the proof. $\square$

Proof of Theorem 4.4. Suppose that ξ is not an equilibrium of Γ_∞. Then there exists some $\delta > 0$ and a deviation for some player in some state (z, i) such that the deviation generates a payoff to that player which is δ higher than the payoff specified by ξ. Choose ε_1, $0 < \varepsilon_1 < \delta$, and the subsequence $n(i)$ such that $d(\xi, \xi_{n(i)}) < \varepsilon_1$ for all $i > \bar{K}$. Let n denote any such $n(i)$. In Γ_n, for any ε_2, $0 < \varepsilon_2 < \varepsilon_1$, there is a z', $|z' - z| < \varepsilon_2$, such that the strategies prescribed by ξ in state (z, i) are identical to those prescribed by ξ_n in state (z', i, n). By the continuity of $(B_k, F_{\theta,k}, G_{\theta,k})$, for any $\varepsilon_3 > 0$, we can choose ε_2 small enough so that

$$\begin{aligned}
|B(z) - B_n(z')| + \sum_\theta |F_\theta(z) - F_{\theta,n}(z')| + |G_\theta(z) - G_{\theta,n}(z')| \leq& \\
d(\xi, \xi_n) + |B(z) - B_n(z')| + \sum_\theta |F_\theta(z) - F_{\theta,n}(z')| + |G_\theta(z) - G_{\theta,n}(z')|& \\
< \varepsilon_1 + \varepsilon_3&
\end{aligned}$$

Let $\varepsilon_1 = \varepsilon_3 = \delta/3$. Then the same deviation in the state (z', i) of Γ_n generates a payoff of at least $\delta/3$ more payoff than the strategy prescribed by ξ_n in the state (z', i, n) contradicting Proposition 4.3. $\square$