Optimal Double Auction

Alex Boulatov[1], Terrence Hendershott[2], and Dmitry Livdan[3]

[1]Center for Advanced Studies, Higher School of Economics, Moscow. email: aboulatov@hse.ru.
[2]Haas School of Business, University of California, Berkeley. email: hender@haas.berkeley.edu.
[3]Haas School of Business, University of California, Berkeley. email: livdan@haas.berkeley.edu.

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[Very Preliminary]

Abstract

We consider a private value double-auction model which can be viewed as an extension of the model of Riley and Samuelson (1981) to the case of multiple sellers. Each seller has one unit of the same asset for sale and buyers compete to purchase one unit of the asset from one of the sellers. In the model sellers costlessly adjust their reserve prices, while sequentially moving buyers optimally choose between sellers to maximize their respective expected payoffs. The reserve price plays the role of a choice variable in the model, for both types of players. The refinement we propose in this paper is based on the following trade-off. From the sellers' perspective the probability of selling the good is increasing in the number of buyers bidding for it and, therefore, sellers have the incentive to decrease their reserve price to attract more buyers. From buyers' perspective, the probability of winning the auction decreases in the number of buyers and therefore buyers with higher private valuations are better off choosing sellers with slightly higher reserve price but fewer number of buyers. We exploit this trade-off to construct the symmetric Nash equilibrium in pure strategies in this double auction. Our first result is that such equilibrium exists and it is unique in the case of two sellers and the even number of buyers and it does not exist when the number of buyers is odd. The equilibrium reserve price is lower than the reserve price set by the monopolist (Riley and Samuelson (1981)) and each seller is visited by the same number of buyers. This is in contrast to the result of Burguet and Sakovics (1999) that there are no symmetric pure-strategy equilibria in the case of two competing sellers when buyers move simultaneously. Our second result is for the case of the number of buyers equal to the integer fraction m of the number of sellers. We show that the symmetric pure-strategy Nash equilibrium exists when m is below an endogenous cut-off m^* which we fully characterize. This equilibrium is not unique as it is supported on the interval of the reserve prices.

1. Introduction

In standard auction theory, the bare bones of the model involves competition on only one side of the market: A single seller of an indivisible good faces a number of potential buyers. In such setting the seller acts as a monopolist by capturing all the informational rents. In practice, however, sellers often do not have monopoly power but instead have to compete against other sellers, giving buyers the opportunity to choose among many auctions. A prime example is online auction sites where many different sellers are selling similar objects, and thus the sellers are in direct competition. Real estate market where current home owners listing their houses for sale compete for buyers is another prime example.

Once the assumption of the monopolist seller is relaxed, several natural questions for analysis arise: First, what form does the competition among the sellers take under the most common auction procedures? For example, does symmetric equilibrium in pure strategies exist and if it does exists then what are the necessary conditions? Second, what amount of the monopoly rents is lost as a function of the number of sellers? In turn, by what means can the sellers minimize loses due to competition? Third, what are the determinants of a sale price? For example, how does the equilibrium reserve price depend on the market size, the buyer to seller ratio, and other market characteristics? Finally, under what conditions is the symmetric equilibrium unique?

These are important questions that have, surprisingly, attracted little attention in the auction literature. McAffee (1993), Peters and Severinov (1997), and Peters (1997), appealing to a large market hypothesis, have looked for an equilibrium in a setting where large number of sellers compete by posting auctions setting the reserve price as they wish, and the buyers decide which seller to visit given the reserve prices posted. In the competitive limit of a large number of sellers, their conclusion is that in equilibrium, assuming zero production costs, sellers set the efficient mechanisms that are equivalent to zero reserve price, second price auctions. While the assumption of perfectly competitive sellers offers several important advantages, for instance it allows for a general mechanism design problem where the sellers compete in direct mechanisms (McAffee (1993)), it also ignores the strategic nature of the sellers' competition in which sellers take into account all the implications of a change in selling mechanism. The imperfect competition among sellers of identical goods can lead to different equilibrium outcomes since it creates a fundamental difference between prices and reserve prices. In the case of a monopoly (see Bulow and Roberts (1989)) or perfect competition, a reserve price is important only when it is to become the price paid in the auction.

When both sellers and buyers act strategically, however, the reserve price matters even when it is not going to determine the transaction price: it always affects the self-selection of buyers and thus the distribution of the "residual" demand.

Burguet and Sákovics (1999) (BS thereafter) have been first to study the imperfect competition between two sellers of one unit of the same good with zero production costs. They find that, unlike the case of perfectly competitive sellers, symmetric pure-strategy equilibrium for the second price sealed bid auction does not exist. This is because the only possible symmetric pure-strategy equilibrium reserve prices are equal to zero and since it is costless to increase the reserve price, since only buyers with the lowest possible valuations (which are zero) are lost, one of the sellers is better off deviating to a small reserve price thus making this equilibrium unstable. Mathematically, this result means that local second order condition of seller optimization fails precisely when reserve prices are zero and, furthermore, the first order condition alone is not sufficient for seller optimization. BS (1999) go on to prove the existence of a mixed-strategy symmetric equilibrium where the reserve prices are positive with probability one, but do not obtain detailed characterization of such an equilibrium.

Virag (2010) extends BS (1999) results to a setting with more than two but a finite number sellers. He follows BS (1999) to show that symmetric pure-strategy equilibrium for the second price sealed bid auction does not exist when the production costs are zero. Virag (2010) then goes on to demonstrate that when the support of the buyers' valuation is finite and bounded away from zero, then the symmetric pure-strategy equilibrium does exists. He also provides intuitive comparative statics results for the equilibrium reserve price and also characterize the rate of convergence of the reserve price to zero.

Models of BS (1999) and Virag (2010) share the same assumption: Buyers move simultaneously. Under this assumption the only decision a buyer needs to make is which seller to visit and both papers concentrate on equilibria in which buyers employ symmetric visiting strategies. This requirement means that the visiting decision of a buyer depends only on his valuation for the object, but not on the name of the buyer and captures the notion that in *large* markets buyers are unable to coordinate their actions and behave in an anonymous manner. While this assumption simplifies the buyers' game, it is quite restrictive to a large number of buyers and not very realistic. In the online auctions the number of buyers can be quite small and they move sequentially thus coordinating their actions. This is also true in the case of other major applications of the double auctions: Real estate sales and takeovers.

In this paper we analyze an imperfect competition in a double auction setting (see, e.g. Das and Sundaram, (1996)) with a finite total number of both buyers and sellers. Our model shares most major features with settings of BS (1999) and Virag (2010); single non-divisible good produced at zero cost is traded and both sellers, who move first by announcing their reserve prices, and buyers have heterogeneous private valuations. We, however, relax the assumption that buyers move simultaneously and allow them to move sequentially in a nature chosen order. In addition, in the spirit of Riley and Samuelson (1981), we allow for examination of alternative forms of auctions[1] or, in other words, optimal double auctions. In our setting this boils down to both buyers and sellers choosing their bidding strategies and reserve prices characterizing a particular market design. When sellers strategically compete for buyers then any change in the rules of the auction forces buyers to choose not only the optimal bid but also the right seller, which in turn forces sellers to set reserve prices by taking into account buyers' optimal bidding strategies. This is not without a price: while BS (1999) and Virag (2010) consider a general distribution function with support on the interval $[0, 1]$ for players' beliefs, we are forced to carry out our analysis using uniform distribution on the same interval.

When buyers move sequentially, they can observe the number of other buyers already visiting each seller and, therefore, condition their visiting strategies on this information by choosing a less crowded auction. Effectively, buyers coordinate their actions, albeit not perfectly. The refinement we propose in this paper utilizes this coordination property of the game and it is based on the following trade-offs: (i) reducing reserve price increases the probability of a sale but reduces the size of the reward; (ii) increasing the reserve price decreases the probability of sale but increases the size of the reward. We use these trade-offs to demonstrate that in equilibrium buyers play cutoff strategies: Buyers with valuations higher than the cutoff participate in the auction while buyers with valuations lower than the cutoff do not participate. The cutoff is endogenous as it depends on the reserve prices. The cutoff exhibits the crucial for our results property: The absolute value of its sensitivity to lower reserve price is less than the absolute value of its sensitivity to higher reserve price. In other words, if a seller with lower reserve price increases it, the loss in the demand due to such increase is less than the gain in the demand when the seller with higher reserve price decreases this reserve price by the same amount. This property ensures the existence of the fixed point in our setting for a wide range of parameters.

We exploit this trade-off and properties of the cutoff to construct the symmetric Nash equi-

[1]Effectively, sellers choose a market design mechanism characterized by the reserve price (see, e.g. Klemperer (2004)).

librium in pure strategies in this double auction. We first consider the case of two sellers and $N_b \geq 2$ buyers. When N_b is even, and contrary to the result of BS (1999), there exists a unique pure-strategy symmetric equilibrium in which both sellers are visited by the same number of buyers all playing the same equilibrium cutoff strategy, $\widehat{\mu} = \widehat{x}$ and set the same reserve price, $\widehat{x} = \frac{3-\sqrt{5}}{2}$. The equilibrium reserve price is independent of the number of buyers and is less than $\frac{1}{2}$, which is the reserve price in the case of the monopolist seller considered by Riley and Samuelson (1981). When N_b is odd we recover the result of BS (1999) that the pure-strategy symmetric equilibrium fails. It is because sellers compete for the extra buyer trying to increases the probability of sale. In the competitive limit $N_b \to \infty$ the coordination between buyers breaks down and, unlike results of Hernando-Veciana (2005) and Virag (2010), the pure-strategy symmetric equilibrium fails.

Lets consider the case of two buyers and two sellers to understand how this equilibrium works. Let sellers set their reserve prices to x_* and $x > x_*$, and consider buyers' game. Each buyer will be calculating his expected payoff from joining either auction. Lets say buyer who moves first chooses the seller with the lower reserve price. The second buyer now faces the following dilemma: (i) He can join the first buyer and face lower winning probability due to competition, but lower reserve price; (ii) Higher probability of winning the auction but higher reserve price. The higher is second bidder's private valuation, the higher is the chance that he is going to choose the seller with the higher reserve price. Then there exists such private valuation at which the second buyer is indifferent between the sellers. This valuation is the cutoff, which is the function of both reserve prices. Since we are looking for symmetric equilibria, this cutoff is the same for both bidders. Indeed, the first bidder did not have to go to the auction with the lower reserve price, he could have used the same cutoff strategy to pick the auction. Once we establish the buyers' optimal strategies, we turn to the sellers' problem. Sellers would like to set the same reserve price to maximize their respective expected utilities by taking into account buyers' cutoff strategies. By construction, the equilibrium requires that neither seller has the incentive to deviate a little either upward or downward from the equilibrium price, $\widehat{x}$. If either seller deviates upward by choosing higher reserve price he may be followed by one of the buyers who's valuation is greater than the cutoff. However, the cutoff's sensitivity to reserve prices discussed above guarantees that gains from increasing the reserve price are offset by the decline in the selling probability, which is decreasing in the reserve price faster than the reserve price itself. If either seller deviates downward by choosing lower reserve price he may be followed by both buyers. In this case, however, gains from increasing the selling probability are offset by the decline in the reserve price, since former is increasing in the reserve price slower than the reserve price itself. Therefore, neither seller has the incentive to deviate from the equilibrium reserve price $\widehat{x}$. It is clear how the equilibrium will work: Both sellers will set the same reserve prices equal to $\widehat{x}$ and buyers will go to the separate sellers. The ability of sellers to coordinate is crucial for our equilibrium since it allows sellers to go to different buyers irrespective of their moving order, i.e. the first buyer can pick either seller while the second buyer makes his pick after observing the pick of the first buyer. In the absence of the coordination this equilibrium breaks down.

Next we consider an arbitrary number of buyers and sellers restricted only by the their ratio being an integer, $m = \frac{N_b}{N_s}$. We show that there exists an endogenously determined "threshold" m beyond which the symmetric Nash equilibrium in pure strategies does not exist. The largest market for which symmetric equilibrium exists is equal to four buyers and four sellers. One important feature of the equilibrium is that it is not unique when the number of sellers is greater than two. It manifests itself in the equilibrium reserve price being on the finite interval, instead of the unique value. For example, in the case of three sellers and $m = 1$, i.e. three buyers, the symmetric equilibrium exists on the interval of reserve prices equal to $[0.382, 0.44]$. Specifically, This result helps explain why absolutely identical properties are often sold at different prices even when the market, i.e. number of sellers and buyers, is the same. The intuition behind multiplicity of equilibria hinges on the properties of sellers optimal strategies in response to one of the sellers deviating setting lower reserve price than other sellers. Sellers set the reserve price higher when the sensitivity of the buyers' cutoff to the reserve price is low, which happens to be the case when majority of the buyers always follow the downward deviating seller.

All multiple equilibria can be ranked based on the seller's expected payoff which helps to take the veil off the old question whether the competition always leads to lower prices. Since this expected payoff increases in the reserve price on the $(0, \frac{1}{2})$ interval, sellers will always pick equilibria with highest reserve prices. As a result we obtain a non-monotone relationship between the number of sellers and the reserve price, i.e. the reserve price in the two buyers and two sellers case is less than the reserve price in the four buyers and four sellers case which in turn is less than the reserve price in the three buyers and three sellers case. The competition between sellers is highlighted by the reserve price in the auction with two sellers. Therefore, the potential buyer is better off in the auction with two sellers. This results is due to the fact that, effectively, strategic sellers take advantage of the buyers who are "too smart for their own good." Sellers recognize that the reserve price is not the only variable in the buyers' decision making as it has to be combined with the buyers' estimate of the degree of competition with other buyers which affects their probability of winning the auction.

Therefore, strategic competition on both sides of the market has an offsetting effect: sellers desire to compete for more buyers to increase the probability of selling is counterweighted by the buyers' recognition that by competing against each other they are facing lower probability of winning the auction. The equilibrium entails where these two forces equalize.

Finally, our model is related to the work of Malenko and Gorbenko (2011) who consider a model with multiple buyers and sellers. In their setting, however, buyers do not observe their private valuations prior to choosing the auction where they participate (characterized by a reserve price), and, therefore, act competitively. Overall, Malenko and Gorbenko (2011) aim to solve the security design problem when there is more than one seller and buyers are allowed to bid using securities.

The rest of the paper is organized as follows. We start by formulating the model in Section 2. We solve the case of two buyers and two sellers in Subsection 2.1 and then extend our analysis to the case of two sellers and three buyers in Subsection 2.2. Subsection 2.3 covers the case of two sellers and N buyers, which is compared to the variant of BS (1999) model in Subsection 2.4. In Subsection 2.5 we extend our analysis to the case of arbitrary number of buyers and sellers. Section 3 concludes.

2. The Model

The model can be viewed as an extension of Riley and Samuelson (1981) to a double auction setting: multiple sellers maximize their expected payoffs by choosing reserve prices subject to buyers' strategies. Under the assumption that reserve prices are announced prior the trade taking place as in Riley and Samuelson (1981), reserve prices become "choice variables" in the double auction. In other words, different sellers may choose different reserve prices and different buyers may trade with different sellers characterized by those reserve prices.

Specifically, we consider the following auction setting. There are N_s identically informed sellers, each with one unit of an indivisible good (e.g., a residential house), and N_b of risk neutral buyers, each with a unit demand for the good. The total number of buyers, N_b, is a public knowledge. Buyers cannot enter more than one auction at a time and all players (buyers and sellers) behave noncooperatively. Each buyer i has a private valuation of the asset v_i and all valuations are independent and identically distributed (i.i.d.) uniform random variables, $v_i \backsim U[0,1]$. The i.i.d. assumption has been first presented by Vickery, and has been frequently employed in the auction literature, including Riley and Samuelson (1981). It implies that each party is uncertain about the others' reservation values, believing that each buyer decides on the maximum amount he is willing to bid independently of other buyers. In addition, all parties (buyers and sellers) share the common priors with respect to each other possible reservation value. Under the i.i.d. assumption the revenue equivalence holds in our setting, as long as the auction mechanism is efficient (i.e. the bidder with the highest valuation wins the auction). Finally, for the sake of simplicity, but without loss of generality, we assume that the each sellers' residual value of the asset is zero.

The timing of the game is simple. First, the N_s sellers simultaneously post their nonnegative reserve prices $\{X_j \in [0,1]\,,\ j = 1,..N_s\}$. After the reserve prices have been posted, buyers choose sellers in a sequential order randomly drawn by the nature. Both buyers and sellers take the bidding order as given. After the bidding order has been established, the buyers decide which seller to visit after observing their private valuations and all the reserve prices posted. At seller j buyers present observe the number of other buyers present and engage in a first price sealed-bid auction with reserve price X_j. After all bids have been submitted the bids are opened and the winner is determined. The winner of the auction is the buyer with the highest bid; in case of a tie, the seller flips a fair coin. The payment of the winner is equal to the reserve price if no other buyer visited seller j and is the highest other bid if there was a competing bid, while losing buyers do not pay. Each seller maximizes his expected revenue. Each buyer i obtains a von Neumann–Morgenstern utility (surplus) equal to his valuation v_i minus his payment b_j if he wins the auction, and zero otherwise. Finally, buyers maximize their expected utility.

Our main goal is to characterize *symmetric* pure strategies in this game. For each seller j, the pure strategy consists of choosing his reserve price $X \in [0,1]$ (the lowest acceptable bid) for the asset. Here we have dropped the seller specific subindex j since we are looking for symmetric pure strategies in the sellers' game. Given his valuation, v_i, each buyer's pure strategy is determined by two variables: the reserve price $X(v_i)$ characterizing a market design, and the optimal bidding strategy $b_i \equiv b(v_i, X)$ defined for any given type of the market X. In what follows we will show that the optimal bidding strategy reduces to choosing appropriate cutoff, μ. By choosing μ_i buyer i effectively picks the auction he is going to participate in and, therefore, the reserve price X. In what follows we show that the choice of μ subsumes buyer i's valuation which we are going to drop from further notations. We proceed in two steps from here by first outlining the optimal bidding strategy when multiple buyers compete with each other to buy the asset from a single seller (Riley and Samuelson (1981) setup). Next, we introduce an additional seller and discuss how competition between sellers changes buyers' optimal strategies.

Once a buyer chooses the auction to visit the game played at this auction is the same as in Reily and Samuelson (1981). In this game buyer i with the reserve price v_i who chooses a seller with the an optimal reserve price, $X = x_* \leq v_i$,[2] and visited by another $N_b - 1$ buyers, plays the optimal bidding strategy

$$b_i^*(v_i, x_*) = \frac{v_i}{N_b}\left[N_b - 1 + \left(\frac{x_*}{v_i}\right)^{N_b}\right], \tag{1}$$

which follows from Proposition 2 in Riley and Samuelson (1981). By playing (1) he gets optimal expected payoff

$$\pi_{x_*}^i(v_i|b_{-i}^*) = \int_{x_*}^{v_i} dv' P_{x_*}^i(v_i|b_{-i}^*), \tag{2}$$

conditional that other buyers choose a bidding strategies $b_{-i}^* = b(v_{-i}, x_*)$, where

$$P_{x_*}^i(v_i|b_{-i}^*) = v_i^{N_b - 1}, \tag{3}$$

is buyer i's winning probability derived in Lemma 1 in the Appendix. Expression (2) can be obtained by substituting equation (8b) into equation (4) in Riley and Samuelson (1981).

Next we outline the building blocks of the sellers' problem we plan to use throughout the paper. Denote by Π_X^i the expected payoff of seller i who chooses the reserve price X. The optimal seller's payoff when both buyers also bid optimally is defined by the envelope theorem.[3] Namely, we take into account that when the buyer i follows the optimal bidding strategy, $b_i^*(v_i, X)$, his optimal payoff if he chooses auction with the reserve price X while taking into account strategies of other buyers is equal to

$$\pi_X^i(v|\mu_{-i}) = P_X^i(v|\mu_{-i})\left(v - b_i^*(v_i, X)\right), \tag{4}$$

Rearranging terms in (4) yields

$$P_X^i(v|\mu_{-i})b_i^*(v_i, X) = P_X^i(v|\mu_{-i})\left(v - \left(v - b_i^*(v_i, X)\right)\right) = vP_X^i(v|\mu_{-i}) - \pi_X^i(v|\mu_{-i}), \tag{5}$$

and hence the optimal seller's i payoff if he sets reserve price to X is equal to

$$\Pi_X^i = \sum_{j=1}^{N_b} \int_X^{\mu_j} dv P_X^j(v|\mu_{-i}) b_i^*(v, X) = \sum_{j=1}^{N_b} \int_X^{\mu_j} dv \left(vP_X^j(v|\mu_{-i}) - \pi_X^i(v|\mu_{-i})\right). \tag{6}$$

It follows from (6) that in equilibrium, $X = \widehat{x}$ and $\mu_j = 1$, $j = 1, ..., N_b$, seller's expected payoff is

[2] It is value below which it is not worthwhile bidding.

[3] Which also leads to universality with respect to the particular auction mechanism (Riley and Samuelson, 1981).

equal to

$$\Pi^i_{X=\widehat{x}} = \underbrace{\frac{N_b-1}{N_b+1}}_{\text{2nd Order Statistic}} \left(1-\widehat{x}^{N_b+1}\right) + (1-\widehat{x})\widehat{x}^{N_b}, \tag{7}$$

and it is an increasing function of the reserve price, $\widehat{x}$, on the interval $\widehat{x} \in (0, \frac{1}{2})$.

To solve for the optimal reserve price, x_*, we consider both upward (increasing reserve price) and downward (decreasing reserve price) deviations by sellers and construct optimal responses by buyers to these deviations. Our refinement relies on the assumption that buyers arrive sequentially and therefore observe the number of other buyers already present at each seller. It follows from (3) that the probability of winning the auction for any buyer is decreasing in the total number of buyers visiting this auction due to the competition between buyers. We also exploit a simple single crossing property: A buyer with higher valuation is more eager to obtain the good and thus he is willing to pay more. Therefore, buyers with valuations higher than some endogenously determined value cutoff μ get higher expected surplus if they go to the seller who deviates upward (i.e. with higher reserve price), since the utility loss due to higher reserve price is offset by higher probability of winning the auction. Likewise, buyers with higher valuations have less incentive than buyers with low valuations to follow a seller who deviates downward (decreases his reserve price).

In the case of a single monopolist seller, $N_s = 1$, our model reduces to the optimal auction model considered in Riley and Samuelson, (1981). In this case, there always exists an optimal pure strategy for the monopolist. In particular, if $v_i \backsim U[0,1]$ and the asset has no "residual value," the optimal reserve price is given by $x_* = \frac{1}{2}$ in the case of two buyers. Since all sellers are ex-ante identical, it is natural to focus on symmetric Nash equilibria (NE). However, the existence of symmetric Nash equilibria in pure strategies with multiple sellers could be problematic, as pointed out by BS (1999). More precisely, this stems from the assumption that the lowest possible valuation of the buyers is equal to zero. As a consequence, the first order condition for seller optimally suggests that the sellers choose a zero reserve price. Then it is costless to increase the reserve price, since only buyers with the lowest possible valuations (which are zero) are lost. So the second order condition fails and a pure strategy equilibrium does not exist as was recognized by BS (1999) for the case of two sellers. To avoid the failure of the second order condition, Virag (2010) considers the "gap case," where the lowest possible valuation is a positive $a > 0$, and provides a sufficient condition for a pure strategy equilibrium to exist. However, Virag (2010) assumes buyers submit bids without observing the number of other buyers present. Our equilibrium concept relies exclusively on the buyers' ability to see the number of other buyers they compete against at each seller.

Since we will be looking for the symmetric pure strategy Nash equilibrium (NE thereafter) in the sellers' game, we have to find the reaction function of each seller with respect to the actions of other players. Before we proceed further, we need to explain what do we mean by the pure strategy NE in our model. In the model, sellers move first by announcing their reserve prices, and in this sense they act more like Stackelberg leaders (because the buyers can condition on their actions, the sellers have to strategically take into account the buyers' reactions.) However, we assume that all sellers (two in this particular example) move simultaneously, and in this sense we can talk about the NE in the sellers' sector. By the nature of the problem, this sector is also the most important, since the main action occurs in the sellers' sector of the market, and the buyers simply optimally react to sellers' actions.

The existence of such pure strategies NE depends on the interrelation of the model parameters. The key condition here is that the optimal boundary, x_d, for the downward deviations from the optimal reserve price, x_*, should be no less than the optimal boundary for the upward deviations, x_u, i.e. we should have

$$x_d \geq x_u, \quad x_{d(u)} \in [x_*, 1]\,, ^4. \tag{8}$$

The intuition behind this is simple. Consider the upward deviations first. According to the above results, no seller would deviate upwards if (and only if) the initial (conjectured) equilibrium is at $x_* \geq x_u$. This means that the symmetric NE are only possible in the range $x_* \geq x_u$, since the NE should be stable w.r.t. the potential upward deviations of the sellers. Analogous, the stability w.r.t. the downward deviations requires that $x_* \leq x_d$.

We use the case of $N_b = N_s = 2$ to illustrate the construct of equilibrium. Next, we review the results of BS (1999), and show that the revenue equivalence result still holds in their setting, provided that the auction mechanism is efficient. Then we extend our analysis to a more general case of $N_b = N$ and $N_s = 2$, and finally, to the case of $N_b = mN_s$.

2.1. Two Buyers and Two Sellers ($N_b = N_s = 2$)

We start by constructing bidding strategies of interest for the case of the upward deviation by one of the sellers to reserve price $x > x_*$. We consider the buyers' expected payoffs if they respond by joining the auctions characterized by the reserve prices x_* and x respectively. We arbitrarily

[4] In fact, we show that $x_u = \frac{3-\sqrt{5}}{2} \approx 0.382$.

and without loss of generality assume that the buyer's 2 valuation is no less than x, $v_2 \geq x$, and therefore he indeed chooses between the two auctions. Lemma 2 summarizes our results.

LEMMA 2: *The optimal buyers' response to one of the sellers deviating from the reserve price x_* to a higher reserve price $x > x_*$ is characterized by the cut-off reaction function:* (1) *If buyer* 1 *stays at x_* then buyer* 2 *stays at x_* if $v_2 \leq \mu_*$, and he moves to $x > x_*$ if $v_2 > \mu_*$;* (2) *If buyer* 1 *moves to $x > x_*$ then buyer* 2 *stays at x_*. The cut-off valuation μ_* is given by $\mu_* = 1 - \sqrt{1 + x_*^2 - 2x}$.*

As Lemma 2 shows, buyers with high enough valuation, $v_2 > \mu_*$, is better off switching to the seller with higher reserve price if the other buyer chooses the seller with low reserve price. Alternatively, Lemma 2 implies that the ex-ante probability that one of the buyers stays at the auction with the lower reserve x_* when the second seller deviates to the higher reserve x, equals exactly the cut-off value μ_*. Note that

$$x_* < x \leq \mu_* \leq 1, \tag{9}$$

which means that the cutoff is higher than the highest reserve price. We are going to use the cut-off strategies from Lemma 2 characterized by cutoffs μ_i to solve for optimal double auction in the case of two buyers and two sellers.

We start by adjusting the timeline of the game for buyers' optimal strategies when $N_b = N_s = 2$. First, sellers simultaneously choose their reserve prices, x_* or $x > x_*$, anticipating the actions of each other and the buyers' reactions. Next, both buyers observe the reserve prices, $\{x_*, x\}$, and their own private valuations, v_i. Next, nature randomly chooses the sequential order in which buyers move after which each buyer uses all available information to calculate his cutoff μ_i, $i = 1, 2$. Both buyers and sellers take the bidding order as given. Finally, each buyer chooses either low reserve price (LRP), x_*, or high reserve price (HRP), x, auctions, bidding process at each auction takes place, and the payoffs are being realized.

Proposition 1 summarizes buyers' optimal strategies.

PROPOSITION 1: *Buyers follow the following cut-off strategies: (i) If both sellers choose the same reserve price, x_*, buyers go to separate sellers; (ii) If one of the sellers deviates upward ($x > x_*$) or downward ($x < x_*$), then one buyer always chooses LRP auction and the other buyer chooses LRP auction if his private valuation is lower than a cutoff $\widehat{\mu}(x, x_*) = \mu_*$, and he otherwise chooses HRP auction.*

Proof: See Appendix.

The intuition behind the Proposition 1 is quite transparent. From the sellers' perspective only two outcomes are possible if one of them chooses to deviate upward or downward: (i) one of the buyers goes to the LRP auction and the other buyer goes to the HRP auction; (ii) both buyers go to the LRP auction. Since both buyers are ex ante identical to the sellers, it really does not matter to the sellers which buyer goes to the HRP auction, as long as either buyer chooses the same cutoff $\widehat{\mu}(x, x_*)$. If the lead buyer plays cutoff strategy $\mu_1 = \widehat{\mu}(x, x_*)$ and goes to the HRP auction, the second buyer always plays $\mu_2 = 1$ and goes to the LRP auction. If the lead buyer goes to the LRP auction, the second buyer plays the cutoff strategy $\mu_2 = \widehat{\mu}(x, x_*)$ and goes either to LRP or HRP auction. *Therefore, from the sellers' perspective the outcome is equivalent to one of the buyers always going to the LRP auction and another buyer using cutoff strategy $\widehat{\mu}(x, x_*)$ to choose between LRP and HRP auctions.* It is also follows from Proposition 1 that optimal bidding strategies of buyers are fully characterized by their cutoffs. Therefore, we are going to use cutoffs when conditioning on bidding strategies.

This buyers' allocation across the two auctions is ex-post optimal. The reason why this works is simple: the main focus of the model is on a competition between the multiple sellers, who only care about their expected payoffs when they choose different reserve prices, which are determined by the aggregate number of potential buyers, and not by the individual strategy of each particular buyer. In other words, the sellers care about their expected payoffs which depend only on the buyers' allocation across different reserve prices, which involves some "aggregation" across the buyers. For example, in the case of the two buyers and two sellers considered above, the buyers' allocation is completely characterized by the rule "one buyer stays at x_*, the other one goes to x_* or to x depending on a cut-off parameter μ, regardless on who of the two buyers chooses the LRP auction." This is because, according to our assumption, the bidding takes place after the buyers choose the auction (reserve price) at which they are willing to participate.

We have two countervailing forces influencing sellers' pricing decision in our setting: (i) reducing reserve price increases the probability of sale but reduces the size of the reward; (ii) increasing the reserve price decreases the probability of sale but increases the size of the reward. For the symmetric NE to exist we need to demonstrate two results. First, when one of the sellers deviates downward from the equilibrium price, the increase in the seller's expected utility due to increase of the probability of sale is completely offset by reduction in the size of the reward. Likewise, when one of the sellers deviates upward from the equilibrium price, the increase in the seller's expected utility due to increase in the size of the reward is completely offset by the decrease of the probability of sale. This is possible in our setting since the cutoff and, therefore, the probability of sales, responds asymmetrically to downward and upward deviations respectively

$$\left|\frac{\partial\widehat{\mu}(x,x_*)}{\partial x_*}|_{x=x_*}\right| = \frac{x_*}{1-x_*} < \frac{\partial\widehat{\mu}(x,x_*)}{\partial x}|_{x=x_*} = \frac{1}{1-x_*},\ 0 \leq x_* < 1. \tag{10}$$

As a result we find that there exists a unique symmetric pure strategy NE as summarized by Theorem 1.

THEOREM 1: *In the market with two sellers and two buyers, there exists a unique symmetric pure strategy NE in the sellers game: Both sellers set reserve prices equal to*

$$\widehat{x} \equiv \frac{3-\sqrt{5}}{2}, \tag{11}$$

and buyers go to separate sellers. Buyers bid

$$\widehat{b}_i\left(v_i,\widehat{x}\right) = \widehat{x}, \tag{12}$$

and the expected payoff of each seller is equal to

$$\Pi^i_{X=\widehat{x}} = \widehat{x}(1-\widehat{x}) = \frac{1}{\sqrt{5}+2}. \tag{13}$$

Proof: See Appendix.

Theorem 1 shows that it is optimal for both sellers to choose the same reserve price $\widehat{x}$ and it is optimal for the buyers to choose separate auctions irrespective of their relative private valuations. At the equilibrium reserve price each seller's probability of selling the item is $1-\widehat{x}$. Optimality means that profits of sellers are maximized and neither seller has incentive to deviate to lower (higher) reserve price. If either seller deviates upward by choosing higher reserve price he may be followed by one of the buyers who's valuation is greater than the cutoff. However, the cutoff's sensitivity to reserve prices discussed above guarantees that gains from increasing the reserve price are offset by the decline in the selling probability, which is decreasing in the reserve price faster than the reserve price itself. If either seller deviates downward by choosing lower reserve price he may be followed by both buyers. In this case, however, gains from increasing the selling probability are offset by the decline in the reserve price, since former is increasing in the reserve price slower than the reserve price itself. Therefore, neither seller has the incentive to deviate from the equilibrium reserve price $\widehat{x}$.

Theorem 1 demonstrates that the competition between sellers erodes their expected[5] and realized profits relative to the case of the monopolist seller considered by Riley and Samuelson (1981). The equilibrium reserve price $\widehat{x}$ is less than $\frac{1}{2}$, which is the reserve price in the case of the monopolist seller.

Before considering the most general case of large markets we need to understand what happens to the buyer's strategies when the number of buyers is different from the number of sellers. The next section illustrates the optimal buyers' strategies in the case of three buyers and two sellers.

2.2. Three Buyers and Two Sellers ($N_b = 3$ and $N_s = 2$)

Following the previous case of two buyers and two sellers we start with the buyers' problem and then continue with the sellers' problem. In this case sellers face two possible outcomes: (i) Two buyers choose the LRP auction and one buyer chooses the HRP auction; (ii) All three buyers choose the LRP auction. Proposition 2 summarizes buyers' optimal strategies.

PROPOSITION 2: *Buyers follow the following cut-off strategies: If one of the sellers deviates upward ($x > x_*$) or downward ($x < x_*$), one of the buyers chooses the LRP auction if his private valuation is lower than a cutoff $\widehat{\mu}(x, x_*)$, which solves*

$$\widehat{\mu}^3 - x_*^3 = 3(\widehat{\mu} - x), \tag{14}$$

and he otherwise chooses the HRP auction, while other buyers always choose the LRP auction.

Proof: See Appendix.

The intuition behind the Proposition 2 is quite similar to the intuition behind Proposition 1. Since all three buyers buyers are ex ante identical to the sellers, it does not matter to the sellers which buyer goes to HRP, as long as all buyer choose the same cutoff $\widehat{\mu}(x, x_*)$. If the lead buyer plays cutoff strategy $\mu_1 = \widehat{\mu}(x, x_*)$ and goes to the HRP auction, both the second and third buyers always play $\mu_2 = \mu_3 = 1$ and go to the LRP auction since for them the effect of lower winning

[5]The expected profits in the monopolistic case with two buyers are equal to $\frac{1}{4} > \frac{1}{\sqrt{5}+2}$.

probability on their expected payoffs is swept by the effect of the lower reserve price. If the lead buyer goes to the LRP auction, $\mu_1 = 1$, the second buyer plays the cutoff strategy $\mu_2 = \widehat{\mu}(x, x_*)$ and goes either to LRP or HRP auction. If the second buyer goes to the HRP auction, the third buyer always goes to the LRP auction, $\mu_3 = 1$. If the second buyer goes to the LRP auction, the third buyer now plays the cutoff strategy $\mu_3 = \widehat{\mu}(x, x_*)$ and goes either to LRP or HRP auction. Therefore, from the sellers' perspective the outcome is equivalent to two of the buyers always going to the LRP auction and the third buyer using cutoff strategy $\widehat{\mu}(x, x_*)$ to choose between LRP and HRP auctions. Once again, optimal bidding strategies of buyers are fully characterized by their cutoffs.

It immediately follows from equation (14) that its solution, $\widehat{\mu}(x, x_*)$ satisfies[6]

$$\widehat{\mu}(x_*, x_*) = x_*, \tag{15}$$

and its partial derivatives with respect to x_* and x are equal to

$$\frac{\partial\widehat{\mu}(x, x_*)}{\partial x_*}|_{x=x_*} = -\frac{x_*^2}{1 - x_*^2} \text{ and } \frac{\partial\widehat{\mu}(x_*, x_*)}{\partial x}|_{x=x_*} = \frac{1}{1 - x_*^2}. \tag{16}$$

We now turn our attention to the seller's problem. Without loss of generality we consider the first buyer playing the cutoff strategy, $\mu_1 = \widehat{\mu}(x, x_*)$.

We start with the upward deviation, i.e. choosing the HRP auction with $X = x$, by one of the sellers. In this case only the first buyer follows and the expected seller's payoff, $\Pi^i_{X=x}$, is given by (121). In order to find the optimal reserve price we optimize $\Pi^i_{X=x}$ with respect to x taking into account that $\widehat{\mu}$ is itself a function of x to obtain

$$\frac{\partial\Pi^i_{X=x}}{\partial x}|_{x=x_*} = 1 - x_* - x_*\frac{\partial\widehat{\mu}(x, x_*)}{\partial x}|_{x=x_*} = \frac{1 - 2x_* - x_*^2 + x_*^3}{1 - x_*} = 0, \tag{17}$$

which has only one solution in the interval $(0, 1)$, $x_u = 0.445$.

Consider now downward deviation, i.e. choosing the LRP auction with $X = x_*$, by one of the

[6] We can dismiss the roots of quadratic equation since one of them is negative and the other is greater than 1 and we are looking for $\mu_1 < 1$.

sellers. The expected payoff of this seller, i, is equal to

$$\Pi^i_{X=x_*} = \sum_{j=1}^{3} \int_{x_*}^{\mu_j} dv(2v-\mu_j) \prod_{k\neq j}^{3} P^j_{X=x_*}(v|\mu_k) = \int_{x_*}^{\mu_1} dv(2v-\mu_1) P^1_{X=x_*}(v|\mu_2)\, P^1_{X=x_*}(v|\mu_3) + \tag{18}$$
$$\int_{x_*}^{1} dv(2v-1) P^2_{X=x_*}(v|\mu_1)\, P^2_{X=x_*}(v|\mu_3) + \int_{x_*}^{1} dv(2v-1) P^3_{X=x_*}(v|\mu_1)\, P^3_{X=x_*}(v|\mu_2)\,,$$

where we have used relations (6) and (116). Substituting probabilities (106) into (18) yields

$$\Pi^i_{X=x_*} = \int_{x_*}^{\widehat{\mu}} dv(2v-\widehat{\mu})v^2 + 2\int_{x_*}^{\widehat{\mu}} dv(2v-1)(1-\widehat{\mu}+v)v + 2\int_{\widehat{\mu}}^{1} dv(2v-1)v = \tag{19}$$

$$= \int_{x_*}^{\widehat{\mu}} dv(2v-\widehat{\mu})v^2 + 2\int_{x_*}^{\widehat{\mu}} dv(2v-1)(v-\widehat{\mu})v + 2\int_{x_*}^{1} dv(2v-1)v. \tag{20}$$

Using expression (19) the first order condition for the optimal LRP reserve price takes the following form

$$\frac{\partial \Pi^i_{X=x_*}}{\partial x_*}|_{x=x_*} = \left(\frac{\partial \widehat{\mu}(x_*,x_*)}{\partial x_*}|_{x=x_*} - 1\right) x_*^3 - 2x_*\left(2x_*-1\right) = \tag{21}$$
$$= -\frac{x_*}{1-x_*^2}\left(x_*^2 - 2(1-x_*^2)(2x_*-1)\right) = 0,$$

which has only one solution in the interval $(0,1)$, $x_d = 0.44$. Since $x_u > x_d$ symmetric NE does not exists. This result attains because the seller with lower reserve price always attracts no less than two buyers. Therefore, each seller's expected gains from the increased winning probability always outweigh expected gains from the higher reserve price.

We now going to build on our results from the above cases and consider the case of N buyers and two sellers.

2.3. $N_b = N$ *and* $N_s = 2$

Once again we start with the buyers' problem and then continue with the sellers' problem. Following previous discussion, two cases should be considered separately: N is even, $N = 2K$, and N is odd, $N = 2K + 1$, where K is a positive integer. We conjecture that buyers play the same cut-off strategy, $\widehat{\mu}_\gamma$, where $\gamma = 1$ when the number of buyers is even and $\gamma = 2$ when the number of buyers is odd, while allowing for the different cut-off strategies across two cases. We will later verify that this conjecture holds true in equilibrium.

One immediate observation is that there always be no less than K buyers in the LRP auction since any buyer who deviates to the HRP faces lower winning probability and higher reserve price. Since all buyers are identical from the sellers perspective, their optimal strategies can be viewed as if K of them always going to the LRP auction and the remaining $N-K$ buyers playing the same cut-off strategy, $\widehat{\mu}_\gamma$. Proposition 3 formalizes this intuition.

PROPOSITION 3: *In the case of N buyers and 2 sellers, K buyers follow the same cutoff strategy*

$$\widehat{\mu}_\gamma(x, x_*) = x_* + \epsilon \frac{1}{1-x_*^\gamma} + O(\epsilon^2),\ \gamma = \begin{cases} 2,\ N = 2K+1, \\ 1,\ N = 2K, \end{cases},\ \epsilon = x - x_* \to 0, \tag{22}$$

by choosing LRP/HRP auction if their private valuation is less/greater than $\widehat{\mu}_\gamma(x, x_*)$, while other $N-K$ buyers always choose LRP auction.

Proof: See Appendix.

One important implication of Proposition 3 is that the optimal cut-off, $\widehat{\mu}_\gamma$, indeed depends on whether the total number of buyers is even, $\gamma = 1$, or odd, $\gamma = 2$. It immediately follows from equation (22) that partial derivatives of $\widehat{\mu}_\gamma(x, x_*)$ with respect to x_* and x are equal to

$$\begin{aligned} \frac{\partial \widehat{\mu}_\gamma(x, x_*)}{\partial x_*}|_{x=x_*} &= -\frac{x_*^\gamma}{1-x_*^\gamma}, \\ \frac{\partial \widehat{\mu}_\gamma(x, x_*)}{\partial x}|_{x=x_*} &= \frac{1}{1-x_*^\gamma}, \end{aligned} \tag{23}$$

where we have used that $\widehat{\mu}_\gamma(x_*, x_*) = x_*$. We now turn our attention to the seller's problem.

We start with the upward deviation, i.e. choosing the LRP auction with $X = x$, by one of the sellers. Using equation (6) we can write deviating seller's expected payoffs as

$$\Pi^i_{X=x} = \sum_{j=1}^{K} \int_{\mu_j}^{1} dv \left(v P^i_{X=x}(v|\mu_j) - \pi^i_{X=x}(v|\mu_j) \right) = K \int_{\widehat{\mu}_\gamma}^{1} dv \Bigg(v \left[P_{X=x}(v|\widehat{\mu}_\gamma)\right]^{K-1} - \int_{\widehat{\mu}_\gamma}^{v} dv' \left[P_{X=x}(v'|\widehat{\mu}_\gamma)\right]^{K-1} - \int_{x}^{\widehat{\mu}_\gamma} dv' \left[P_{X=x}(v'|\widehat{\mu}_\gamma)\right]^{K-1} \Bigg). \tag{24}$$

After simple algebra we obtain

$$\Pi^i_{X=x} = K \left(\int_{\widehat{\mu}_\gamma}^{1} dv (2v-1) v^{K-1} - (1-\widehat{\mu}_\gamma)(\widehat{\mu}_\gamma - x) \left(\widehat{\mu}_\gamma\right)^{K-1} \right). \tag{25}$$

Using (17) the FOC for x_u takes the following form

$$x_*^{\gamma}(1-x_*)+2x_*-1=0,$$

which in the case of N being even yields $x_u^{even}=\widehat{x}=\frac{3-\sqrt{5}}{2}$, and it yields $x_u^{odd}=0.445$ in the case of N being odd.

We now consider downward deviation, i.e. choosing the LRP auction with $X=x_*$, by one of the sellers. The expected payoff of this seller is equal to

$$\begin{aligned}
\Pi^i_{X=x_*} &= \sum_{j=1}^{N}\int_{x_*}^{\mu_j} dv(2v-\mu_j)\prod_{k\neq j}^{N} P^j_{X=x_*}(v|\mu_k) = \\
&= K\int_{x_*}^{\widehat{\mu}_\gamma} dv(2v-\widehat{\mu}_\gamma)\left[P_{X=x_*}(v|\mu=\widehat{\mu}_\gamma)\right]^{K-1}\left[P_{X=x_*}(v|\mu=1)\right]^{N-K}+ \\
&+(N-K)\int_{x_*}^{\widehat{\mu}_\gamma} dv(2v-1)\left[P_{X=x_*}(v|\mu=\widehat{\mu}_\gamma)\right]^{K}\left[P_{X=x_*}(v|\mu=1)\right]^{N-K-1}+ \\
&+(N-K)\int_{\widehat{\mu}_\gamma}^{1} dv(2v-1)\left[P_{X=x_*}(v|\mu=1)\right]^{N-K-1} \\
&= K\int_{x_*}^{\widehat{\mu}_\gamma} dv(2v-\widehat{\mu}_\gamma)(1-\widehat{\mu}_\gamma+v)^{K-1}v^{N-K}+(N-K)\int_{\widehat{\mu}_\gamma}^{1} dv(2v-1)v^{N-K-1}+ \\
&+(N-K)\int_{x_*}^{\widehat{\mu}_\gamma} dv(2v-1)(1-\widehat{\mu}_\gamma+v)^{K}v^{N-K-1}.
\end{aligned} \tag{26}$$

Using expression (23) the first order condition for the optimal reserve price for the seller who deviates downward takes the following form

$$\begin{aligned}
\frac{\partial\Pi^i_{X=x_*}}{\partial x_*}|_{x=x_*} &= K\left(\frac{\partial\widehat{\mu}_\gamma(x,x_*)}{\partial x_*}|_{x=x_*}-1\right)x_*^{N-K+1}-(N-K)\left(2x_*-1\right)x_*^{N-K-1} \\
&= -\frac{x_*^{N-K-1}}{1-x_*^{\gamma}}\left(Kx_*^2-(N-K)(1-2x_*)\left(1-x_*^{\gamma}\right)\right)=0,
\end{aligned} \tag{27}$$

from where it immediately follows that when N is even the optimal reserve price solves

$$x_*^2-3x_*+1=0,$$

and it solves

$$x_*^2 - \frac{N+1}{N-1}(1-x_*^2)(2x_*-1) = 0, \tag{28}$$

when N is odd. In the case when N is even, x_d^{even} is the same as in the case of two buyers and two sellers, and it is a function of number of buyers in the case when N is odd. Specifically, equation (28) reduces to (17) for $N = 3$. Therefore, pure-strategy NE exists only in the case when the number of buyers is even, i.e. when $x_u^{even} = x_d^{even} = \widehat{x} = \frac{3-\sqrt{5}}{2}$. When the number of buyers is odd, the solution is $x_u^{odd} = 0.445 > x_d^{odd}(N)$ for all N and the pure-strategy symmetric NE does not exist. In the competitive case, $N \to \infty$, x_d^{odd} is equal to 0.403 and the pure-strategy symmetric NE also does not exist.

Theorem 2 summarizes these results.

THEOREM 2: *In the market with N buyers and two sellers, there exists a unique symmetric pure-strategy NE in the sellers game when the number of buyers is even. In this case half of the buyers, $K = \frac{N}{2}$, go to either seller and both sellers choose the same reserve price, $\widehat{x} = \frac{3-\sqrt{5}}{2}$. Buyers bid*

$$\widehat{b}_i\left(v_i, \widehat{x}\right) = \frac{v_i}{K}\left[K - 1 + \left(\frac{\widehat{x}}{v_i}\right)^K\right], \tag{29}$$

and the expected payoff of each seller is equal to

$$\Pi^i_{X=\widehat{x}} = \frac{K-1}{K+1}\left(1-\widehat{x}^{K+1}\right) + (1-\widehat{x})\widehat{x}^K. \tag{30}$$

Pure-strategy symmetric NE does not exists when the number of buyers is odd and in the competitive limit, $N \to \infty$.

When the number of buyers is even we recover the same result as in the case of two buyers and two sellers, thus confirming that the imperfect competition between the buyers alone does not affect the prices in the symmetric NE. However, when buyers become perfectly competitive, the symmetric NE does not exist. This is in contrast to the case of a monopolist seller considered by Riley and Samuelson (1981), where the optimal reserve price does not depend on the number of buyers. Overall, our result can be interpreted as having two Riley and Samuelson (1981) games played in each auction with the reserve prices reflecting the competition between the auction and therefore been lower than $\frac{1}{2}$, which is the reserve price in the case of the monopolist seller.

Contrast to the result of BS (1999) the symmetric NE exists in our setting. The key assumption for the existence in our case relative to BS (1999) is that buyers can condition their strategies on the number of other buyers visiting each seller. Therefore in our setting each buyer can implicitly incorporate the competition with other buyers into his bidding strategy, while in buyers form the expectation about the number of other buyers visiting each seller. We, therefore, return to the auction setup considered by BS (1999) and use technique developed in this section to illustrate that symmetric NE fails does not exists in BS (1999) setting. We also show that the revenue equivalence result still holds in BS setting, provided that the auction mechanism is efficient.

2.4. Equilibrium concept of Burguet and Sakovics (1999)

BS (1999) consider two competing sellers[7] and $N_b \geq 1$ buyers who bid for one unit of the same good, which is exactly our setup from the previous section. However, unlike our model where buyers move sequentially, buyers move simultaneously in BS (1999). Therefore, instead of observing the number of other buyers at each seller, buyers in BS (1999) have to form expectations about the number of buyers per seller or, in other words, play mixed strategy when choosing the seller. The sellers act by announce auction mechanisms before buyers decide on the auction to participate in and the bidding strategy. This assumption turns sellers into "Stackelberg leaders" and, therefore, they have to anticipate buyers' strategies before choosing the auction mechanisms[8]. In what follows we show that, analogous to the monopolist case of Riley and Samuelson (1981), the extended revenue equivalence result holds, sellers are indifferent between auction mechanisms, as long as auction mechanisms are efficient, and the reserve prices remain fixed. Next we show that pure strategies NE in the sellers' subgame does not exist, because for each conjectured symmetric NE with a strictly positive reserve price, one of the sellers will always have an incentive to "undercut" the other seller by deviating to a slightly smaller reserve price. Formally, this is related to the fact that the downward deviating seller's expected payoff has a square root singularity as a function of the reserve price x and its first derivative becomes negative and infinitely large in its absolute value at the conjectured equilibrium reserve x_*, as pointed out in BS (1999). However, it is important to note that, as we show below, such "singularity" occurs precisely because buyers randomize in a certain range of their types. For instance, this is not the case in our setting, when buyers play deterministic asymmetric strategies. For this reason, one should expect that in our setting, the equilibrium becomes more "robust" and still exists, at least on some range of the parameter space.

[7]In this setting, it is straightforward to extend to arbitrary number of sellers.

[8]As we have already mentioned above, we will show that a particular auction mechanism is irrelevant as long as it is efficient, and in this sense the only relevant "choice variables" are the reserve prices. This is analogous to the situation in a monopolist setting considered in Riley and Samuelson (1991).

Following BS (1999), we consider the pure strategies NE in the seller's subgame, assuming that the buyer's subgame has a symmetric NE, but the buyers may randomize in some ranges of their types. Then, as pointed out by Pai (2014), the buyers decide in which auction to participate based on the auction mechanisms announced by the sellers. Analogous to the previous subsection, we conjecture a pure strategy equilibrium for sellers with a positive equilibrium reserve price $x_* > 0$. Therefore, we first need to consider the buyers' reaction when of the sellers deviates to the LRP auction. Suppose that one of the sellers deviates to x, $0 < x < x_*$. What would be the buyers' reaction to this? If we assume a symmetric NE in the buyers' subgame, the result is that there is an equilibrium in this subgame when buyers randomize in a certain range of their valuations. Namely, conjecture a symmetric NE in the buyers' subgame, when the buyers randomize in the range of their private valuations $v_i \in [\mu, 1]$, $i = 2, ..N_b$, with a "cut-off" parameter $\mu \in [0, 1]$. Namely, in this range of valuations, the buyers participate in the HRP auction with probability $p \in [0, 1]$, while they participate in the LRP auction with probability $1-p \in [0, 1]$. Assuming that all buyers $N_b - 1$ except for the first one follow this strategy, the probability of winning the HRP auction with the reserve price x_* is given by

$$P^i_{X=x_*}(v|\mu) = \begin{cases} (1-p+p\mu)^{N_b-1}, & v \in [0,\mu], \\ (1-p+pv)^{N_b-1}, & v \in [\mu,1], \end{cases} \tag{31}$$

while the probability of winning the LRP auction with the reserve price x is given by

$$P^i_{X=x}(v|\mu) = \begin{cases} (v+p(1-\mu))^{N_b-1}, & v \in [0,\mu], \\ ((1-p)v+p)^{N_b-1}, & v \in [\mu,1]. \end{cases}, \tag{32}$$

where we have taken into account that in symmetric NE all buyers play the same cut-off strategy with the cut-off parameter μ. The optimal cut-off strategy, $\widehat{\mu}$, is defined from the revenue indifference between HRP and LRP auctions

$$\int_{x_*}^{\widehat{\mu}} dv\,(1-p+p\widehat{\mu})^{N_b-1} = \int_{x}^{\widehat{\mu}} dv\,(v+p(1-\widehat{\mu}))^{N_b-1}. \tag{33}$$

Now, we check the validity of indifference condition, which involves verification of conjectured equilibrium strategies. We should have

$$\int_{\widehat{\mu}}^{v} dv'\left(1-p+pv'\right)^{N_b-1} = \int_{\widehat{\mu}}^{v} dv'\left(p+(1-p)v'\right)^{N_b-1}, \tag{34}$$

which should hold for any $v \in [\widehat{\mu}, 1]$. Since (34) holds identically in the interval $v \in [\widehat{\mu}, 1]$, the integrands on the r.h.s. and the l.h.s. should be also identical, and therefore

$$p = 1 - p = \frac{1}{2}. \tag{35}$$

Then, the winning probabilities are given by

$$P^{i}_{X=x_*}\left(v|\mu=\widehat{\mu}\right) = \begin{cases} \left(\frac{1+\widehat{\mu}}{2}\right)^{N_b-1}, & v \in [0,\widehat{\mu}]\,, \\ \left(\frac{1+v}{2}\right)^{N_b-1}, & v \in [\widehat{\mu},1]\,, \end{cases} \tag{36}$$

for the HRP and

$$P^{i}_{X=x}\left(v|\mu=\widehat{\mu}\right) = \begin{cases} \left(v+\frac{1-\widehat{\mu}}{2}\right)^{N_b-1}, & v \in [0,\widehat{\mu}]\,, \\ \left(\frac{1+v}{2}\right)^{N_b-1}, & v \in [\widehat{\mu},1]\,, \end{cases} \tag{37}$$

for the LRP auctions, respectively, and the condition (33) becomes

$$\left(\widehat{\mu} - x_*\right)\left(\frac{1+\widehat{\mu}}{2}\right)^{N_b-1} = \int\limits_{x}^{\widehat{\mu}} dv \left(v + \frac{1-\widehat{\mu}}{2}\right)^{N_b-1}. \tag{38}$$

In the limit $x_* = x + \epsilon,\ \epsilon \to 0$, we conjecture the following expression for $\widehat{\mu}$

$$\widehat{\mu} = x_* + \Delta = x + \epsilon + \Delta, \tag{39}$$

where $\Delta \to 0$ when $\epsilon \to 0$. This makes sense, since we know that in the limit $x_* \to x$, we should also have $\mu \to x$, but the optimal cut-off $\widehat{\mu}$ should remain grater than x_* for any finite distance $\epsilon = x_* - x$ between the reserve prices. Substituting (39) into (38), we obtain in the limit $\epsilon \to 0$

$$\Delta = \sqrt{\epsilon}\sqrt{\frac{1+x}{2}}, \quad \epsilon \to 0, \tag{40}$$

which has an infinite derivative w.r.t. ϵ at the point $\epsilon = 0$.

Now, we analyze a symmetric pure strategy NE, following BS (1999). Namely, conjecture an equilibrium at the reserve x_*. Consider the "undercutting" strategy when one of the sellers deviates "down" to the LRP auction with $x \leq x_*$. The expected seller's payoffs in the LRP auction are

$$\Pi^i_{X=x} = \int\limits_x^{\widehat{\mu}} dv\,(2v-1)\,P^i_{X=x}\,(v|\mu = \widehat{\mu}) + \frac{1}{2}\int\limits_{\widehat{\mu}}^{1} dv\,(2v-1)\,P^i_{X=x}\,(v|\mu = \widehat{\mu}) +$$

$$+\left(\frac{1-\widehat{\mu}}{2}\right)\int\limits_x^{\widehat{\mu}} dv P^i_{X=x}\,(v|\mu = \widehat{\mu})\,. \tag{41}$$

Substituting (39) and (39) into (38), we obtain

$$\Pi^i_{X=x} = (N_b-1)\int\limits_x^{\widehat{\mu}} dvv\left(1-\left(v+\frac{1-\widehat{\mu}}{2}\right)\right)\left(v+\frac{1-\widehat{\mu}}{2}\right)^{N_b-2} +$$

$$+ (N_b-1)\int\limits_{\widehat{\mu}}^{1} dv\frac{v}{2}\left(1-\frac{1+v}{2}\right)\left(\frac{1+v}{2}\right)^{N_b-1} \tag{42}$$

$$+ x\left(x+\frac{1-\widehat{\mu}}{2}\right)^{N_b-1}\left(1-\left(x+\frac{1-\widehat{\mu}}{2}\right)\right).$$

Note that (42) converts into the result of BS (1999) (first unnumbered equation on p.237) specialized to our case when the prior distribution is uniform. Recall that, as demonstrated in BS (1999), the reason for non-existence of the symmetric pure strategy equilibria is that the derivative of the expected payoffs (42) of the seller who deviated down with respect to his reserve price $x = x_* - \epsilon$, is negative and has a singularity in the limit $\epsilon \to 0$. Indeed, differentiating (42) with respect to and taking a limit $\epsilon \to 0$, we obtain

$$\left(\frac{\partial \Pi^i_{X=x}}{\partial x}\right)_{x=x_*} = \frac{1}{2}x\left(\frac{1+x}{2}\right)^{N_b-1}\left(\frac{\partial \widehat{\mu}}{\partial x}\right)_{x=x_*}. \tag{43}$$

With the use of (40) , (43) yields

$$\left(\frac{\partial \Pi^i_{X=x}}{\partial x}\right)_{x=x_*} \sim -\frac{1}{4\sqrt{\epsilon}}x\left(\frac{1+x}{2}\right)^{N_b-\frac{1}{2}} \to -\infty,\ \epsilon \to 0, \tag{44}$$

confirming the BS (1999) result.

Now we make use of the following general relation that holds for any differentiable function

$F(\cdot)$, any $\overline{x} \geq \underline{x}$ and $n \in N \geq 1$

$$
\begin{aligned}
&\int_{\underline{x}}^{\overline{x}} dv F^{n-1}(v)\left(vF'(v) + F(v) - 1\right) \\
&= (n-1)\int_{\underline{x}}^{\overline{x}} dv v F'(v) F^{n-2}(v)(1 - F(v)) \\
&+ \underline{x}F^{n-1}(\underline{x})(1 - F(\underline{x})) - \overline{x}F^{n-1}(\overline{x})(1 - F(\overline{x})) .
\end{aligned} \tag{45}
$$

Substituting $\underline{x} = x$, $\overline{x} = 1$, $n = N_b$, and $F(\cdot)$ consistent with (37)

$$
F(v) = \begin{cases} v + \frac{1-\widehat{\mu}}{2}, & v \in [0, \widehat{\mu}], \\ \frac{1+v}{2}, & v \in [\widehat{\mu}, 1], \end{cases} \tag{46}
$$

we observe that

$$
\Pi_x^i = \int_x^1 dv P_{X=x}^i(v|\mu = \widehat{\mu})\left(vF'(v) + F(v) - 1\right), \tag{47}
$$

with $P_{X=x}^i(v|\mu = \widehat{\mu}) = (F(v))^{N_b - 1}$, which is identical to the result of Riley and Samuelson (1991) when the effective distribution is given by (46), and indicates that the revenue equivalence still holds, provided that the auction mechanism is efficient and the winning probabilities are defined consistently with bidding strategies.

We are going to turn our attention to the case of more than two sellers.

2.5. Markets with $N_b = mN_s$

Consider that the numbers of buyers $N_b = mN_s$ with m being an integer. Following previous sections, we conjecture that the symmetric NE exists at some reserve price $x_* \in [0, 1]$, and then analyze whether or not one of the sellers (say seller 1) would deviate to some different reserve price $x \neq x_*$. Analogous to what was done in the previous sections, we must start by analyzing buyers' reaction functions to the seller's deviation, since sellers move first by announcing their reserve prices to the buyers. We conjecture that cut-off strategies, $\{\mu_i, i = 1, 2, 3, ..., N_b\}$, exist and satisfy the following "ladder" structure

$$
0 \leq x_* < x \leq \mu_1 \leq \mu_2 \leq ... \leq \mu_{N_b} \leq 1. \tag{48}
$$

This conjecture applies to both upward and downward deviations. One important complication in this case is the absence of symmetry between the upward and downward deviations. To see this consider the case of two sellers with the same number of buyers K and the same reserve price. If one of the sellers deviates downward, i.e. chooses lower reserve price, he attracts buyers, whose valuations are below the cut-off, from the second seller. This is equivalent to the second seller deviating upward, since the same buyers deviate to the seller with the lower reserve price. Let us now consider three sellers with K buyers each and the same reserve prices. If one of the sellers deviates downward, buyers from both sellers with higher reserve price whose valuations are below the cut-off deviate to this seller. If the same seller deviates upward, then his own his buyers whose valuations are below the cut-off deviate to sellers with lower reserve price. As a result, the outcomes of the downward and upward deviations by one of the sellers are not equivalent. Therefore, we have to analyze the upward and downward deviations separately.

We start with upward deviation by one of the sellers. Suppose that all N_s sellers start with the conjectured equilibrium reserve price x_* (LRP). Now suppose that one of the sellers deviates to a reserve price $x > x_*$ (HRP). Then, there are $N_s - 1$ sellers at the LRP and one seller at the HRP auctions. The deviated seller has m buyers who can potentially deviate to the LRP auction once the seller moves to the HRP auction. Since winning probability decreases in the number of buyers, each of the deviating buyers would pick a separate seller. Since all sellers staying at the LRP auction are identical, deviating buyers simply "roll the dice" when choosing a particular seller in the LRP auction, i.e. play mixed strategy. In the case when all m buyers stay in the LRP auction when their original seller deviates to the HRP auction there will be m sellers with $m+1$ buyers and $N_s - m - 1$ sellers with m buyers in the LRP auction. Therefore we have two groups of buyers: (i) buyers coming from sellers with $m+1$ buyers, total of m sellers; (ii) buyers coming from sellers with m buyers, total of $N_s - m - 1$ sellers. We conjecture that k_1 buyers per each seller from the first group and k_2 buyers per each seller from the second group play the cutoff strategy with the same cutoff, $\mu < 1$. Therefore, the total number of buyers playing the cutoff strategy is equal to

$$K' = k_1 m + k_2 (N_s - m - 1). \tag{49}$$

Consider now a seller of the first type who has a total of $m+1$ buyers bidding for his item with k_1 of them using the same cut-off strategy, μ, while other $m + 1 - k_1$ buyers always stay in the LRP auction. The indifference between the LRP and HRP auctions for a buyer with valuation in the interval $v \in [x_*, \mu]$ who has chosen this seller takes the following form

$$\int_{x_*}^{\mu} dv \underbrace{(1-\mu+v)}_{P_{X=x_*}(v|\mu=\mu_{k_1})}{}^{k_1-1} \underbrace{v}_{P_{X=x_*}(v|\mu=1)}{}^{m+1-k_1} = \int_{x}^{\mu} dv \underbrace{\mu}_{P_{X=x}(v|\mu=\mu_{K'})}{}^{K'-1}, \ \forall K' \geq 1. \tag{50}$$

A buyer who plays a cut-off strategy with $\mu < 1$ would stay at the HRP auction when his valuation is $v \in [\mu, 1]$, i.e. would not deviate, if the following condition holds

$$\int_{\mu}^{v} dv' \left(v'\right)^{m+1-k_1} \leq \int_{\mu}^{v} dv' \left(v'\right)^{K'-1}. \tag{51}$$

Likewise a buyer who plays a cut-off strategy with with a cut-off equal to 1 would stay at the LRP auction when his valuation is $v \in [\mu, 1]$, i.e. would not deviate, if the following condition holds

$$\int_{\mu}^{v} dv' \left(v'\right)^{m-k_1} \geq \int_{\mu}^{v} dv' \left(v'\right)^{K'}. \tag{52}$$

Since both equations (51) and (52) are satisfied for $v \in [\mu, 1]$, they are equivalent to

$$m + 1 - k_1 > K' - 1, \tag{53}$$

and

$$m - k_1 \leq K', \tag{54}$$

respectively. Combining equations (53) and (54) yields

$$m - 1 \leq K' + k_1 - 1 < m + 1. \tag{55}$$

Next consider now a seller of the second type who has total of m buyers bidding for his item with k_2 of them using the same cut-off bidding strategy, μ, while other $m - k_2$ buyers always stay in the LRP auction. The indifference between the LRP and HRP auctions for a buyer with valuation in $v \in [x_*, \mu]$ who has chosen this seller takes the following form

$$\int_{x_*}^{\mu} dv \underbrace{(1-\mu+v)}_{P_{X=x_*}(v|\mu=\mu_{k_2})}{}^{k_2-1} \underbrace{v}_{P_{X=x_*}(v|\mu=1)}{}^{m-k_2} = \int_{x}^{\mu} dv \underbrace{\mu}_{P_{X=x}(v|\mu=\mu_{K'})}{}^{K'-1}, \ \forall k_2 \geq 1. \tag{56}$$

A buyer who plays a cut-off strategy with $\mu < 1$ would stay at the HRP auction when his valuation is $v \in [\mu, 1]$, i.e. would not deviate, if the following condition holds

$$\int_{\mu}^{v} dv' \left(v'\right)^{m-k_2} \leq \int_{\mu}^{v} dv' \left(v'\right)^{K'-1}. \tag{57}$$

Likewise a buyer who plays a cut-off strategy with with a cut-off equal to 1 would stay at the LRP auction when his valuation is $v \in [\mu, 1]$, i.e. would not deviate, if the following condition holds

$$\int_{\mu}^{v} dv' \left(v'\right)^{m-1-k_2} \geq \int_{\mu}^{v} dv' \left(v'\right)^{K'}. \tag{58}$$

Since both equations (57) and (58) are satisfied for $v \in [\mu, 1]$, they are equivalent to

$$m - k_2 > K' - 1, \tag{59}$$

and

$$m - 1 - k_2 \leq K', \tag{60}$$

respectively. Combining equations (59) and (60) yields

$$m - 1 \leq K' + k_2 < m + 1. \tag{61}$$

Comparing inequalities (55) and (61) leads to

$$k_1 = k_2 + 1, \tag{62}$$

which after substitution into (49) yields

$$K' + k_2 = m + k_2 N_s, \tag{63}$$

which in turn helps to rewrite the inequality (61) as

$$m - 1 \leq m + k_2 N_s < m + 1. \tag{64}$$

It immediately follows from the inequality (64) that $k_2 N_s \in [0, 1]$ and hence $k_2 = 0$ for $N_s \geq 2$ and, correspondingly, $k_1 = 1$ and $K' = m$. Therefore, only buyers who switch from the deviated seller play $\mu < 1$ strategy, i.e. may deviate to the HRP auction. Condition $k_2 = 0$ means that buyers from the second group of sellers with m buyers per seller always stay in the LRP auction (i.e. play $\mu = 1$) strategy. Therefore, only buyers from the first group of sellers with $m + 1$ buyers per seller play $\mu < 1$ strategy and can deviate to the HRP auction. The total number of buyers playing $\mu < 1$ is equal to m.

After establishing that there exists a finite average number (even in the competitive limit) of buyers per seller playing the same strategy $\mu < 1$, we are going to solve for μ when upward deviations are small, $x = x_* + \epsilon$, $\epsilon \to 0$. We conjecture that optimal cut-off is takes the form $\widehat{\mu} = x_* + \epsilon + \Delta$ and use it in equation (50), which reduces in this case to

$$\int_{x_*}^{\widehat{\mu}} dv v^m = \int_{x}^{\widehat{\mu}} dv \widehat{\mu}_m^{m-1}, \tag{65}$$

to obtain

$$(\epsilon + \Delta)\, x_*^m = \Delta x_*^{m-1}, \tag{66}$$

which yields

$$\Delta = \epsilon \frac{x_*}{1 - x_*}. \tag{67}$$

Proposition 4 summarizes these results.

PROPOSITION 4: *In the case of $N_b = mN_s$, when one of the sellers deviates to the HRP auction only his buyers follow the same cutoff strategy*

$$\widehat{\mu}(x, x_*) = x_* + \epsilon \frac{1}{1 - x_*}, \ \epsilon = x - x_* \to 0, \tag{68}$$

by choosing LRP/HRP auction if their private valuation is less/greater than $\widehat{\mu}(x, x_*)$. Other buyers always choose the LRP auction.

We illustrate the intuition behind Proposition 4 for the case of three buyers and three sellers (3B3S thereafter), i.e. $m = 1$ and $N_s = 3$. Since all three buyers buyers are ex ante identical to the sellers, it does not matter to the sellers which buyer goes to HRP, as long as all buyer choose the same cutoff $\widehat{\mu}(x, x_*)$. If the lead buyer plays cutoff strategy $\mu_1 = \widehat{\mu}(x, x_*)$ and goes to the HRP auction, both the second and third buyers always play $\mu_2 = \mu_3 = 1$ and go to the LRP auctions. If the lead buyer goes to the LRP auction, $\mu_1 = 1$, the second buyer plays the cutoff strategy $\mu_2 = \widehat{\mu}(x, x_*)$ and goes either to LRP or HRP auction. If the second buyer goes to the HRP auction, the third buyer always goes to the LRP auction, $\mu_3 = 1$. If the second buyer goes to the LRP auction, the third buyer now plays the cutoff strategy $\mu_3 = \widehat{\mu}(x, x_*)$ and goes either to LRP or HRP auction. Therefore, from the sellers' perspective the outcome is equivalent to two of the buyers always going to the LRP auctions and the third buyer using cutoff strategy $\widehat{\mu}(x, x_*)$ to choose between LRP and HRP auctions. Once again, buyers' optimal bidding strategies are fully characterized by their cutoffs.

We now turn our attention to the sellers' problem. Using equation (6) we can write deviating seller's expected payoffs as

$$\Pi_{X=x} = m\int_{\widehat{\mu}}^{1} dv \left(v\left[P_{X=x}(v|\widehat{\mu})\right]^{m-1} - \pi_{X=x}(v|\widehat{\mu})\right) =$$

$$= m\int_{\widehat{\mu}}^{1} dv \left(v\left[P_{X=x}(v|\widehat{\mu})\right]^{m-1} - \int_{\widehat{\mu}}^{v} dv' \left[P_{X=x}(v'|\widehat{\mu})\right]^{m-1} - \int_{x}^{\widehat{\mu}} dv' \left[P_{X=x}(v'|\widehat{\mu})\right]^{m-1}\right) = \quad (69)$$

$$= m\int_{\widehat{\mu}}^{1} dv \left(v^{m-1} - \int_{\widehat{\mu}}^{v} dv' \left(v'\right)^{m-1} - \int_{x}^{\widehat{\mu}} dv \left(\widehat{\mu}\right)^{m-1}\right) =$$

$$= m\left(\int_{\widehat{\mu}}^{1} dv \left(2v-1\right) v^{m-1} - \left(1-\widehat{\mu}\right)\left(\widehat{\mu}-x\right)\left(\widehat{\mu}\right)^{m-1}\right). \quad (70)$$

The optimal reserve price in the HRP auction, x, is set by maximizing sellers' expected payoff averaged over all sellers

$$\frac{\partial \Pi_{X=x}}{\partial x}|_{x_*=x} = 0,$$

Taking derivative with respect to x yield

$$\frac{\partial \Pi_{X=x}}{\partial x}|_{x_*=x} = m\frac{x^{m-1}}{1-x}\left(1-3x+x^2\right) = 0, \quad (71)$$

where we have used that

$$\frac{\partial \widehat{\mu}_m(x,x)}{\partial x}|_{x_*=x} = \frac{1}{1-x},$$

which has the following solution on the interval $x \in [0,1]$

$$x_u = \widehat{x} = \frac{3-\sqrt{5}}{2} \approx 0.382. \quad (72)$$

We now consider downward deviation by one of the sellers. Suppose that all N_s sellers start with the conjectured equilibrium reserve price x. Now suppose that one of the sellers deviates to a reserve price $x_* < x$ (LRP) leading to $N_s - 1$ sellers at the HRP and one seller at the LRP auctions. Let K be the number (integer) of buyers playing $\mu < 1$ strategy and k' be the number of buyers per seller in the HRP auction. Since at least m buyers play $\mu = 1$ strategy by "permanently" following the deviating seller to the LRP auction, there are potentially $N_b - m = m(N_s - 1)$ buyers playing $\mu < 1$, i.e. $0 \le K \le m(N_s - 1)$, and, correspondingly, $N_b - K$ buyers playing $\mu = 1$ strategy. There exists an additional intrinsic relation between K and k'. When all buyers play $\mu = 1$, i.e. $K = 0$, then all buyers go to the LRP auction, i.e. $k' = 0$. When K is in the interval $1 \le K \le N_s - 1$, then there is exactly one buyer per seller in the HRP auction, i.e. $k' = 0$. This is true since buyers strategically join those HRP auctions without other buyers present. Following the same logic we have

$$k' = \left\lceil \frac{K}{N_s - 1} \right\rceil, \tag{73}$$

where $\lceil x \rceil$ stands for the ceiling function. The maximum value of K is equal to $m(N_s - 1)$ implying that the maximum value of k' is equal to m. We need to find equilibrium values of μ, K, and k'. Just as we have done before, we are going to use the stability conditions to pin down K and k', and seller's indifference between the LRP and HRP auctions condition to find μ.

The indifference between the LRP and HRP auctions for a buyer with valuation in $v \in [x_*, \mu]$ who has chosen this deviating seller takes the following form

$$\int_{x_*}^{\mu} dv\,(1 - \mu + v)^{K-1}\, v^{N_b - K} = \int_{x}^{\mu} dv \mu^{k'-1}. \tag{74}$$

A buyer who plays a cut-off strategy with $\mu < 1$ would stay at the HRP auction when his valuation is $v \in [\mu, 1]$, i.e. would not deviate, if the following condition holds

$$\int_{\mu}^{v} dv' \left(v'\right)^{N_b - K} \le \int_{\mu}^{v} dv' \left(v'\right)^{k'-1}. \tag{75}$$

Likewise a buyer who plays a cut-off strategy with with a cut-off equal to 1 would stay at the LRP auction when his valuation is $v \in [\mu, 1]$, i.e. would not deviate, if the following condition holds

$$\int_{\mu}^{v} dv' \left(v'\right)^{N_b - K - 1} > \int_{\mu}^{v} dv' \left(v'\right)^{k'}. \tag{76}$$

Since both equations (75) and (76) are satisfied for $v \in [\mu, 1]$, they are equivalent to

$$N_b - K > k' - 1 \Leftrightarrow N_b + 1 \ge k' + K, \tag{77}$$

and

$$N_b - K - 1 < k' \Leftrightarrow N_b - 1 \le k' + K, \tag{78}$$

respectively. We can now combine inequalities (77) and (78) to obtain

$$N_b - 1 \leq k' + K < N_b + 1, \tag{79}$$

in turn equivalent to $K+k' = \{N_b - 1, N_b\}$. Using relation (73) we have that $K = \{N_b - m - 1, N_b - m\}$ and $k' = \{m, m\}$ when $K + k' = \{N_b - 1, N_b\}$ respectively. When $N_b = 2$, the only viable value for $K + k'$ is 2, since in this case one of the buyers is playing the cut-off strategy and thus there is one buyer per seller in the HRP auction.

After finding the number of buyers per seller playing the same strategy $\mu < 1$, we are going to solve for μ when upward deviations are small, $x = x_* + \epsilon$, $\epsilon \to 0$. We conjecture that optimal cut-off takes the form $\widehat{\mu}_\gamma = x_* + \epsilon + \Delta$ and use it in the equation (74) to obtain

$$(\epsilon + \Delta)\, x_*^{N_b - K} = \Delta x_*^{k' - 1}, \tag{80}$$

which can be solved to yield

$$\Delta = \epsilon \frac{x_*^\gamma}{1 - x_*^\gamma}, \;\; \gamma = \{1, 2\}\,, \tag{81}$$

where we have used the notation $\gamma \equiv N_b + 1 - K - k' = \{1, 2\}$ since $K + k' = \{N_b - 1, N_b\}$ from the inequality (79). Proposition 5 summarizes these results.

PROPOSITION 5: *In the case of $N_b = mN_s$, when one of the sellers deviates to the LRP auction his buyers always follow him, while $K = \{N_b - m - 1, N_b - m\}$ of the remaining buyers play the same cutoff strategy*

$$\widehat{\mu}_\gamma(x, x_*) = x_* + \epsilon \frac{1}{1 - x_*^\gamma}, \;\; \gamma = \{1, 2\}\,, \;\; \epsilon = x - x_* \to 0, \tag{82}$$

by choosing LRP/HRP auction if their private valuation is less/greater than $\widehat{\mu}_\gamma(x, x_*)$.

We illustrate the intuition behind Proposition 5 for the same 3B3S case. If the lead buyer plays cutoff strategy $\mu_1 = \widehat{\mu}(x, x_*)$ and goes to the HRP auction, the second buyer can either play the same cut-off strategy as the first buyer, $\mu_2 = \widehat{\mu}(x, x_*)$ and in this case $K = 2$, or always pick the LRP auction, $\mu_2 = 1$. The decision of the third buyer depends on whether on the previous two buyers went to the LRP auction. If one of the buyers is already at the LRP auction the third buyer either plays either $\mu_3 = \widehat{\mu}(x, x_*)$ ($K = 2$) or $\mu_3 = 1$ ($K = 1$). If both of the previous two buyers went to the HRP auction the third buyer always goes to the LRP auction, i.e. $\mu_3 = \widehat{\mu}(x, x_*)$ and $K = 2$. If the lead buyer goes to the LRP auction, $\mu_1 = 1$, the second buyer plays either

$\mu_2 = \widehat{\mu}(x, x_*)$ or $\mu_2 = 1$ ($K = 1$). If the second buyer goes to the HRP auction, the third buyer plays either $\mu_3 = \widehat{\mu}(x, x_*)$ ($K = 2$) or $\mu_3 = 1$ ($K = 1$). Therefore, from the sellers' perspective the outcome is equivalent either to two of the buyers always going to the LRP auction or $K = 1$ while the third buyer playing the cutoff strategy $\widehat{\mu}(x, x_*)$ to choose between LRP and HRP auctions, or just one of the buyers always going to the LRP auction and the other two buyers playing the cutoff strategy $\widehat{\mu}(x, x_*)$. These two outcomes are very different from the point of view of the seller who deviates to the LRP auction since the probability of him selling the item is higher when two of the buyers always follow him. As we show below, the deviating seller can use the reserve price x_* to steer buyers into the optimally preferred scenario.

We now turn our attention to sellers' problem. Using equation (6) we can write deviating seller's expected payoffs as

$$\Pi^i_{X=x_*} = \sum_{j=1}^{N_b} \int_{x_*}^{\mu_j} dv(2v - \mu_j) \prod_{k \neq j}^{N_b} P^j_{X=x_*}(v|\mu_k) = \tag{83}$$

$$= K \int_{x_*}^{\widehat{\mu}_\gamma} dv(2v - \widehat{\mu}_\gamma) \left[P_{X=x_*}(v|\mu = \widehat{\mu}_\gamma)\right]^{K-1} \left[P_{X=x_*}(v|\mu = 1)\right]^{N_b - K} +$$

$$+ (N_b - K) \int_{x_*}^{\widehat{\mu}_\gamma} dv(2v - 1) \left[P_{X=x_*}(v|\mu = \widehat{\mu}_\gamma)\right]^{K} \left[P_{X=x_*}(v|\mu = 1)\right]^{N_b - K - 1} +$$

$$+ (N_b - K) \int_{\widehat{\mu}_\gamma}^{1} dv(2v - 1) \left[P_{X=x_*}(v|\mu = 1)\right]^{N_b - K - 1}$$

$$= K \int_{x_*}^{\widehat{\mu}_\gamma} dv(2v - \widehat{\mu}_\gamma)(1 - \widehat{\mu}_\gamma + v)^{K-1} v^{N_b - K} + (N_b - K) \int_{x_*}^{\widehat{\mu}_\gamma} dv(2v - 1)(1 - \widehat{\mu}_\gamma + v)^{K} v^{N_b - K - 1} +$$

$$+ (N_b - K) \int_{\widehat{\mu}_\gamma}^{1} dv(2v - 1) v^{N_b - K - 1}.$$

The optimal reserve price in the LRP auction, x_*, is set by maximizing sellers' expected payoff averaged over all sellers

$$\frac{\partial \Pi_{X=x_*}}{\partial x_*}|_{x=x_*} = 0,$$

Taking derivative with respect to x_* yields

$$\frac{\partial \Pi^i_{X=x_*}}{\partial x_*}|_{x=x_*} = K\left(\frac{\partial \widehat{\mu}_\gamma(x,x_*)}{\partial x_*}|_{x=x_*} - 1\right) x_*^{N_b-K+1} - (N_b-K)\,(2x_*-1)\,x_*^{N_b-K-1} \quad (84)$$
$$= -\frac{x_*^{N_b-K-1}}{1-x_*^\gamma}\left(Kx_*^2 - (N_b-K)(1-2x_*)\,(1-x_*^\gamma)\right) = 0,$$

that partial derivatives of $\widehat{\mu}_\gamma(x,x_*)$ with respect to x_* and x are equal to

$$\frac{\partial \widehat{\mu}_\gamma(x,x_*)}{\partial x_*}|_{x=x_*} = -\frac{x_*^\gamma}{1-x_*^\gamma}, \quad (85)$$

It can be rewritten in the case of $\gamma = 1$ as[9]

$$(N_s-3)\,x_*^2 + 3x_* - 1 = 0. \quad (86)$$

The root of (86) on the interval $x_* \in [0,1]$ satisfies

$$x_d \le \frac{1}{3} < x_u, \quad (87)$$

in the case when $N_s \ge 3$. In this case the symmetric NE does not exist. When the number of sellers is equal to 2 we recover the result from Theorem 3 that there exists a unique symmetric NE with the reserve price $\widehat{x} = \frac{3-\sqrt{5}}{2}$.

When $\gamma = 2$ equation (84) takes the following form[10]

$$\frac{m(N_s-1)-1}{m+1}x_*^2 - (1-2x_*)\left(1-x_*^2\right) = 0, \quad (88)$$

and it can be rewritten in a more convenient form as

$$\left[\frac{m(N_s-1)-1}{m+1} - 1 - x_*\right]x_*^2 - (1+x_*)(x_*^2 - 3x_* + 1) = 0. \quad (89)$$

This case applies only when $N_s \ge 3$ since when $\gamma = 2$ and $N_s = 2$ we have $K = m-1$, which is inconsistent with the sellers' optimal strategy from Theorem 2. It is easy to verify that the root of the equation (88) lying in the $[0,1]$ interval is decreasing with $\frac{m(N_s-1)-1}{m+1}$. It follows from (89) that when $\frac{m(N_s-1)-1}{m+1} = 1+\widehat{x}$, the desired root is equal precisely to $\widehat{x}$ and, therefore, equilibrium does not exists when $\frac{m(N_s-1)-1}{m+1} > 1+\widehat{x}$. Therefore we have that the symmetric NE exists when

[9] In this case $K = m(N_s-1) = N_b - m$.

[10] In this case $K = m(N_s-1) - 1 = N_b - m - 1$.

the following inequality is satisfied

$$m \leq m^* = \left\lfloor \frac{2+\widehat{x}}{N_s - \widehat{x} - 2} \right\rfloor, \tag{90}$$

where $\lfloor x \rfloor$ stands for the floor function. For example, in the case of the same number of buyers and sellers, $m = 1$, the symmetric NE exists up to $N_s = 4$ when $m^* = 1$ with the interval of equilibrium reserve prices equal to $[0.382, 0.403]$. This is the same equilibrium as in the case of when $m = 2$ and $N_s = 3$, since $\frac{m(N_s-1)-1}{m+1} = 1$ when $m = 2$ and $N_s = 3$ as well as when $m = 1$ and $N_s = 4$.

Theorem 3 summarizes these results.

THEOREM 3: *In the case of $N_b = mN_s$, the symmetric pure strategy NE exists when $m \leq m^* = \left\lfloor \frac{2+\widehat{x}}{N_s-\widehat{x}-2} \right\rfloor$, and it does not exist when $m > m^*$.*

Lets consider $m = 1$ and $N_s = 3$, i.e. three buyers and three sellers. The solution of the equation (88) in this case is equal to $x_d = 0.44 > x_u$ and, therefore, symmetric NE exists on the interval of reserve prices $[0.382, 0.440]$.

We can now compare this case to the one with two buyers and two sellers (2B2S) where symmetric equilibrium is unique. From Proposition 5 we have two separate scenarios: (i) two buyers play $\mu < 1$ strategy, i.e. $K = 2$, $k' = 1$, and $\gamma \equiv N_b + 1 - K - k' = 1$; (ii) one buyer plays $\mu < 1$ strategy, i.e. $K = 1$, $k' = 1$, and $\gamma \equiv N_b + 1 - K - k' = 2$.

In the first scenario the deviating seller is followed by a single buyer, and, therefore, in order to increase his probability of selling he needs to lower his reserve price below $\widehat{x}$, optimally chosen by the seller who deviates to LRP in the 2B2S case. It can be seen by comparing the expected profits of the downward deviating seller between 2B2S

$$\Pi^{2B2S}_{X=x_*} == \int_{x_*}^{\widehat{\mu}} dv\,(2v - \widehat{\mu})\,v + \int_{x_*}^{\widehat{\mu}} dv\,(2v-1)\,(1-\widehat{\mu}+v) + \int_{\widehat{\mu}}^{1} dv\,(2v-1)\,, \tag{91}$$

and 3B3S (from Eq. (83) with $K = 2$ and $\gamma = 1$) cases

$$\Pi^{3B3S}_{X=x_*} = 2\int_{x_*}^{\widehat{\mu}_1} dv(2v-\widehat{\mu}_1)(1-\widehat{\mu}_1+v)v + \int_{x_*}^{\widehat{\mu}_1} dv(2v-1)(1-\widehat{\mu}_1+v)^2 + \int_{\widehat{\mu}_1}^{1} dv(2v-1). \tag{92}$$

At the optimum where $x \to x_*$ we have both thresholds converging to the optimal reserve price. Also the derivatives of the threshold with respect to x_* are the same for both cases. Therefore, the main difference between $\Pi^{3B3S}_{X=x_*}$ and $\Pi^{2B2S}_{X=x_*}$ around their maxima, i.e. when the reserve price is close to the optimum, comes from the first term, derivative of which with respect to x_* at the optimal reserve price is less negative in the 3B3S case, thus leading to the lower reserve price.

In the second scenario the deviating seller is followed by two buyers, while third buyer plays the cutoff strategy. As a result, the probability of selling is already high and it is higher than in the 2B2S case. In addition, the threshold $\widehat{\mu}_1$ is less sensitive to x_* than in the 2B2S case

$$\left|\frac{\partial\widehat{\mu}_1(x,x_*)}{\partial x_*}|_{x=x_*}\right| = \frac{x_*^2}{1-x_*^2} = \frac{x_*}{1+x_*}\frac{x_*}{1-x_*} \leq \frac{\partial\widehat{\mu}(x,x_*)}{\partial x_*}|_{x=x_*} = \frac{x_*}{1-x_*},\ 0 \leq x_* < 1, \tag{93}$$

which makes it easier for the seller to set the reserve price more aggressively than in the first scenario as well as 2B2S case. Therefore, the downward deviating seller sets the reserve price higher than $\widehat{x}$.

However, once the number of buyers playing the cut-off strategy, K, becomes large enough it creates additional incentives for the deviating seller to further lower his reserve price, since the first term his expected payoff

$$\Pi^i_{X=x_*} = K\int_{x_*}^{\widehat{\mu}_\gamma} dv(2v-\widehat{\mu}_\gamma)(1-\widehat{\mu}_\gamma+v)^{K-1}v^{N_b-K} + (N_b-K)\int_{x_*}^{\widehat{\mu}_\gamma} dv(2v-1)(1-\widehat{\mu}_\gamma+v)^{K}v^{N_b-K-1}+ \\ + (N_b-K)\int_{\widehat{\mu}_\gamma}^{1} dv(2v-1)v^{N_b-K-1}, \tag{94}$$

increases with K. Also, lowering x_* leads to the increase in the optimal threshold, $\widehat{\mu}_\gamma$, thus increasing the integration range and the overall value of the first term in (94). To summarize, lowering the reservation price increases expected payoff from buyers playing the cut-off strategy. Therefore when the number of buyers per seller increases and so naturally does the number of buyers playing the cut-off strategy, thus leading to low optimal reserve price and destruction of the symmetric NE.

2.5.1 Discussion of multiple equilibria

All multiple equilibria can be ranked based on the seller's expected payoff. Since seller's expected payoff given by equation (7) increases in the reserve price on the $(0, \frac{1}{2})$ interval, sellers will always pick equilibria with highest reserve prices. As a result we obtain a non-monotone relationship between the number of sellers and the reserve price

$$\widehat{x}^{2B2S} = 0.382 < \widehat{x}^{4B4S} = 0.403 < \widehat{x}^{3B3S} = 0.440. \tag{95}$$

On the other hand, given a choice to join an auction with either two, three or four sellers, the potential buyer is better off in the auction with two sellers. The competition between sellers is highlighted by the reserve price in the auction with two sellers. The lower bound on the equilibrium reserve price is set by the upward deviating seller. Sellers compete for buyers by lowering their respective reserve prices until one of the sellers is better off in expectation by offering a higher reserve price since in this case there will be a buyer with high enough valuation who will trade off higher cost of the good for the higher probability of acquiring the good. Such strategy requires a proper coordination between the buyers who need to be able to estimate their winning probabilities, which is achieved in our model by allowing buyers to move sequentially.

Effectively, strategic sellers take advantage of the buyers who are "too smart for their own good." Sellers recognize that the reserve price is not the only variable in the buyers' decision making as it has to be combined with the buyers' estimate of the degree of competition with other buyers which affects their probability of winning the auction. Therefore, strategic competition on both sides of the market has an offsetting effect: sellers desire to compete for more buyers to increase the probability of selling is counterweighted by the buyers' recognition that by competing against each other they are facing lower probability of winning the auction. The equilibrium entails where these two forces equalize.

TABLE 1: Reserve Prices

		N_b					
		1	2	3	4	5	6
N_s	1	0.500	0.500	0.500	0.500	0.500	0.500
	2	∅	0.382	∅	0.382	∅	0.382
	3	∅	∅	0.440	∅	∅	0.403
	4	∅	∅	∅	0.403	∅	∅

3. Conclusion

We consider a private value double-auction model which can be viewed as an extension of the model of Riley and Samuelson (1981) to the case of multiple sellers. Each seller has one unit of the same asset for sale and buyers compete to purchase one unit of the asset from one of the sellers. In the model sellers costlessly adjust their reserve prices, while sequentially moving buyers optimally choose between sellers to maximize their respective expected payoffs. The reserve price plays the role of a choice variable in the model, for both types of players. The refinement we propose in this paper is based on the following trade-off. From the sellers' perspective the probability of selling the good is increasing in the number of buyers bidding for it and, therefore, sellers have the incentive to decrease their reserve price to attract more buyers. From buyers' perspective, the probability of winning the auction decreases in the number of buyers and therefore buyers with higher private valuations are better off choosing sellers with slightly higher reserve price but fewer number of buyers. We exploit this trade-off to construct the symmetric Nash equilibrium in pure strategies in this double auction. Our first result is that such equilibrium exists and it is unique in the case of two sellers and the even number of buyers and it does not exist when the number of buyers is odd. The equilibrium reserve price is lower than the reserve price set by the monopolist (Riley and Samuelson (1981)) and each seller is visited by the same number of buyers. This is in contrast to the result of Burguet and Sakovics (1999) that there are no symmetric pure-strategy equilibria in the case of two competing sellers when buyers move simultaneously. Our second result is for the case of the number of buyers equal to the integer fraction m of the number of sellers. We show that the symmetric pure-strategy Nash equilibrium exists when m is below an endogenous cut-off m^* which we fully characterize. This equilibrium is not unique as it is supported on the interval of the reserve prices.

APPENDIX

LEMMA 1: *Suppose we have a single seller and an arbitrary number of buyers $N_b \geq 1$ each with a private valuation $v_i \sim U[0,1]\ \forall\ i = 1,..N_b$. Suppose that seller sets a reserve price x and each buyer observes x before choosing his bidding strategy (i.e. optimal bid). Then the expected probability of winning the auction for buyer i when other $N_b - 1$ buyers bid b_{-i} is equal to*

$$P_x^i(v_i|b_{-i}) = v_i^{N_b-1}. \tag{96}$$

Proof of Lemma 1: Note that since the reserve price is posted, the number of buyers that actually bid in the auction become a random variable, since their valuations are not known to the seller, and some of them may actually be below the reserve x. For this reason, the winning probability in the auction also becomes a random variable described by a binomial distribution. However, since all buyers are risk neutral, the only important quantity is the *expected* winning probability. The expected winning probability is given by

$$P_{N_b}(v) = \sum_{k=0}^{N_b-1} C_{N_b-1}^k x^{N_b-1-k}\left((1-x)F(v|x)\right)^k, \tag{97}$$

where $F(v|x)$ is a distribution for buyers conditional on their participation in the auction with the reserve x, which is given by $F(v|x) = \frac{v-x}{1-x}$. The above expression has a simple interpretation. The factor $C_{N_b-1}^k \equiv \begin{pmatrix} N_b-1 \\ k \end{pmatrix} = \frac{(N_b-1)!}{(N_b-1-k)!k!}$ takes into account that all buyers are ex ante identical and describes the number of ways to separate N_b-1 buyers in two groups with k buyers in one group and the rest of them in the other group. Then, with probability x, each buyer may have a valuation below the reserve and therefore would not participate in bidding. With the probability $(1-x)$, each buyer participates. Since all private valuations are i.i.d., for each of the outcome when k buyers participate in the auction, the winning probability is $(F(v|x))^k$. Using the Newton's binomial law we obtain that $P_{N_b}(v) = (x + (1-x)F(v|x))^{N_b-1}$ from which equation (96) follows immediately. *QED*

Proof of Lemma 2: We start by calculating expected payoffs of buyer 2 conditional on the strategy of buyer 1. If the buyer 1 stays at x_*, then if buyer 2 also stays at x_*, there are two competing buyers at x_*, and, therefore, the optimal expected payoff of buyer 2 is given by

$$\pi_{x_*}^2(v_2|x_*) = \int_{x_*}^{v_2} dv' P_{x_*}^2(v'|x_*) = \int_{x_*}^{v_2} dv' v' = \frac{1}{2}\left(v_2^2 - x_*^2\right). \tag{98}$$

Analogous, if buyer 1 stays at x_* and buyer 2 goes to $x_* + \epsilon$, then buyer 2 has no one to compete with and hence $P_{x_*}^2(v|x) = 1$. Therefore, we have

$$\pi_{x_*}^2(v_2|x) = \int_x^{v_2} dv' P_{x_*}^2(v'|x) = \int_x^{v_2} dv' = v_2 - x. \tag{99}$$

Payoffs $\pi_x^2(v_2|x_*)$ and $\pi_x^2(v_2|x)$ can be calculated analogously. All payoffs are shown in Table 2.

TABLE 2

	buyer 1 at x_*	buyer 1 at $x > x_*$
buyer 2 at x_*	$\pi^2_{x_*}(v_2\vert x_*) = \frac{1}{2}\left(v_2^2 - x_*^2\right)$	$\pi^2_{x_*}(v_2\vert x) = v_2 - x_*$
buyer 2 at x	$\pi^2_x(v_2\vert x_*) = \pi^2_{x_*}(v_2\vert x) - (x - x_*)$	$\pi^2_x(v_2\vert x) = \pi^2_{x_*}(v_2\vert x_*) - \frac{1}{2}\left(x^2 - x_*^2\right)$

In order to evaluate the optimal reaction function of buyer 2, compare the payoffs (**??**) and (99). We have

$$\Delta\pi^2_{x_*\to x}(v_2|x_*) = v_2 - x - \frac{1}{2}\left(v_2^2 - x_*^2\right) = \frac{1}{2}\left(\mu_*^- - v_2\right)\left(v_2 - \mu_*^+\right), \tag{100}$$

with

$$\begin{aligned}\mu_*^- &= 1 - \sqrt{(1-x_*)^2 - 2(x - x_*)} \in (x_*, 1]\,, \\ \mu_*^+ &= 1 + \sqrt{(1-x_*)^2 - 2(x - x_*)} \geq 1.\end{aligned} \tag{101}$$

It is easy to show that $x \leq \mu_*^- \leq 1$ and $\mu_*^+ \geq 1$. From (100), it follows that $\Delta\pi^2_{x_*\to x}(v_2|x_*) \leq 0$ if $v_2 \leq \mu_*^-$ or $v_2 \geq \mu_*^+$, and $\Delta\pi^2_{x_*\to x}(v_2|x_*) \geq 0$ when $v_2 \in [\mu_*^-, \mu_*^+]$. Since $\mu_*^+ \geq 1$ while $v_2 \backsim U[0,1]$ and therefore bounded by one, the larger root μ_*^+ is irrelevant. Changing the notation for the smaller root as

$$\mu_* \equiv \mu_*^- = 1 - \sqrt{(1-x_*)^2 - 2(x - x_*)}, \tag{102}$$

we conclude that

$$\begin{aligned}\Delta\pi^2_{x_*\to x}(v_2|x_*) \leq 0, &\qquad v_2 \in [0, \mu_*]\,, \\ \Delta\pi^2_{x_*\to x}(v_2|x_*) \geq 0, &\qquad v_2 \in [\mu_*, 1]\,.\end{aligned} \tag{103}$$

Therefore, the reaction function of buyer 2 to the situation when buyer 1 stays at x_* depends on the valuation of buyer 2, v_2, and can be characterized as a *cut-off strategy*: buyer 2 stays at x_* if $v_2 \leq \mu_*$, and he moves to $x > x_*$ if $v_2 \geq \mu_*$.

Now, suppose that buyer 1 moves to x. Since $\pi^2_x(v_2|x) < \pi^2_{x_*}(v_2|x_*) \leq \pi^2_{x_*}(v_2|x)\ \forall v_2 \in [v_*, 1]$, the optimal reaction of buyer 2 in this case is to always stay at x_*, regardless his valuation v_2 (but as long as $v_2 \geq x$). *QED*

Proof of Proposition 1: We conjecture that cut-off strategies $\{\mu_i, i = 1, 2\}$ exist and, without loss of generality, satisfy the following condition

$$0 \leq x_* < x \leq \mu_1 \leq \mu_2 \leq 1. \tag{104}$$

We consider this condition to be without loss of generality since both cut-off will be defined endogenously. The optimal reaction of each buyer is found by comparing his expected payoffs across different bidding strategies. Strategy of buyer $i \in \{1,2\}$ can be characterized as follows: buyer i chooses LRP auction if his valuation satisfies $v \leq \mu_i$ and he chooses HRP auction if $v > \mu_i$. In order to calculate the expected payoffs we need to calculate winning probabilities for each buyer. Denote by $P^i_X(v|\mu_{j\neq i})$ the probability of buyer $i \in \{1,2\}$ winning the auction with the reserve price $X \in \{x_*, x\}$, conditional on the strategy of buyer j, $\mu_{j\neq i}$.

Consider buyer i. We first calculate his probability of winning the LRP auction which depends on the strategy of the buyer $j \neq i$, μ_j. Buyer j chooses LRP auction with probability μ_j (i.e. when $v \leq \mu_j$) and he chooses HRP auction with probability $1 - \mu_j$. Therefore, buyer i wins the LRP auction with probability 1 conditional on buyer j choosing HRP auction, while his probability of winning conditional on buyer j choosing LRP auction is equal to $\frac{1}{\mu_j} v$. In summary, the c.d.f. of buyer i winning LRP auction conditional on second buyer's strategy, μ_j, takes the following form

$$F^i_{X=x_*}(v|\mu_{j\neq i}) = \begin{cases} \frac{1}{\mu_{j\neq i}} v, & v \leq \mu_{j\neq i} \\ 1, & v > \mu_{j\neq i} \end{cases}, \ i \in \{1,2\}. \tag{105}$$

Using conditional c.d.f. (105) we can find the total probability of winning LRP auction for buyer i as

$$\begin{aligned} P^i_{X=x_*}(v|\mu_{j\neq i}) &= (1-\mu_{j\neq i}) \cdot 1 + \mu_{j\neq i} F^i_{x_*}(v|\mu_{j\neq i}) \\ &= \begin{cases} 1 - \mu_{j\neq i} + v, & v \leq \mu_{j\neq i} \\ 1, & v > \mu_{j\neq i} \end{cases}, \ i \in \{1,2\}. \end{aligned} \tag{106}$$

Next, we are going to calculate buyer i's probability of winning HRP auction. If buyer i's valuation is in the interval $\mu_i < v \leq \mu_{j\neq i}$ then buyer i chooses HRP auction only if buyer j chooses LRP auction, which happens with probability μ_j (since with probability $1-\mu_j$ buyer j chooses HRP auction and buyer i is better off choosing LRP auction which he wins in this case with probability 1). In this case buyer i wins HRP auction with probability 1 thus resulting in the conditional probability of winning HRP auction equal to μ_j on the interval $\mu_i < v \leq \mu_{j\neq i}$. If buyer i has valuation greater than μ_j, he wins the HRP auction with probability 1 conditional on buyer j choosing LRP auction, i.e. with probability μ_j, while his probability of winning conditional on buyer j choosing HRP auction, which happens with probability $1-\mu_j$, is equal to $\frac{v-\mu_j}{1-\mu_j}$. Thus the total probability of winning HRP auction for buyer i when his valuation is greater than μ_i is equal to

$$P^i_{X=x}(v|\mu_{j\neq i}) = \mu_{j\neq i} \cdot 1 + (1-\mu_{j\neq i}) \frac{v-\mu_{j\neq i}}{1-\mu_{j\neq i}} = v, \ v > \mu_{j\neq i}, \ i \in \{1,2\}, \tag{107}$$

which leads to the following expression for $P^i_{X=x}(v|\mu_{j\neq i})$

$$P^i_{X=x}(v|\mu_{j\neq i}) \begin{cases} \mu_{j\neq i}, & \mu_i < v \leq \mu_{j\neq i} \\ v, & v > \mu_{j\neq i} \end{cases}, \ i \in \{1,2\}. \tag{108}$$

The optimal expected payoff of buyer i with valuation $v \leq \mu_{j\neq i}$ if he chooses LRP auction, $\pi^i_{x_*}(v|\mu_{j\neq i})$, is given by

$$\pi^i_{x_*}(v|\mu_{j\neq i}) = \int_{x_*}^{v} dv' P^i_{X=x_*}(v'|\mu_{j\neq i}) = \int_{x_*}^{v} dv' \left(1 - \mu_{j\neq i} + v'\right), \ v \leq \mu_{j\neq i}, \tag{109}$$

while his optimal expected payoff from choosing HRP auction is equal to

$$\pi^i_{x}(v|\mu_{j\neq i}) = \int_{x}^{v} dv' P^i_{X=x}(v'|\mu_{j\neq i}) = \int_{x}^{v} dv' \mu_{j\neq i}, \ v \leq \mu_{j\neq i}. \tag{110}$$

Our goal is to find $\mu_i < \mu_{j\neq i} \leq 1$ such that the following two optimality conditions are satisfied

$$\pi^i_{x_*}(\mu_i|\mu_{j\neq i}) = \pi^i_x(\mu_i|\mu_{j\neq i}),\ i,j \in \{1,2\}, \tag{111}$$

$$\pi^i_{x_*}(v|\mu_{j\neq i}) \leq \pi^i_x(v|\mu_{j\neq i}),\ \mu_i < v \leq \mu_{j\neq i},\ i,j \in \{1,2\}. \tag{112}$$

The first condition is the optimality of cut-off strategies, i.e. buyer i is indifferent between LRP and HRP auctions at the cut-off μ_i, while the second condition ensures the *optimality of deviation to HRP auction if buyer i's valuation is above the cut-off* $\mu_i, i = 1,2$. Let us rewrite equation (111) separately for each buyer taking into account that $\mu_i < \mu_{j\neq i} \leq 1$

$$\int_{x_*}^{\mu_i} dv\,(1 - \mu_{j\neq i} + v) = \int_{x}^{\mu_i} dv \mu_{j\neq i}, \tag{113}$$

$$\int_{x_*}^{\mu_i} dv\,(1 - \mu_i + v) + \int_{\mu_i}^{\mu_{j\neq i}} dv = \int_{x}^{\mu_i} dv \mu_i + \int_{\mu_i}^{\mu_{j\neq i}} dv v. \tag{114}$$

If condition (113) holds it is easy to verify that condition (114) cannot be satisfied.[11] Moreover, since the left hand side of condition (114) is greater or equal to the right hand side of the same condition we can conclude that buyer j is always better off choosing LRP auction if (113) holds or, equivalently, it means that optimal cut-off strategy for buyer j is $\mu_{j\neq i} = 1$.

We can now use equation (113) together with $\mu_{j\neq i} = 1$ to solve for the optimal cut-off strategy of buyer i. It immediately follows from (113) that

$$\frac{1}{2}\left(\mu_i^2 - x_*^2\right) = \mu_i - x, \tag{115}$$

leading to

$$\mu_i = 1 - \sqrt{1 + x_*^2 - 2x} = \widehat{\mu}(x, x_*).$$

QED

Proof of Theorem 1: We conjecture that the symmetric equilibrium exists for some reserve price $x_* \in [0,1]$, and then analyze whether one of the sellers (say seller 1) would be better off by deviating to $x \neq x_*$. It requires us to first analyze buyers' reaction functions to the seller's deviation, which has been done in the previous section and can be summarized as follows: One of the buyers (say buyer 2) always stays in the LRP auction, while the other buyer stays in LRP auction if his private valuation is lower than cutoff $\mu_1 = \widehat{\mu}$ and goes to the HRP auction if his valuation is higher than the cutoff. Since we are looking for symmetric NE in reserve prices we first solve for the optimal reserve price in the LRP auction while requiring that $x = x_*$, and then solve for the optimal reserve price in the HRP using the same requirement.

[11]Indeed, since $\mu_i \leq \mu_j$ the integral on the left hand side of condition (113) is less than the first integral on the left hand side of condition (114), while the integral on the right hand side of condition (113) is greater than the first integral on the right hand side of condition (114). Since all valuations are less or equal to one, the second integral on the left hand side of condition (114) is greater than the second integral on the right hand side of condition (114), thus implying that condition condition (114) should be inequality rather than equality.

In the case when $X = x_*$ we can use condition (2) to obtain

$$\int_{x_*}^{\mu_j} dv \pi^j_{X=x_*}(v|\mu_{-j}) = \int_{x_*}^{\mu_j} dv \int_{x_*}^{v} dv' P^j_{X=x_*}(v'|\mu_{-j}) = \int_{x_*}^{\mu_j} dv(\mu_j - v) P^i_{X=x_*}(v|\mu_{-i}). \tag{116}$$

where we have changed an order of integration in the second line and relabeled the integration variable in the third line. Substituting (**??**) into (6), we obtain in the case of $X = x_*$

$$\Pi^i_{X=x_*} = \int_{x_*}^{\widehat{\mu}} dv(2v - \widehat{\mu}) P^1_{X=x_*}(v|\mu_2 = 1) + \int_{x_*}^{1} dv(2v-1) P^2_{X=x_*}(v|\mu_1 = \widehat{\mu}) \tag{117}$$
$$= \int_{x_*}^{\widehat{\mu}} dv\,(2v - \widehat{\mu})\,v + \int_{x_*}^{\widehat{\mu}} dv\,(2v-1)\,(v + 1 - \widehat{\mu}) + \int_{\widehat{\mu}}^{1} dv\,(2v-1)\,.$$

The optimal reserve price in the LRP auction, x_*, is set by maximizing seller i's expected payoff

$$\frac{\partial \Pi^i_{X=x_*}}{\partial x_*}|_{x=x_*} = 0,$$

while taking into account that the cut-off $\widehat{\mu}$ depends on both reserve prices x and x_*

$$\frac{\partial \widehat{\mu}(x, x_*)}{\partial x_*}|_{x=x_*} = -\frac{x_*}{1-x_*} \text{ and } \frac{\partial \widehat{\mu}(x, x_*)}{\partial x}|_{x=x_*} = \frac{1}{1-x_*}, \tag{118}$$

as well as that $\widehat{\mu}(x_*, x_*) = x_*$. The first order condition for the optimal LRP reserve price reduces to the following quadratic equation

$$\frac{\partial \Pi^i_{X=x_*}}{\partial x_*}|_{x=x_*} = \frac{1 - 3x_* + x_*^2}{1 - x_*} = 0, \tag{119}$$

which has the following solution on the interval $x_* \in [0, 1]$

$$x_d = \frac{3 - \sqrt{5}}{2} \approx 0.382. \tag{120}$$

Since $\frac{\partial \Pi^i_{X=x_*}}{\partial x_*}|_{x=x_*} > 0$ for $x_* < x_d$ and $\frac{\partial \Pi^i_{X=x_*}}{\partial x_*}|_{x=x_*} < 0$ for $x_* > x_d$ we immediately conclude that $\Pi^i_{X=x_*}$ has a local maximum at $x_* = x_d$. Therefore NE may exists on the interval $[0, x_d]$, i.e. since $\Pi^i_{X=x_*}$ is increasing on the interval $[0, x_d]$, seller i would choose to deviate downward by picking a reserve price in this interval.

We now consider HRP auction. If one of the sellers, i, stays in LRP auction and the second seller, $-i$, deviates to HRP auction, he will be followed by the buyer 1 if his valuation is greater than than cutoff $\mu_1 = \widehat{\mu}$. In this case buyer 1 wins HRP auction with probability $P^1_{X=x}(v|1) = \mu_2 = 1$ by submitting a winning bid, $b_1^*(v)$, equal to x (Riley and Samuelson (1981)), thus yielding the following expected payoff to the seller who deviates to HRP auction

$$\Pi_{X=x}^{-i} = \int_{\widehat{\mu}}^{1} dv' P_{X=x}^{1}\left(v'|1\right) b_1^*\left(v'\right) \tag{121}$$

$$= \int_{\widehat{\mu}}^{1} dv' x = \underbrace{x}_{\text{winning bid}} \underbrace{(1-\widehat{\mu})}_{\Pr(N_b=1)}.$$

In order to find the optimal reserve price we optimize Π_x^{-i} with respect to x taking into account that μ_* is itself a function of x to obtain

$$\frac{\partial \Pi_{X=x}^{-i}}{\partial x}|_{x=x_*} = 1 - x_* - x_* \frac{\partial \widehat{\mu}(x, x_*)}{\partial x}|_{x=x_*} = \frac{1-3x_*+x_*^2}{1-x_*} = 0, \tag{122}$$

which is the same as (119) and, therefore, has the same solution as (120), $x_u = \frac{3-\sqrt{5}}{2}$. *QED*

Proof of Proposition 2: Analogous to the case of two sellers and two buyers, we conjecture that cut-off strategies $\{\mu_i, i = 1, 2, 3\}$ exist and satisfy the following condition

$$0 \le x_* < x \le \mu_1 \le \mu_2 \le \mu_3 \le 1. \tag{123}$$

The optimal expected payoff of buyer i with valuation $v \le \mu_{-i}$ if he chooses LRP auction, $\pi_{x_*}^i(v|\mu_{-i})$, is given by

$$\pi_{x_*}^i(v|\mu_{-i}) = \int_{x_*}^{v \le \min(\mu_{-i})} dv' \prod_{j \neq i}^{3} P_{X=x_*}^i\left(v'|\mu_j\right), \tag{124}$$

while his optimal expected payoff from choosing HRP auction is equal to

$$\pi_{x}^i(v|\mu_{-i}) = \int_{x}^{v \le \min(\mu_{-i})} dv' \prod_{j \neq i}^{3} P_{X=x}^i\left(v'|\mu_j\right), \tag{125}$$

where winning probabilities are given by (106) and (108) respectively. We are going to use the above expressions to find the cut-off thresholds $\{\mu_i, i = 1, 2, 3\}$.

Consider buyer 1. His optimal expected payoffs are

$$\pi_{x_*}^1(v|\mu_{-1}) = \int_{x_*}^{v \le \mu_2} dv' \prod_{j=2}^{3} P_{X=x_*}^1\left(v'|\mu_j\right) = \int_{x_*}^{v \le \mu_2} dv' \left(1-\mu_2+v'\right)\left(1-\mu_3+v'\right), \tag{126}$$

and

$$\pi_{x}^1(v|\mu_{-1}) = \int_{x}^{v \le \mu_2} dv' \prod_{j=2}^{3} P_{X=x}^1\left(v'|\mu_j\right) = \int_{x}^{v \le \mu_2} dv' \mu_2 \mu_3. \tag{127}$$

The cut-off strategy $\mu_1$is defined by conditions (111) and (112) which in this case take the following form

$$\int_{x_*}^{\mu_1} dv\,(1-\mu_2+v)\,(1-\mu_3+v) = \int_{x}^{\mu_1} dv\mu_2\mu_3, \tag{128}$$

and

$$\int_{\mu_1}^{v\le\mu_2} dv'\left(1-\mu_2+v'\right)\left(1-\mu_3+v'\right) \le \int_{\mu_1}^{v\le\mu_2} dv'\mu_2\mu_3. \tag{129}$$

Since inequality (129) has to be satisfied point-by-point it reduces to

$$(1-\mu_2+v)\,(1-\mu_3+v) \le \mu_2\mu_3,\ \ \mu_1 \le v \le \mu_2, \tag{130}$$

which after simple algebra reduces to

$$1+v \le \mu_2+\mu_3,\ \ \mu_1 \le v \le \mu_2, \tag{131}$$

and since it has to be satisfied on the interval $\mu_1 \le v \le \mu_2$, it leads to $\mu_3 \ge 1$ as the sufficient condition for that. As a result we have that μ_3 must be equal to 1.

We consider buyer 2 next. His optimal expected payoffs are

$$\pi^2_{x_*}(v|\mu_{-2}) = \int_{x_*}^{\mu_1} dv'\left(1-\mu_1+v'\right)\left(1-\mu_3+v'\right) + \int_{\mu_1}^{v\le\mu_3} dv'\left(1-\mu_3+v'\right), \tag{132}$$

and

$$\pi^2_{x}(v|\mu_{-2}) = \int_{x}^{\mu_1} dv'\mu_1\mu_3 + \int_{\mu_1}^{v\le\mu_3} dv'v'\mu_3, \tag{133}$$

which immediately implies the following form of the condition (111)

$$\begin{aligned}\int_{x_*}^{\mu_1} dv\,(1-\mu_1+v)\,(1-\mu_3+v) + \int_{\mu_1}^{\mu_2} dv\,(1-\mu_3+v)\\ = \int_{x}^{\mu_1} dv\mu_1\mu_3 + \int_{\mu_1}^{\mu_2} dvv\mu_3,\ \ \mu_1 \le v \le \mu_3 = 1.\end{aligned} \tag{134}$$

Comparing equalities (134) and (128) and taking into account that

$$\int_{x_*}^{\mu_1} dv\,(1-\mu_1+v)\,(1-\mu_3+v) \ge \int_{x_*}^{\mu_1} dv\,(1-\mu_2+v)\,(1-\mu_3+v)\,, \tag{135}$$

and

$$\int_{x}^{\mu_1} dv\mu_1\mu_3 \le \int_{x}^{\mu_1} dv\mu_2\mu_3, \tag{136}$$

since $\mu_1 \leq \mu_2$, we observe that the condition (134) can only be satisfied as inequality with its right hand side greater or equal to its left hand side. Equivalently, it means that buyer 2 always better off choosing LRP auction to HRP auction and thus that $\mu_2 = 1$.

Finally, having established that $\mu_3 = \mu_2 = 1$ we can use equation (128) which reduces to

$$\int_{x_*}^{\mu_1} dv v^2 = \int_{x}^{\mu_1} dv, \tag{137}$$

to find $\widehat{\mu}_1$ which solves

$$\mu_1^3 - x_*^3 = 3(\mu_1 - x). \tag{138}$$

QED

Proof of Proposition 3: Analogous to the previous two cases, we conjecture that cut-off strategies, $\{\mu_i, i = 1, 2, 3, ..., N\}$, exist and satisfy the following "ladder" structure

$$0 \leq x_* < x \leq \mu_1 \leq \mu_2 \leq ... \leq \mu_N \leq 1. \tag{139}$$

These cutoffs are determined for each buyer i by the "indifference" condition

$$\pi^i_{x_*}(\mu_i|\mu_{-i}) = \pi^i_x(\mu_i|\mu_{-i}), \; i \in \{1, ..., N\}, \tag{140}$$

which with the help of the expressions for buyers' expected payoffs (124) and (125) takes the following form

$$\begin{aligned}
&\int_{x_*}^{\mu_1} dv \prod_{j \neq i}^{N} P^i_{X=x_*}(v|\mu_j) + \sum_{m=2}^{i} \int_{\mu_{m-1}}^{\mu_m} dv \prod_{j \neq i}^{N} P^i_{X=x_*}(v|\mu_j) = \\
&\int_{x}^{\mu_1} dv \prod_{j \neq i}^{N} P^i_{X=x}(v|\mu_j) + \sum_{m=2}^{i} \int_{\mu_{m-1}}^{\mu_m} dv \prod_{j \neq i}^{N} P^i_{X=x}(v|\mu_j) \Longrightarrow \\
&\int_{x_*}^{\mu_1} dv \prod_{j \neq i}^{N} (1 - \mu_j + v) + \sum_{m=2}^{i} \int_{\mu_{m-1}}^{\mu_m} dv \prod_{(j=m) \neq i}^{N} (1 - \mu_j + v) = \\
&\int_{x}^{\mu_1} dv \prod_{j \neq i}^{N} \mu_j + \sum_{m=2}^{i} \int_{\mu_{m-1}}^{\mu_m} dv v^{i-1} \prod_{(j=m) \neq i}^{N} \mu_j.
\end{aligned} \tag{141}$$

The second set of conditions for cut-off strategies states that buyer i is better off in the HRP auction if his valuation is in the interval $\mu_i < v \leq \mu_{i+1}$ when his expected payoffs in the LRP and HRP auctions satisfy the following condition

$$\pi^i_{x_*}(v|\mu_{-i}) \leq \pi^i_x(v|\mu_{-i}), \; \mu_i < v \leq \mu_{i+1}, \tag{142}$$

while he is better off in the LRP auction if his valuation is in the interval $\mu_{i-1} < v \leq \mu_i$ when his expected payoffs in the LRP and HRP auctions satisfy the following condition

$$\pi^i_{x_*}(v|\mu_{-i}) \geq \pi^i_x(v|\mu_{-i}), \; \mu_{i-1} < v \leq \mu_i, \tag{143}$$

where the expected payoffs are

$$\pi^i_{x_*}(v|\mu_{-i}) = \int\limits_{x_*}^{\mu_1} dv \prod_{j\neq i}^{N} (1-\mu_j+v) + \sum_{m=2}^{i-1} \int\limits_{\mu_{m-1}}^{\mu_m} dv \prod_{(j=m)\neq i}^{N} (1-\mu_j+v) + \\ + \int\limits_{\mu_{i-1}}^{v} dv' \prod_{j>i}^{N} \left(1-\mu_j+v'\right),$$

$$\pi^i_{x}(v|\mu_{-i}) = \int\limits_{x}^{\mu_1} dv \prod_{j\neq i}^{N} \mu_j + \sum_{m=2}^{i-1} \int\limits_{\mu_{m-1}}^{\mu_m} dv v^{i-1} \prod_{(j=m)\neq i}^{N} \mu_j + \int\limits_{\mu_{i-1}}^{v} dv' \left(v'\right)^{i-1} \prod_{j>i}^{N} \mu_j.$$

Consider now difference in the expected payoffs in LRP and HRP for buyer i when his private valuation is greater than μ_{i-1}

$$\pi^i_{x_*}(v|\mu_{-i}) - \pi^i_{x}(v|\mu_{-i}) = \int\limits_{x_*}^{\mu_1} dv \left(\prod_{j\neq i}^{N} (1-\mu_j+v) - \prod_{j\neq i}^{N} \mu_j\right) + \\ + \sum_{m=2}^{i-1} \int\limits_{\mu_{m-1}}^{\mu_m} dv \left(\prod_{(j=m)\neq i}^{N} (1-\mu_j+v) - v^{i-1} \prod_{(j=m)\neq i}^{N} \mu_j\right) + \\ + \int\limits_{\mu_{i-1}}^{v} dv' \left(\prod_{j>i}^{N} \left(1-\mu_j+v'\right) - \left(v'\right)^{i-1} \prod_{j>i}^{N} \mu_j\right).$$

Since $\mu_{i-1} \leq \mu_i$ we have that

$$\prod_{j\neq i-1}^{N} (1-\mu_j+v) \leq \prod_{j\neq i}^{N} (1-\mu_j+v), \tag{144}$$

$$\prod_{j\neq i-1}^{N} \mu_j \geq \prod_{j\neq i}^{N} \mu_j.$$

These two conditions together with relation (141) imply

$$\int\limits_{x_*}^{\mu_1} dv \left(\prod_{j\neq i}^{N} (1-\mu_j+v) - \prod_{j\neq i}^{N} \mu_j\right) + \sum_{m=2}^{i-1} \int\limits_{\mu_{m-1}}^{\mu_m} dv \left(\prod_{(j=m)\neq i}^{N} (1-\mu_j+v) - v^{i-1} \prod_{(j=m)\neq i}^{N} \mu_j\right) \geq 0,$$

thus leading to the following inequality on the interval $v > \mu_{i-1}$

$$\pi^i_{x_*}(v|\mu_{-i}) - \int\limits_{\mu_{i-1}}^{v} dz \prod_{j>i}^{N} (1-\mu_j+z) \geq \pi^i_{x}(v|\mu_{-i}) - \int\limits_{\mu_{i-1}}^{v} dz z^{i-1} \prod_{j>i}^{N} \mu_j. \tag{145}$$

We now postulate that there exists such buyer's number $i = K$ that all buyers with indices $i \geq K+1$ and private valuations in the interval $\mu_{i-1} < v \leq \mu_i$ would prefer LRP auction. It immediately follows from the condition (145) that $\pi^i_{x_*}(v|\mu_{-i}) \geq \pi^i_x(v|\mu_{-i})$ if

$$\prod_{j>i}^{N}(1-\mu_j+v) \geq v^{i-1}\prod_{j>i}^{N}\mu_j,\ \forall i \geq K+1,\ v \in [\mu_{i-1}, \mu_i], \tag{146}$$

Since $\mu_i = 1\ \forall i \in [K+1, N]$ we can simplify the above condition to

$$v^{N-K-1}\prod_{j>i}^{K+1}(1-\mu_j+v) \geq v^{i-1}\prod_{j>i}^{K+1}\mu_j,\ \forall i \geq K+1,\ v \in [\mu_{i-1}, \mu_i]. \tag{147}$$

Condition (147) must hold for $i = K+1$ yielding

$$v^{N-K-1} \geq v^K \Leftrightarrow K \geq \frac{N-1}{2},\ \forall v \in [\mu_K, \mu_{K+1}]. \tag{148}$$

We now consider buyers with indices $i \leq K$ and private valuations in the interval $\mu_i < v \leq \mu_{i+1}$. By construction, these buyers prefer HRP auction, from where it immediately follows that $\pi^i_{x_*}(v|\mu_{-i}) < \pi^i_x(v|\mu_{-i})$ if

$$\prod_{j>i}^{N}(1-\mu_j+v) < v^{i-1}\prod_{j>i}^{N}\mu_j,\ \forall i \leq K,\ v \in [\mu_i, \mu_{i+1}], \tag{149}$$

Since $\mu_i = 1\ \forall i \in [K+1, N]$ we can simplify the above conditions to

$$v^{N-K}\prod_{j>i}^{K}(1-\mu_j+v) < v^{i-1}\prod_{j>i}^{K}\mu_j,\ \forall i \leq K,\ v \in [\mu_i, \mu_{i+1}]. \tag{150}$$

Specifically, condition (151) must hold for $i = K$ immediately yielding

$$v^{N-K} < v^{K-1} \Leftrightarrow K < \frac{N+1}{2},\ \forall v \in [\mu_K, \mu_{K+1}]. \tag{151}$$

Conditions (148) and (151) provide lower and upper bounds on the critical buyer's index above which all buyers always choose LRP auction

$$\frac{N-1}{2} \leq K < \frac{N+1}{2}. \tag{152}$$

The reason for the bound instead of a single threshold is quite simple: the number of buyers can be either even or odd, in which case we pick the threshold which gives an integer

$$K = \begin{cases} \frac{2n+1-1}{2} = n,\ N = 2n+1, \\ \frac{2n}{2} = n,\ N = 2n \end{cases}. \tag{153}$$

Next, we assume that high, x, and low, x_*, reserve prices satisfy

$$x = x_* + \epsilon,\ \epsilon \to 0, \tag{154}$$

meaning that we consider small upward deviations. We then propose a candidate solution for the cut-off thresholds of first K buyers in the following form

$$\widehat{\mu}_i = x_* + \epsilon + \Delta_i,\ \Delta_i \to 0,\ \epsilon \to 0,\ i = 1, ..., K, \tag{155}$$

and then substitute it into Eq. (141) to obtain

$$\left(\epsilon + \Delta_1 + \sum_{m=2}^{i} \left(\Delta_m - \Delta_{m-1}\right)\right) x_*^{N-K} = \Delta_1 x_*^{K-1} + ... + \left(\Delta_i - \Delta_{i-1}\right) x_*^{i-1} x_*^{K-i} \Rightarrow \tag{156}$$

$$\left(\epsilon + \Delta_i\right) x_*^{N-2K+1} = \Delta_i \Rightarrow \Delta_i = \epsilon \frac{x_*^{N-2K+1}}{1 - x_*^{N-2K+1}} = \epsilon \begin{cases} \frac{x_*^2}{1-x_*^2},\ N = 2n+1, \\ \frac{x_*}{1-x_*},\ N = 2n \end{cases}.$$

It immediately follows from (156) that all first K buyers follow the same cutoff strategy. *QED*

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