A Theory of Repurchase Agreements, Collateral Re-use, and Repo Intermediation*

Piero Gottardi
European University Institute

Vincent Maurin
European University Institute

Cyril Monnet
University of Bern, SZ Gerzensee
& Swiss National Bank

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Abstract

This paper characterizes repurchase agreements as equilibrium contracts starting from first principles. We show that repos trade-off the borrower's desire to augment its consumption today with the lender's desire to hedge against future market risk. As a result, safer assets will command a lower haircut and a higher liquidity premium relative to riskier assets. Haircuts may also be negative. When lenders can re-use the asset they receive in a repo, we show that collateral constraints are relaxed and borrowing increases. Re-usable assets should command low haircuts. Finally, endogenous credit intermediation arises whereby trusty counterparties re-use collateral and borrow on behalf of riskier counterparties. These findings are helpful to rationalize chains of trade observed on the repo market.

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1 Introduction

According to Gorton and Metrick (2012), the financial panic of 2007-08 started with a run on the market for repurchase agreements (repos). Their paper was very influential in shaping our understanding of the crisis. It was quickly followed by many attempts to understand repo markets more deeply, both empirically and theoretically as well as calls to regulate these markets.[1]

A repo is the sale of an asset combined with a forward contract that requires the original seller to repurchase the asset at a given price. Repos are different from simple collateralized loans in (at least) one important way. A repo lender obtains the legal title to the pledged collateral and can thus use the collateral during the length of the forward contract. This practice is known as re-use or re-hypothecation.[2] With standard collateralized loans, borrowers must agree to grant the lender similar rights. This special feature of repos has attracted a lot of attention from economists and regulators alike.[3]

Repos are extensively used by market makers and dealer banks as well as other financial institutions as a source of funding, to acquire securities that are on specials, or simply to obtain a safe return on idle cash. As such, they are closely linked to market liquidity and so they are important to understand from the viewpoint of Finance. The Federal Reserve Bank, the central bank in the United States, and other central banks use repos to steer the short term nominal

[1] See Acharya (2010) "A Case for Reforming the Repo Market" and (FRBNY 2010)

[2] There is a subtle difference between US and EU law. Under EU law, a repo is a transfer of the security's title to the lender. However, as Comotto (2014) explains, a repo in the US falls under New York law, and "Under the law of New York, which is the predominant jurisdiction in the US, the transfer of title to collateral is not legally robust. In the event of a repo seller becoming insolvent, there is a material risk that the rights of the buyer to liquidate collateral could be successfully challenged in court. Consequently, the transfer of collateral in the US takes the form of the seller giving the buyer (1) a pledge, in which the collateral is transferred into the control of the buyer or his agent, and (2) the right to re-use the collateral at any time during the term of the repo, in other words, a right of re-hypothecation. The right of re-use of the pledged collateral (...) gives US repo the same legal effect as a transfer of title of collateral."

[3] Aghion and Bolton (1992) argue that securities are characterize by cash-flow rights but also control rights. Collateralized loans grant neither cash-flow rights nor control rights over the collateral to the lender. To the contrary, a repo that allows re-use, gives the lender full control rights over the security as well as over its cash-flows (the interest rate usually controls for cash-flows that accrue during the length of the contract). There is a legal difference between re-use and rehypothecation, but economically they are equivalent, see e.g. Singh (2011).

interest rate. The Fed's newly introduced reverse repos are considered an effective tool to increase the money market rate when there are large excess reserves. As such, repos are essential to the conduct of monetary policy. Finally, firms also rent capital and use collateralized borrowing and some forms of repos to finance their activities or hedge exposures (notably interest rate risk, see BIS, 1999). This affects real activities, and so repos are also an important funding instrument for the macroeconomy.

Most existing research papers study specific aspects of the repo markets, e.g. exemption from automatic stay, fire sales, etc., taking the repo contract and most of its idiosyncrasies as given. These theories leave many fundamental questions unanswered, such as why are repos different from collateralized loans? What is the nature of the economic problem solved by the repo contract? If one is to answer these questions, to understand the repo market and the effect of regulations, one cannot presume the existence or the design of repo contracts. In this paper we thus study repos from first principles.

More precisely, starting from simple preferences and technology, we explain why repos is the funding instrument of choice, the impact of repos on asset prices and their liquidity premium, as well as collateral re-use. We do not take the structure of financial instruments as given, but we borrow techniques from security design to show how repos arise endogenously in equilibrium. In our model the equilibrium contract shares many features with repos: The borrower sells an asset spot combined with a forward contract promising a re-purchase at an agreed price.

The model has three periods and has two types of agents, a natural borrower and a natural lender. The borrower is risk neutral and more or less trustworthy, while the lender is risk averse. The borrower is endowed with an asset that yields an uncertain payoff in the last period. The payoff realization becomes known in the second period. Therefore, the second period price of the asset will fully reflect its payoff. In the first period the borrower wants to consume more than his resources. He could sell the asset to the lender, but the lender will not pay much for it, as it exposes him to price risk. Instead, the borrower can obtain resources from the lender by selling the asset combined with a forward contract promising to repurchase the asset in period 2. The lender prefers this contract to an outright sale, whenever the forward contract offers him some insurance. However, a constant repurchase price may trigger default in the lowest states since the value of collateral could fall below the promised repayment[4]. In other words, the no-default constraint of the borrower puts an upper bound on the repurchase price of the repo contract. In our model, a borrower's trustworthiness depends positively on the size of a non-pecuniary cost for default and determines the tightness of this constraint.

As a result, agents may not be able to finance the first-best level of borrowing in the lowest states. The forward price of the equilibrium repo contract is a flat schedule only when the asset is abundant or the borrower is trustworthy enough. When the asset is scarce and/or the borrower is not so trustworthy, the borrowing constraint binds and the lender is not perfectly hedged against price risk. In general, the repurchase price schedule thus consists of an increasing part for lowest states and a constant part for highest states of the world.

Using this equilibrium contract we derive equilibrium properties of haircuts, and liquidity premium. We show that haircuts are increasing with asset risk. Interestingly, haircuts may even be negative when the borrower is a good counterparty but the asset is scarce. Also, we find that safer assets command higher liquidity premium than riskier assets. This is intuitive: safer assets are better collateral as the forward contract will better insure the lender. This property increases the equilibrium value of safe assets relative to risky ones.

In Section 4, we introduce collateral re-use, to account for a natural feature of the repo contract. The lender indeed acquires ownership of the asset used as collateral in the repo transaction. Agents value this automatic right to re-use collateral when the asset supply is scarce. A borrower can ultimately leverage up and increase his borrowing capacity with respect to a situation where re-use is prohibited. To fix ideas, suppose indeed the collateral commands a negative haircut. This means that the borrower can borrow \$110 say, for an asset which can be sold for \$100. So let us suppose the borrower conducts this trade and so gets \$110 from the lender. With this \$110, the borrower can then purchase the asset for \$100

[4] In practice, even in the absence of outright default, traders opportunistically delay the settlment of transactions, as documented by Fleming and Garbade (2005). In our model, the repurchase price can be made state-contigent which rules out default in equilibrium. The state-contigency somehow plays a similar role to margin adjustment in actual transactions which serves a similar purpose.

leaving him with \$10 and the asset. But then he can re-pledge the asset with the lender (or another one), to get another \$110, etc. The borrower can then generate more resources this way and get closer to his goals. As we explained, negative haircuts may arise endogenously in the model when the borrower is trustworthy and the asset is scarce. When haircuts are low but positive, the lender pockets these gains from re-use. Interestingly, in the limit case where the non-pecuniary penalty is zero, our results are similar to Maurin (2015) and re-use does not affect the equilibrium allocation. Overall, one implication of our model is that collateral re-use should be more prevalent for assets that command low haircuts.

Finally, Section 4.2 analyzes collateral re-use in connection with repo intermediation. We argue that accounting for the right to re-use collateral sheds light on the microstructure of repo markets. In practice, dealer banks indeed make for a significant share of this market where they play a role as intermediaries between natural borrowers and lenders. To frame our argument in a realistic setting, consider a Hedge Fund in need for cash and a cash rich Money Market Fund (MMF). The Hedge Fund would typically borrow from a dealer bank through a repo. The dealer bank would then tap in the MMF cash pool through another repo to finance the transaction. We show that an intermediation equilibrium may exist when the dealer bank is more trustworthy than the hedge fund. Our analysis showed indeed that borrowing capacity increases in the quality of the counterparty to the MMF. Hence, intermediation allows to combine the asset of the risky hedge fund with the higher trustowrthiness of the dealer bank. Remarkably, dealer bank arises endogenously as a repo intermediary although the Hedge Fund would be free to trade with the MMF[5].

Relation to the literature

Gorton and Metrick (2012) argue that the recent crisis started with a run on repo whereby funding dropped dramatically for many financial institutions. Subsequent studies by Krishnamurty et al. (2014) and Copeland et al. (2014) have qualified this finding by showing that the run was specific to the - large - bilateral segment of the repo market. Recent theoretical works indeed highlighted some features of repo contracts as sources of funding fragility. As a short-term debt

[5]Our analysis thus extends Infante (2015) and Muley (2015) which assume intermediation exogenously.

instrument to finance long-term assets, Zhang (2014) and Martin et al. (2014) show that repos are subject to roll-over risk. Antinolfi et al. (2015) emphasize the trade-off from the exemption from automatic stay for repo collateral. Lenders easy access to collateral from defaulted loans enhance market liquidity but creates the potential for fire sales, a point also made by Infante (2013). These papers take repurchase agreements as given while we want to understand why they emerge as a funding instrument. In this respect, we share the objective of Narajabad and Monnet (2012), Tomura (2013) and Parlatore (2015) to explain why repos prevail over outright sale of the collateral. In these papers, agents choose collateralized loans or repos because lenders might not be able to extract the full value of the asset due to bargaining frictions on spot markets[6]. In our competitive model, repos are essential because the repurchase price provides insurance to the initial buyer. Our theory thus rationalizes haircuts since the hedging components explains why borrowers choose repos when they could obtain more cash in the spot market[7]. To derive the repo contract, we follow Geanakoplos (1996) , Araújo et al. (2000) and Geanakoplos and Zame (2014) where collateralized promises traded by agents are selected in equilibrium. Our model differs from theirs as we allow for an extra non-pecuniary penalty for default in the spirit of Dubey et al. (2005).

In the second part of the paper, we explore a crucial feature of repurchase agrements whereby the borrower tranfers the legal title to the collateral to the lender. Singh and Aitken (2010) and Singh (2011) argue that collateral re-use or rehypothecation lubricates transactions in the financial system[8]. However rehypothecation may entail risks for collateral pledgors as explained by Monnet, 2011. While Bottazzi et al. (2012) or Andolfatto et al. (2014) abstract from the limited commitment problem of the collateral receiver, Maurin (2015) shows in a general environment that re-use risk seriously mitigates the benefits from circulation. In our model indeed, re-use relaxes collateral constraints only thanks to the extra penalty for default for borrowers besides the collateral loss. Asset re-use then plays a role similar to pyramiding (see Gottardi and Kubler, 2015) whereby lenders pledge the face value of a debt owed rather than re-pledging the asset backing the

[6]In Parlatore (2015), cash in the market pricing delivers the same result.

[7]In particular, we do not need transactions costs as suggested by Duffie (1996).

[8]Fuhrer et al. (2015) estimate an average 5% re-use rate in the Swiss repo market over 2006-2013.

debt. Muley (2015) rationalizes the choice of rehypothecation over pyramiding by the difference in income pledgeability across types of agents. We highlight the connecton between collateral re-use and repo market intermediation as in Infante (2015) and Muley (2015). Unlike these papers, intermediation arises endogenously as trustworthy agents re-use the collateral from risky counterparties to borrow on their behalf. Endogenous credit intermediation is efficient since the intermediation rent-seeking behavior in Farboodi (2015) is absent from our competitive model.

The structure of the paper is as follows. We present the model and the complete market benchmark in Section 2. We analyze the optimal repo contracts, including properties for haircuts, liquidity premiums, and repo rates in Section 3. In Section 4, we allow for collateral re-use and study intermediation. Finally, Section 5 concludes.

2 The Model

2.1 Setting

The economy lasts three dates, $t = 1, 2, 3$. There are two agents $i = 1, 2$ and only one good each period. Agent 1 is endowed with ω units of the good in each of the three periods, while agent 2 is endowed with ω in all but the last period. Agent 1 is also endowed with a units of an asset while agent 2 has none. This asset pays dividend s in date 3. The dividend is distributed according to a cumulative distribution function $F(s)$ with support $\mathcal{S} = [\underline{s}, \bar{s}]$ and with mean $E[s] = 1$. In date 2, all agents receive a perfect signal on the realization of s in date 3. This is an easy way to model price risk at date 2.

Let c_t^i denote agent i consumption in period t. Preferences from consumption profile (c_1, c_2, c_3) for agent 1 and 2 respectively are

$$
\begin{aligned}
v^1(c_1, c_2, c_3) &= c_1 + \delta(c_2 + c_3) \\
v^2(c_1, c_2, c_3) &= c_1 + u(c_2)
\end{aligned}
$$

where $\delta < 1$ and $u(.)$ is a strictly concave function that satisfies Inada conditions. We assume $u'(\omega) > \delta$ and $u'(2\omega) < \delta$, so that there are gains from trade in date

2 and the optimal allocation is interior. These preferences contain two important elements: First, as $\delta < 1$, agent 1 prefers to consume in date 1 and would like to borrow from agent 2. Second, agent 2 does not care for consumption in date 3 and dislikes consumption variability in date 2.

Agents might not be able to fully commit to future promised payments. When an agent defaults, he incurs a non-pecuniary penalty proportional to the value of the default, that is he suffers a utility cost θr when he defaults on a repayment r, where $\theta \in [0,1]$. There are no other stigma attached to default, and while he incurs a penalty, a defaulting agent still has market access. As will be clear below, $\theta = 1$ corresponds to the case with full-commitment for agent 1, and $\theta = 0$ is the case with no commitment, when loans are non-recourse (See Geanakoplos, 1996). We assume throughout that $\theta_2 = 0$ while $\theta_1 \in [0,1]$. Setting $\theta_2 = 0$ is innocuous since agent 2 will be a natural lender. Hence, only the limited commitment of the borrower will shape equilibrium outcomes.

The environment is a simple set-up where a repo contract arises naturally: Limited commitment implies that the value of the borrowers' debt must be in line with the (uncertain) market value of their collateral. But the lender dislikes this market risk and will try to hedge it using a forward contract. A repo is different from a standard collateralized loans, as the latter does not (necessarily) grant the lender the right to re-use the collateral. When agents can re-use the asset, we will show that they strictly prefer to use a repo to a collateralized loan.

2.2 Full commitment

In this section we study the benchmark allocation when agents can fully commit to future promises. An Arrow-Debreu equilibrium of this economy is a system of consumption prices $(q_2, \{q_3(s)\}_{s\in\mathcal{S}})$ and allocations $(c_1^i, c_2^i, c_3^1(s)\}_{s\in\mathcal{S}})^{i=1,2}$ such that agents solve their maximization problem,

$$\max_{c_1^i, c_2^i, \{c_3^i(s)\}_{s\in\mathcal{S}} \in \mathbb{R}^+} \int_s v^i(c_1^i, c_2^i, c_3^i(s))dF(s)$$

subject to

$$c_1^i + q_2 c_2^i + \int_s q_3(s)c_3^i(s)dF(s) \le (1+q_2)\omega + \int_s q_3(s)(\mathbb{I}_{\{i=1\}}\omega + a_0^i s)dF(s),$$

and markets clear

$$\begin{aligned} c_t^1 + c_t^2 &= 2\omega \qquad \text{for } t = 1, 2, \\ c_3^1(s) + c_3^2(s) &= \omega + as. \end{aligned}$$

Clearly agent 2 will not consume at date 3. Considering the case with strictly positive consumptions, the necessary and sufficient first order conditions for both agents give $q_2 = q_3(s) = \delta$, for all s as well as $c_2^2 = c_{2,*}^2$ and $c_2^1 = c_{2,*}^1$ where $(c_{2,*}^1, c_{2,*}^2)$ is the unique solution to

$$\begin{cases} u'(c_{2,*}^2) = \delta \\ c_{2,*}^1 + c_{2,*}^2 = 2\omega \end{cases} \tag{1}$$

Since $u'(\omega) > \delta$ and $u'(2\omega) < \delta$ both agents consume positive amounts in date 2. The date 1 consumption of agent 1 is given by his budget constraint, $c_1^1 = \omega + \delta(c_{2,*}^2 - \omega)$, and $c_1^1 \geq 0$ follows from $c_{2,*}^2 > \omega$. Turning to c_1^2, and using the BC of agent 2 we obtain $c_1^2 = \omega - \delta(c_{2,*}^2 - \omega)$, so that $c_1^2 > 0$ requires

$$\frac{(1+\delta)}{\delta}\omega \geq c_{2,*}^2 \tag{2}$$

We maintain this assumption in the rest of the paper, which implicitly defines a restriction on δ. In the following, we introduce limited commitment and we show that a repo dominates simple spot trades to implement this allocation.

2.3 Limited commitment, spot trades, and repos

In this section, we introduce limited commitment so that agents cannot write unsecured contracts. Agents cannot achieve the full commitment benchmark allocation by using only spot trades, as it exposes agent 2 to market risk. We thus allow agents to trade repurchase agreements and show how these contracts improve over a sucession of spot trades.

2.3.1 Spot Transactions and Efficiency

Here we show that agents cannot reach the full commitment benchmark allocation if they can only trade the asset in a spot market. Intuitively, with only spot market trades, agent 2 who purchased a_1^2 units of the asset in period 1 will consume $c_2^2(s) = \omega + p_2(s)a_1^2$, as he is endowed with ω and holds a_1^2 units of the asset with unit-price $p_2(s)$. So the dividend risk at date 3 generates price risk at date 2 when agent 2 sells back his assets. Thus, he cannot achieve the efficient allocation which is independent of s. We further characterize the equilibrium where only spot markets are available in the Appendix. If we let p_1 be the equilibrium price of the asset in period 1 then we can define the asset liquidity premium in the standard way,

$$\mathcal{L} = p_1 - \delta\mathbb{E}[p_2(s)] \tag{3}$$

where $p_2(s) = s$ is the equilibrium spot price of the asset at date 2, and so $\delta E[s]$ is the fundamental value of the asset at date 1. Proposition 1 states the equilibrium properties of the liquidity premium and the importance of asset availability for the agents' ability to borrow and save. Recall that agent 1 is endowed with a units of the asset.

Proposition 1. *When agents can only trade spot, the equilibrium is as follows:*

1. Low asset quantity: Suppose (a, ω) is such that $\mathbb{E}\{s[u'(\omega + sa) - \delta]\} > 0$ then agent 1 sells his entire asset holdings at date 1. The liquidity premium is $\mathcal{L} = \mathbb{E}\{s[u'(\omega + sa) - \delta]\} > 0$.

2. High asset quantity: Suppose (a, ω) is such that $\mathbb{E}\{s[u'(\omega + sa) - \delta]\} \leq 0$ then agent 1 sells less than a at date 1. The liquidity premium is $\mathcal{L} = 0$.

As we observed, agent 1 wants to consume at date 1. When agent 1's asset endowment is low, he sells his entire asset holding and then buys it all back at date 2. Agent 1 would like to sell more, but he cannot short the asset. In this case the asset commands a liquidity premium which is related to the severity of agent 1's short-selling constraint. When a is high enough, agent 1 does not sell everything in date 1 and the asset price is determined by agent 1 according to its fundamental value. So in this case there is no liquidity premium.[9]

[9] If agent 2 was alive in period 3, he may want to save whenever $\omega + sa > c_{2*}^2$. However, we

As we will show below the sale in a repo will allow agent 1 to borrow at date 1 while the commitment to a repurchase price insures agent 2 against price risk.

2.3.2 Trading in Spot and Repo Markets

In this section, we specify the agents' problem when they can trade spot and repo contracts. A repo is the combination of a sale of an asset with a forward contract to buy it back. Intuitively, the forward leg allows the borrower to provide insurance against future price risk to the lender. However, under limited commitment, agents may not always make good on the promises implicit to that forward contract. In the following, we thus derive the space of feasible repo contracts. Agents may trade any such feasible repo and equilibrium will select the contract agents actually trade.

Repos

Consider two agents i and j where i is the borrower or seller and j is the lender or buyer. A repo contract at date 1 between i and j is given by $F_{ij} = (\{\bar{p}_{ij}(s)\}_{s\in\mathcal{S}}, \nu_{ij})$ where $\bar{p}_{ij}(s)$ is the price i agrees to pay j at date 2 to repurchase the asset in state s and $\nu_{ij} \in [0,1]$ the share of collateral that agent j can re-use. We only restrict $s \mapsto \bar{p}_{ij}(s)$ to be continuous. The contract price $p_{F,ij}$ is the size of the loan. Re-use is the only non-standard feature in our analysis. To be clear, if agent j obtains a units of the asset from *entering a repo* with agent i, he can re-use at most an amount $\nu_{ij}a$ from that collateral in later trades. We say that the remaining fraction $(1-\nu_{ij})a$ is segregated[10]. In the first part of the analysis, we will set $\nu_{ij} = 0$. In Section 4, the repo contract will differ from a collateralized loan when we let $\nu_{ij} > 0$.

Collateral Constraint

For each such repo contract F_{ij}, we denote by $b^{ij} \geq 0$ the quantity of assets that agent i sells or pledges as collateral if he borrows from j, and $\ell^{ij} \geq 0$ the collateral agent i receives if he lends to j. The analysis will show that we need only to consider two contracts F^{ij} and F^{ji} per pair of agents, that is other available contracts will not be traded in equilibrium. Having differentiated borrowing from

should emphasize that allowing for this possibility would not modify the result, even if agent 2 had risk neutral preferences for period 3 consumption.

[10]Naturally, if agent j *purchased* a from agent i in the spot market, agent j can naturally use all of a later.

lending positions, we can write the agents' collateral constraint as

$$a_1^i + \nu_{ji}\ell^{ij} \geq b^{ij}. \tag{4}$$

The collateral constraint (4) has a simple interepretation. The LHS is the asset available to agent i and can come from two sources: asset purchase or re-usable collateral. It must cover the needs of agent i who borrows b^{ij} (the RHS). Re-usable collateral is indistinguishable from the asset owned by agent i and as such, can also be sold in the spot market. This implies in particular that agent i position on the spot market a_1^i can be negative.[11]

Borrower and Lender Default

When entering a repo contract, the borrower promises to repurchase the asset at a pre-agreed price while the lender promises to return the re-usable collateral. Upon default, an agent loses his entitlement (the collateral or the cash). In addition, he incurs the non-pecuniary penalty[12] proportional to the size of the default and his commitment power θ. In line with market practice, a default by one agent cancels the obligation of the counterparty. Finally, a default does not impair an agent's ability to access the spot market in period 2.

Formally, consider a repo $F_{ij} = (\{\bar{p}_{ij}(s)\}_{s\in\mathcal{S}}, \nu_{ij})$ between borrower i and lender j. If borrower i defaults on a repo with lender j, he loses the b^{ij} units of the asset pledges and incurs a penalty equal to $\theta_i \bar{p}(s)\tilde{a}$. Therefore, given $b^{ij} > 0$ and $\theta_2 = 0$, borrower i does not default on his promise whenever $p_2(s)b^{ij} + \theta_i \bar{p}_{ij}(s)b^{ij} \geq \bar{p}_{ij}(s)b^{ij}$, where the LHS is the cost of defaulting comprising the loss of the asset market value plus the non-pecuniary cost where again $\theta_2 = 0$, while the RHS is the cost of repaying the loan[13]. Therefore, the borrower makes good on his promise to pay

[11] When only re-pledging is possible, the collateral constraints breaks into two parts

$$a_1^i \geq 0,$$
$$a_1^i + \nu_{ji}\ell^{ij} \geq b^{ij}.$$

[12] The penalty is also lost to the counterparty. Without this assumption, the penalty would be enough to transfer consumption across time and collateral would play no role. We might also want to see the penalty as a non-pecuniary cost to avoid hitting the constraint of positivity of consumption. In that case the actual penalty would be $\min\{\theta\bar{p}(s)\tilde{a}, \omega\}$.

[13] Again, we may use this unified formulation only because $\theta_2 = 0$. If agent 2 has some

$\bar{p}_{ij}(s)$ in state s whenever,

$$\bar{p}_{ij}(s) \leq \frac{p_2(s)}{1-\theta_i} \tag{5}$$

Conveniently, $\theta_1 = 1$ corresponds to the case with full commitment for agent 1 and any repayment schedule $\bar{p}(s)$ is incentive compatible. Setting $\theta_1 = 0$ corresponds to the case with no commitment since the borrower does not incur any loss besides that of the collateral. Hence, as long as $\theta_1 > 0$ selling the asset in a repo potentially allows agent 1 to increase his borrowing capacity with respect to a standard spot sale. Since the penalty is lost to the lender, there is no loss in generality in focusing on forward contracts satisfying no-default constraint (5).

Similarly, if lender j does not return collateral νl^{ji} in state s, he loses the promised repayment $\bar{p}_{\text{ij}}(s)l^{ji}$ and incurs a penalty $\theta_j \nu \tilde{a} p_2(s)$, equal to a fraction θ_j of the value of the collateral he should return. Therefore, given $l^{ij} > 0$ and $\theta_2 = 0$, lender i does not default on his promise whenever $\bar{p}_{ij}(s)l^{ij} + \theta_i \nu_{ji} l^{ij} p_2(s) \geq \upsilon_{ji} l^{ij} p_2(s)$. The LHS is the cost of defaulting including the missed payment from the borrower and the penalty. The RHS is the cost of returning the re-usable units of collateral at market value[14]. Importantly, upon default, the lender does not have access to the segregated collateral which is automatically returned to the borrower. The lender no-default constraint writes:

$$\bar{p}_{ij}(s) \geq \nu_{ij}(1-\theta_j)p_2(s) \tag{6}$$

To simplify the rest of the exposition, observe that the price of the asset at date 2 is $p_2(s) = s$. Indeed, agent 2 has no lust for consumption at date 3. Hence, agent 3 will hold the asset between period 2 and 3 so that the price must reflect his marginal valuation equal to $1.s$ at date 2. We can thus write explcitly the set

commitment power $\theta_2 > 0$, then his no default constraint would write

$$u\left(X - [\bar{p}_{21}(s) - p_2(s)]\, b^{21}\right) \geq u(X) - \theta_2 \bar{p}_{21}(s) b^{21}$$

However, we observe again that agent 2 is a natural lender. Hence he never faces such a decision.

[14] A lender might re-use collateral and not have in on his balance sheet when he must return it to the lender. However, observe that he can always purchase the relevant quantity of the asset in the spot market to satisfy his obligation.

of feasible repo contracts $\mathcal{F}_{ij}$ between two agents i and j as

$$\mathcal{F}_{ij} = \left\{ \bar{p}_{ij} \in \mathcal{C}^0 \left[\underline{s}, \bar{s}\right] \mid \forall\, s \in \left[\underline{s}, \bar{s}\right],\ \nu_{ij}(1-\theta_j)s \leq \bar{p}_{ij}(s) \leq \frac{s}{1-\theta_i} \right\}$$

Agents' optimization problem.

We can now write the agent's optimization problem. Although they are free to trade any repo, the analysis will show that only two contracts F_{12} and F_{21} may be traded in equilibrium. With this in mind, we can now write the problem of agent i in the following concise way:

$$\max_{a_t^i, \{b^{ij}\}, \{\ell^{ij}\}} \omega + p_1(a_0^i - a_1^i) - p_{F,ji}\ell^{ij} + p_{F,ij}b^{ij} \tag{7}$$

$$+\mathbb{E}\left[u_i\left(\omega + p_2(s)(a_1^i - a_2^i) + \bar{p}_{ji}(s)\ell^{ij} - \bar{p}_{ij}(s)b^{ij}\right)\right]$$

$$+\mathbb{E}\left[\mathbb{I}_1(i)u_i\left(\omega + sa_2^i\right)\right] \tag{8}$$

subject to:

$$a_1^i + \nu\ell^{ij} \geq b^{ij} \qquad (\gamma_1^i) \tag{9}$$

$$b^{ij} \geq 0 \qquad (\xi_b^{ij}) \tag{10}$$

$$\ell^{ij} \geq 0 \qquad (\xi_\ell^{ij}) \tag{11}$$

where $u_1(x) = \delta x$, $u_2(x) = u(x)$ and $\mathbb{I}_1(i)$ equals 1 when $i = 1$ and zero otherwise. At date 1, agents have resources $\omega + p_1 a_0^i$, where $a_0^1 = a$ and $a_0^2 = 0$, and choose their asset holding a_1^i, lending ℓ^{ij} using contract F^{ji} and borrowing b^{ij} using contract F^{ij} . Given these decisions, their resources at date 2 is their endowment ω and the value of their asset holdings $p_2(s)a_1^i$ as well the net value of their forward contracts positions $\bar{p}_{ij}(s)\ell^{ij} - \bar{p}_{ji}(s)b^{ij}$. Agent 2 has no lust for consumption at date 3 and so will not save, i.e. $a_2^2 = 0$. Agent 1 however will save a_2^1 to consume at date 3 and market clearing implies $a_2^1 = a$. Finally, notice that (9) and (10) imply a no-short sale constraint on the asset when $\nu = 0$.

Definition 2. *Repo equilibrium*

An equilibrium is a pair of repo contracts $(F_{12}, F_{21}) \in \mathcal{F}_{12} \times \mathcal{F}_{21}$, prices $(p_{F,12}, p_{F,21})$ for these contracts, a spot price p_1, Lagrange multipliers $\{\gamma_1^i, \xi_b^{ij}, \xi_l^{ij}\}^{i=1,2.j\neq i}$ and allocations $\{c_t^i(s), a_t^i, \ell^{ij}, b^{ij}\}_{t=1..3}^{i=1,2.j\neq i}$ such that

1. $\{c_t^i(s), a_t^i, \ell^{ij}, b^{ij}\}_{t=1..3}^{j\neq i}$ solves agent $i = 1, 2$ problem (7)-(11).

2. Markets clear, that is $a_2^1 + a_1^2 = a$ and $b^{ij} = l^{ji}$ for $i = 1, 2$ and $j \neq i$

3. $\forall\, \tilde{F}_{ij} \in \mathcal{F}_{ij} \backslash \{F_{ij}\}$,

$$E[\tilde{p}_{ij}(s)u_i'(c_2^i(s))] + \gamma_1^i - \xi_b^{ij} \geq E[\tilde{p}_{ij}(s)u_j'(c_2^j(s))] + \nu_j\gamma_1^j + \xi_l^{ji} \qquad (12)$$

The last requirement means that agents do not want to trade any other feasible contract that the equilibrium contracts. Precisely, there should be a price $\tilde{p}_{F,ij}$ for any contract F_{ij} such that the marginal willingness to sell contract F_{ij} of agent i (the LHS of equation (12)) lies above $\tilde{p}_{F,ij}$ while agent j marginal willingness to buy this contract (the RHS of equation (12)) lies above $\tilde{p}_{F,ij}$.

3 Equilibrium contracts with no re-use ($\nu = 0$)

In this section, we characterize the equilibrium when agents cannot re-use collateral ($\nu = 0$) and repo contracts are indistinguishable from collateralized loans. It is clear from the set of feasible contracts that any combination of repos can be replicated by a single contract. We conjecture that agents do not trade spot and agent 1 only sells a repo $\bar{p} = \{\bar{p}(s)\}_{s\in\mathcal{S}}$ to agent 2. When agents are collateral constrained, they strictly prefer to trade the repo contract than trading spot. Otherwise they are indifferent and could use both contracts in equilibrium.

The construction of the equilibrium repo contract relies on a trade-off between a borrowing motive and a risk-sharing motive. A forward contract smoothes consumption variability at date 2 when the repurchase price is constant. But limited commitment imposes the no default-constraint (5) which puts an upper bound on the repurchase price. This hinders the borrowing capacity of the repo seller. The optimal contract reflects this tension. Let us first we define s^* as the state where agent 2 can reach $c_{2,*}^2$ when he repos the entire stock of assets, that is s^* satisfies

$$u'\left(\omega + \frac{s^*}{1-\theta}a\right) = \delta. \qquad (13)$$

Clearly, s^* is decreasing with a, θ, or δ. Therefore, it is easier to achieve the first best level of consumption the larger the stock of asset, the more agent 1 is able to commit, or the more agent 1 values the asset in period 2. We have the following result (all proofs are in the Appendix).

Proposition 3. *There is a unique equilibrium allocation supported by the repo contract $F = \{\bar{p}(s)\}_{s\in\mathcal{S}}$ where:*

1. *If $s^* \geq \bar{s}$ (a is low), $\bar{p}(s) = s/(1-\theta)$ for all $s \in \mathcal{S}$*
2. *If $s^* \in [\underline{s}, \bar{s}]$ (a is intermediate),*

$$\bar{p}(s) = \begin{cases} \frac{s}{1-\theta} & \text{for } s \leq s^* \\ \frac{s^*}{1-\theta} & \text{for } s \geq s^* \end{cases} \tag{14}$$

3. *If $s^* \leq \underline{s}$ (a is high), $\bar{p}(s) = p^*$ for all $s \in \mathcal{S}$ where $p^* \in [s^*/(1-\theta), \bar{s}/(1-\theta)]$.*

In equilibrium, agents strictly prefer to trade repo over any combination of repo and spot trades in cases 1 and 2. They weakly prefer to trade repo in case 3.

In the first two cases, agents trade all the available asset in a repo and they do not achieve the first best allocation. However, when the asset is plentiful, agents can achieve the first best allocation by using some of the asset in a repo. In this case, the equilibrium is essentially unique since agents can choose several p^* in the specified interval to attain the same allocation. If p^* is higher than the lower bound, agents just pledge less asset.

The feasible contracts are all the continuous locus below the line $s/(1-\theta)$. The blue line in Figure 1 shows the equilibrium repo contract when the asset level is in the intermediate region. The increasing part of the repo contract corresponds to the asset payoffs where the borrowing constraint of agent 1 binds and exposes agent 2 to price risk. Above s^*, the contract is flat as agents can finance the optimal level of borrowing. Hence limited commitment decreases the borrowing capacity of agent 1 in the lowest states and reduce hedging for the lender.

Now, it may be that agents prefer to trade other contracts.

As part of our equilibrium requirements, agents must not be willing to trade any other feasible repo contract $\tilde{p} \neq \bar{p}$. Below, we provide the inuition for the argument

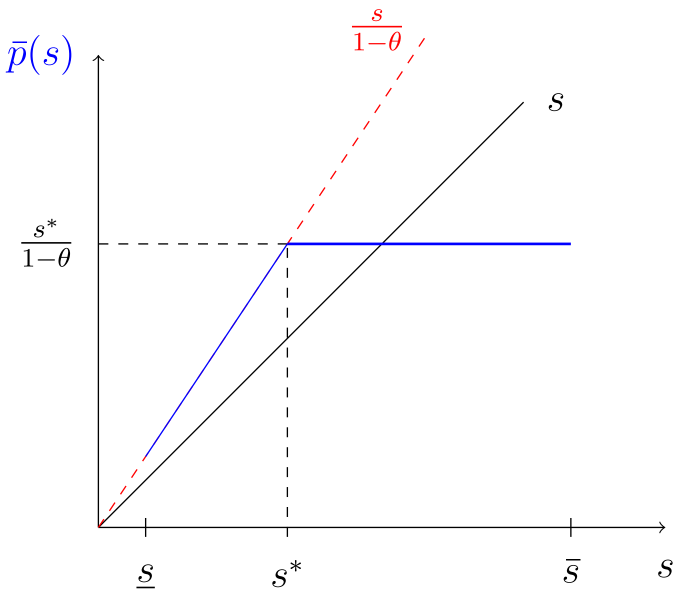

Figure 1: Equilibrium repo contract

of the proof and we refer the interested reader to the Appendix for details. In order to derive agents' marginal willingness to buy and sell other contracts, we can use the equilibrium condition. From the first order conditions of agent 1 with respect to b^{12}, we obtain

$$p_F = \delta \mathbb{E}\left[\bar{p}(s)\right] + \gamma_1^1, \tag{15}$$

where γ_1^1 is the Lagrange multiplier on the collateral constraint of agent 1. From the first order condition of agent 2 with respect to l^{21}, we obtain:

$$p_F = \mathbb{E}\left[\bar{p}(s)u'(\omega + \bar{p}(s)a)\right]. \tag{16}$$

where we have used the equilibrium $\ell^{21} = a$. The left hand side of equation (15) (resp. equation(16)) is the marginal willingness to sell (resp. to buy) contract F for agent 1 (resp. agent 2). Similarly, in equilibrium, the willingness of agent 2 to buy another contract $\tilde{p}$ is $\tilde{p}_F^2 = E\left[\tilde{p}(s)u'(\omega + \bar{p}(s)a)\right]$, while the willingness of agent 1 to sell this contract is $\tilde{p}_F^1 = \delta E\left[\tilde{p}(s)\right] + \gamma_1^1$. Therefore agents are not willing

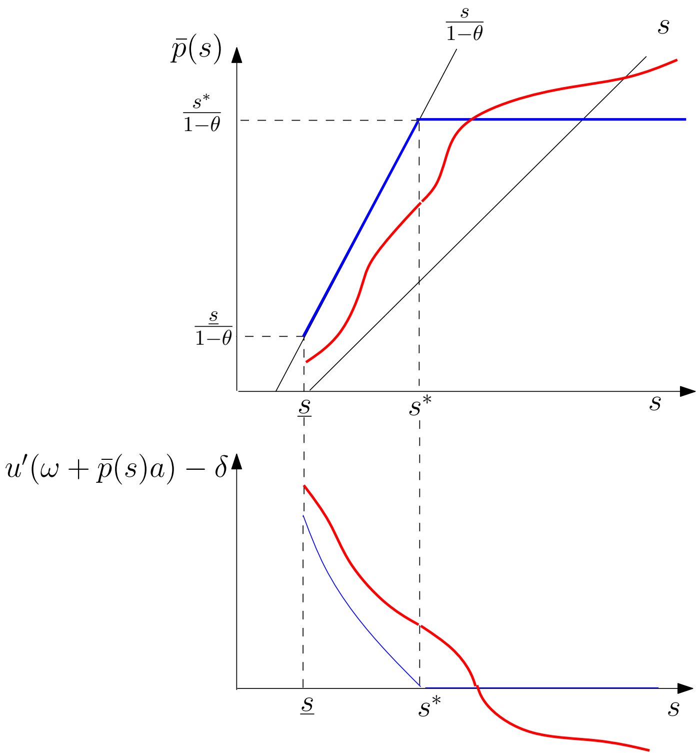

Figure 2: Non-equilibrium repo contract

to trade this contract iff [15] $\tilde{p}^2_F \leq \tilde{p}^1_F$, or:

$$\mathbb{E}\left[\big(\bar{p}(s) - \tilde{p}(s)\big)\big(u'(\omega + \bar{p}(s)a) - \delta\big)\right] \geq 0 \tag{17}$$

where we used (15)-(16) to obtain the expression for γ^1_1. Condition (17) is a necessary and sufficient no-trade condition for contract $\tilde{p}$ when contract $\bar{p}$ is available.

First, it should be clear that the equilibrium contract $\bar{p}(s)$ represented by the blue line in 1 satisfies (17). When $u'(\omega + \bar{p}(s)a) > \delta$, we have $\bar{p}(s) = s/(1-\theta)$ so that there is no other contract such that $\bar{p}(s) < \tilde{p}(s)$. Also, when $p(s) < s/(1-\theta)$, we have $u'(\omega + \bar{p}(s)a) = \delta$, so that (17) is satisfied for any other feasible repo contract $\tilde{p}(s)$.

Figure 2 helps to understand why the equilibrium contract should precisely be that of Proposition (3). Let indeed be $\hat{p}$ a candidate price schedule as depicted by the red curve on Figure 2. This $\hat{p}$ is feasible as it lies below the default line but

[15] In the Appendix, we also check this when agent 1 is a buyer and agent 2 is a seller.

it does not satisfy (17). At the margin, agent 1 is happy to increase borrowing as long as the gross interest rate does not exceed $1/\delta$. Agent 2 is marginally willing to lend at a (state contingent) interest rate of $R(s) = 1/u'(c_2^2(s))$. Hence, for a repo to be an equilibrium contract it must be that either $R(s) = 1/\delta$ or that the no-default constraint $\bar{p}(s) = s/(1-\theta)$ binds. In particular, for $s < s^*$, agents gains from increasing borrowing since $R(s) < 1/\delta$. Hence the repayment schedule should move up from the red line to the blue line. Symmetrically, the repurchase price schedule cannot exceed the blue line as agents gain from reducing borrowing since $R(s) > 1/\delta$ then. The proof in the Appendix formalizes this intuition using explictly condition (17).

The last part of the Proposition states that when agents are not attaining the first best allocation, they strictly prefer to repo the entire quantity of asset to any combination of spot and repo trades. This is intuitive as the payoff profile of a spot transaction $p_2(s) = s$ is a feasible repurchase price for the repo contract. When the asset is not abundant, the equilibrium uniquely selects the repo contract traded. Alternatively, consider a portfolio change whereby agents uses $a - \epsilon$ units of the asset in a repo instead of a and sells the other ϵ units. Agents' marginal utility gains can be written:

$$\frac{\partial v^1}{\partial \epsilon}_{|\epsilon=0} = -p_F + p_1 + \delta E[\bar{p}(s) - s] = 0$$
$$\frac{\partial v^2}{\partial \epsilon}_{|\epsilon=0} = p_F - p_1 + E\left[(s - \bar{p}(s))\, u'(c_2^2(s))\right] = -\gamma_1^2 < 0$$

Agent 2 loses from the portfolio change because he ends up holding the asset while he would like to sell it to agent 1 so that the latter can increase borrowing.

3.1 Haircuts, liquidity premium, and repo rates

In this section, we derive the equilibrium properties of the liquidity premium and repo haircuts. We compare the haircuts and liquidity premiums of two assets with different risk profile in a given economy, and we do comparative statics with respect to counterparty risk, as measured by θ.

The liquidity premium is given by (3). In our economy, the liquidity premium is positive because the short sales constraint is binding. When there is only spot trades, agent 1 would like to sell more of the asset than what he has, while with repo, his collateral constraint is binding. The repo haircut is

$$\mathcal{H} \equiv p_1 - p_F = \delta\left(\mathbb{E}[s] - \mathbb{E}[\bar{p}(s)]\right), \tag{18}$$

as it costs p_1 to obtain 1 unit of the asset, which can be pledged as collateral to borrow p_F. So to purchase 1 unit of the asset, an agent needs $p_1 - p_F$ which is the downpayment or haircut.[16] Finally, the repo rate is

$$1 + r = \frac{\mathbb{E}[\bar{p}(s)]}{p_F} = \frac{1}{\delta}\frac{\delta\mathbb{E}[s] - \mathcal{H}}{\mathcal{L} + \delta\mathbb{E}[s] - \mathcal{H}}. \tag{19}$$

Observe that the repo rate lies below the rate prevalent under the first best $1/\delta$ whenever the asset price bears a liquidity premium $\mathcal{L} > 0$. Remember that a positive liquidity premium indicates asset scarcity so that agents are (collateral) constrained. Hence in equilibrium, it must be that the interest rate o falls so that agent 1 finds it optimal to borrow in a repo rather than collecting the liquidity premium by selling the asset. Observe that lower haircuts also increase the repo rate. Again when $\mathcal{L} > 0$, agents are constrained and want to borrow as much as possible. Hence, if the loan to value ratio p_F/p_1 decreases the repayment rate must increase. Interestingly, repo rates r could even be negative for assets with large haircuts and large liquidity premium. This is consistent with market data as reported in ICMA (2013).[17] We can derive the haircut and liquidity premium for the optimal price schedule $\bar{p}(s)$.

Corollary 4. *The equilibrium haircut and liquidity premium are given by:*

[16] An alternative definition is $(p_1 - p_F)/p_1$.

[17] The ICMA (2013) reports that "The demand for some assets can become so strong that the repo rate on that particular asset falls to zero or even goes negative. The repo market is the only financial market in which a negative rate of return is not an anomaly." (p.12) and in footnote 6 "negative repo rates have been a frequent occurrence and can be deeply negative." Also, see Duffie (1996), or Vayanos and Weill (2008).

1. *If* $s^* \geq \bar{s}$ *(a is low),*

$$\mathcal{H} = -\frac{\delta\theta}{1-\theta}\mathbb{E}[s],$$
$$\mathcal{L} = \delta\mathbb{E}\left[\frac{s}{1-\theta}\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right)\right].$$

2. *If* $s^* \in [\underline{s}, \bar{s}]$ *(a is intermediate),*

$$\mathcal{H} = \delta\left[-\int_{\underline{s}}^{s^*}\frac{\theta}{1-\theta}s\, dF(s) + \int_{s^*}^{\bar{s}}\left(s - \frac{s^*}{1-\theta}\right)\mathrm{d}F(s)\right],$$
$$\mathcal{L} = \delta\int_{\underline{s}}^{s^*}\frac{s}{1-\theta}\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right) dF(s).$$

3. *If* $s^* \leq \underline{s}$ *(a is high),*

$$\mathcal{H} \in \left[\delta\mathbb{E}[s] - \delta\frac{\underline{s}}{1-\theta}, \delta\mathbb{E}[s] - \delta\frac{s^*}{1-\theta}\right],$$
$$\mathcal{L} = 0.$$

The corollary implies that haircuts must be negative when the asset is sufficiently scarce, and can be either negative or positive otherwise. As Figure 3 shows, the borrowing and hedging motives contribute in opposite ways to the size of the haircut. First, the borrowing motive contributes negatively to the haircut. The reason is that in low states, the repo contract pays more to the asset holder than a spot sale. So if the repo lender anticipated a low state, he would be willing to pay more for the repo (p_F) than the period 1 spot price, implying a negative haircut. In the opposite, hedging contributes positively to the haircut. Again, hedging limits the payment to agent 2 in high states, and if the repo lender expects s to fall in this region, he is willing to pay less than the period 1 spot price to acquire the asset in a repo, hence implying a positive haircut.

To summarize, if $\theta = 0$ the borrower cannot borrow more than the present value of the asset and the haircut is positive. As θ increases, the borrower increases its borrowing capacity and the haircut declines. Finally, observe that the haircut is not pinned down when $s^* \leq \underline{s}$ since several (constant) repurchase prices $\bar{p}$ are

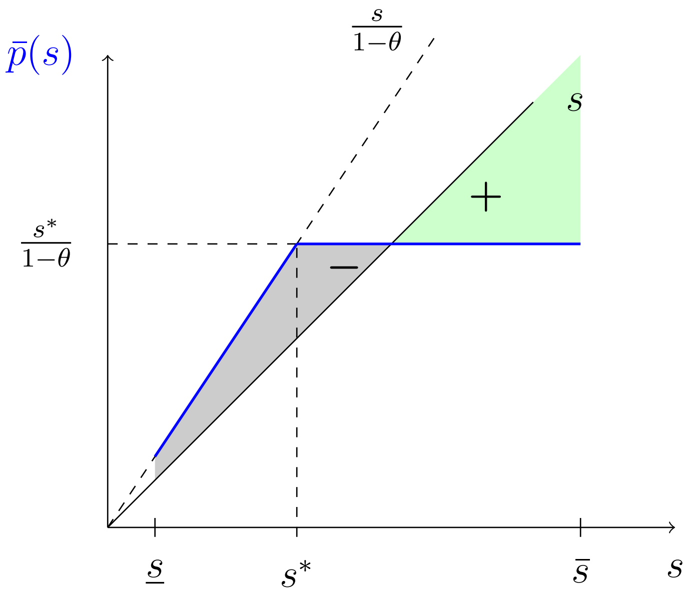

Figure 3: Repo haircuts

possible in equilibrium.

3.1.1 Counterparty risk

We now perform a comparative static exercise varying θ a proxy for counterparty risk. The risk that the borrower defaults is higher as θ is low since the non-pecuniary default loss is proportional to θ. Of course, there is never default in equilibrium, but the equilibrium contract reflects default risk. In particular, s^* depends negatively on θ. Using the expression for the haircut $\mathcal{H}$ derived in Corollary 4, we obtain that haircuts increase with counterparty risk, as

$$\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{\delta}{(1-\theta)^2} \int_{\underline{s}}^{s^*} s dF(s) \leq 0$$

Indeed, as Figure 4 shows, a higher θ increases the amount a borrower can raise per unit of the asset pledged. This naturally leads to a decrease in the haircut, by increasing the size of the region where $\bar{p}(s) > s$ while leaving the other region unchanged.

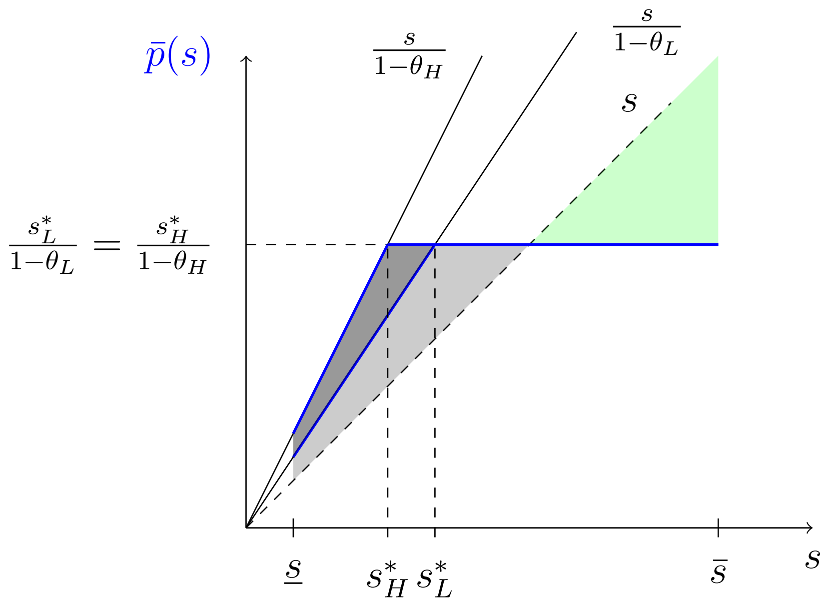

Figure 4: Influence of θ

Turning now to liquidity, it would natural to expect that an increase in θ has the same effect as an increase in a, as they both raise the borrowing capacity of agent 1. Hence, we would expect the liquidity premium to decrease since agents become less constrained. This argument however does not fully take into account the optimal contract design. Indeed, the slope of the optimal repurchase schedule, $\bar{p}(s)$, increases with θ for $s \in [\underline{s}, s^*]$ where the agent is constrained. This has the adverse effect of raising agent 2 consumption variability in the lowest states. We have, with $c_2^2(s) = \omega + \frac{s}{1-\theta}a$,

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{(1-\theta)^2} \int_{\underline{s}}^{s^*} s \left[u'\left(c_2^2(s)\right) - \delta + \frac{u''\left(c_2^2(s)\right)}{1-\theta} \right] dF(s)$$

The first term captures the positive effect of θ on the liquidity premium: a higher borrowing capacity for agent 1 brings agent 2 consumption closer to the first best level of consumption in states $s \in [\underline{s}, s^*]$. The second term shows the effect of the steeper slope of the repurchase profile $\bar{p}(s)$. Therefore, the overall effect of a higher θ on the liquidity premium is ambiguous and depends on the risk aversion of agent 2, as the expression above shows

3.1.2 Asset risk

In this section, we compare repo terms given a richer asset structure[18]. To do this, we introduce two assets with different risk profiles but perfectly correlated payoffs. We compute the liquidity premium of the safer asset relative to the riskier, the haircuts that both assets carry, and the repo rate for both assets.

As before, $s \sim F[\underline{s}, \bar{s}]$ but there are now two assets $i = A, B$ with payoffs $\rho_i(s)$:

$$\rho_i(s) = s + \alpha_i(s - \mathbb{E}[s]),$$

where $\alpha_i \geq -1$, and $\alpha_B > \alpha_A$. When $\alpha_i = -1$, the asset is perfectly safe and pays off $\mathbb{E}[s]$. An asset with a higher α has the same mean but a higher variance than an asset with a lower α. Indeed $Var[\rho_\alpha] = (1 + \alpha)Var[s]$. Therefore, asset B is relatively riskier than A. We choose to consider two assets with perfectly correlated payoff, as in this way the structure of the repo contract does not change. Otherwise holding both assets may offer some risk sharing which will modify the structure of the optimal contract we derived above. Agent 1 is endowed with a units of asset A and b units of asset B, while agent 2 does not hold any of the assets.

It is relatively straightforward to extend the equilibrium analysis of the previous section to this new economy with two assets. The set of available contracts consists of feasible repos using assets A and B. The structure of the repo contracts is essentially the same as before, defining the threshold s^{**} above which agent 2 consumes $c^2_{2,*}$, as

$$u'\left(\omega + \frac{a\rho_A(s^{**}) + b\rho_B(s^{**})}{1 - \theta}\right) = \delta.$$

Then the repayment schedule for asset i is

$$\bar{p}_i(s) = \begin{cases} \frac{\rho_i(s)}{1-\theta} & \text{for } s \leq s^{**}, \\ \frac{\rho_i(s^{**})}{1-\theta} & \text{for } s \geq s^{**}. \end{cases}$$

Then we have the following result.

[18] In the Appendix we consider the effect of an increase in risk, in the sense that the support $[\underline{s}, \bar{s}]$ changes when s is uniformly distributed.

Proposition 5. *The safer asset A always has a higher liquidity premium and a lower haircut than the riskier asset B. The repo rate spread is indeterminate.*

The riskier asset carries a larger haircut because its repo contract contains a larger hedging component for high states. This means that the loan to value ratio is smaller or equivalently that the haircut is bigger. Intuitively, agent 2 is willing to pay less to accept a risky asset in a repo. Also, because the safer asset has a lower haircut, agent 1 is willing to pay a higher price to acquire this asset to later pledge it as collateral. This implies that the liquidity premium is higher for the safe asset.

The (gross) repo rate $1 + r_i$ defined in (19) may be lower or higher for the safe asset. Indeed, we showed that a higher liquidity premium reduces the interest rates while a lower haircut increases this rate. Our model thus suggests that repo rates may not be a good statistics to determine the risk profile of an asset.

4 Collateral re-use and intermediation

4.1 Collateral re-use

In this section, we analyze the equilibrium implications of allowing agents to re-use a fraction of the repo collateral. Re-use has been very much under scrutiny following the crisis (see Singh and Aitken, 2010) since a default on re-used collateral may affect several agents along a credit chain. While we do not model the consequence of such default cascades, we provide the foundations for this analysis by highlighting the benefits of allowing re-use.

To solve this problem we will guess and verify the structure of the equilibrium contracts by building on the intuition from the case without re-use. It also helps to think of the implicit sequence of trades that yields the result. Without re-use, agent 2 (the lender) is constrained: he would like to sell some of the asset to agent 1 but he only holds collateral pledged by agent 1. When we allow re-use, agent 2 has to segregate at least a fraction $1 - \nu$ of agent 1's collateral a, but he can sell a fraction ν back to agent 1 (or some other agent of type 1). Agent 1 can then use νa as collateral to borrow from agent 2 again. It is important to notice that agent 1 can use his entire asset holding at this stage, νa, as he *bought* it from agent 2. Due to segregation, after each round of such trades, the asset available to agent 1 decreases. So the trading process might go on until it extinguishes the available asset. It should now be intuitive that allowing re-use relaxes both agents' constraints.

We now guess that agents use only one repo contract $F^{12} = (\bar{p}(\nu), \nu)$ and agent 2 resells the fraction ν of the collateral he obtains on the spot market. Let $\bar{p}(\nu)$ be defined by

$$\bar{p}(s, \nu) = \begin{cases} \frac{s}{1-\theta} & \text{if } s < s^*(\nu) \\ \frac{s^*(\nu)}{1-\theta} + \nu(s - s^*(\nu)) & \text{if } s \geq s^*(\nu) \end{cases} \tag{20}$$

where we define $s^*(\nu)$ implicitly as follows

$$u'\left(\omega + \frac{as^*(\nu)}{1-\nu}\left[\frac{1}{1-\theta} - \nu\right]\right) = \delta.$$

Then we show in the Appendix

Proposition 6. ***Collateral Re-use.***

Suppose agent 2 can re-use at most a fraction $\nu \in [0,1]$ of the collateral. The (essentially) unique equilibrium forward contract underlying the repo is $\bar{p}(\nu)$ defined in (20). Collateral re-use is essential whenever $\theta > 0$ and the complete market benchmark allocation is achieved for any $\nu \geq \nu^*$ defined as

$$\nu^* = \frac{s^* - \underline{s}}{s^* - (1-\theta)\underline{s}}.$$

The proof makes clear that the benefit of re-use is to allow agent 2 to short-sell the asset. Looking back at (9) suppose that agents are constrained, so that their collateral constraint binds. We thus have

$$a_1^1 = b^{12} \tag{21}$$

$$a_1^2 = -\nu\ell^{21} \tag{22}$$

and agent 2 goes short in the asset whenever $\ell^{21} > 0$, that is, he sells the collateral that he obtained from lending. Naturally, even with re-use, there is a constraint on the amount of physical asset available since the spot market must clear, that is

$a_1^1 + a_1^2 = a$. Clearing on the repo market implies $b^{12} = \ell^{21}$. Hence, summing (21) and (22) we obtain

$$a_1^1 = \frac{a}{1-\nu} = b^{12}$$
$$a_1^2 = -\frac{\nu}{1-\nu} a = -\nu b^{12}$$

To understand these expressions, let us decompose again the pledging and re-using process. In the first step, agent 1 pledges a units of collateral. Agent 2 must segregate a fraction $1-\nu$ but resells a fraction ν to agent 1 who repeats the first operation. Through this mechanism, agent 1 can leverage his asset holdings a by a factor $1/(1-\nu)$. Agent 2 effectively has a short position in the asset because he sold collateral received on his loan to agent 1. Ultimately, re-use allows agents that are borrowing constrained to leverage up their position by engaging in several rounds of borrowing with the same piece of asset. Although outlining these steps helps understand the re-use process, they take place simultaneously in our environement.

Notice that re-use is only essential when agents are to some extent trustworthy, i.e. $\theta > 0$. To see this, let us assume that $\theta = 0$, and allow re-use $\nu = 1$. Since $\theta = 0$ the repo repayment schedule is $\bar{p}(s)$ lies below s for all s. This implies that in any state s, agent 2 will only get less than s from the repo loan, which is what he has to pay s for each unit of assets he sold short at date 1. Thus agent 2 is unable to increase his date 2 consumption beyond what is feasible with no re-use. Hence re-use is redundant when $\theta = 0$.[19] However, with $\theta > 0$ agent 2 can increase his consumption in the low states as agent 1 can promise $s/(1-\theta)$ per unit of the asset, while it only costs s to agent 2 to cover a unit short position.

Haircuts can again be positive or negative but the effect of re-use is ambiguous. Using (18) and replacing the repayment schedule (20), we find

$$\mathcal{H}/\delta = \mathbb{E}[s] - \int_{\underline{s}}^{s^*(\nu)} \frac{s}{1-\theta} dF(s) - \int_{s^*(\nu)}^{\bar{s}} \left[\frac{s^*(\nu)}{1-\theta} + \nu \left(s - s^*(\nu) \right) \right] dF(s)$$

Re-use decreases the interval of spaces $[\underline{s}, s^*(\nu)]$ where agents are constrained

[19] Maurin (2015) provides a general proof for this result in a more general environment.

and use the full borrowing capacity of the collateral $s/(1-\theta)$. This effect increases haircuts. However, the repayment schedule has a slope of ν on the rest of the state space $[s^*(\nu), \bar{s}]$ - to correct for the short asset position of the lender - which tends to decrease haircuts. An interesting result however is that haircuts are always negative when $\nu = 1$. There indeed, for each round of sale and re-pledge, the borrower nets $p_F - p_1 = -\mathcal{H}$. If the haircut was positive, he would actually decrease his consumption in date 1. Our model predicts that re-use is most helpful when collateral is most scarce and there is evidence that this is indeed the case (see Fuhrer et al. (2015)). When it comes to the liquidity premium, observe that agents may reach the first-best allocation when $\nu \geq \nu^*$ so that $\mathcal{L} = 0$. However, $\mathcal{L}$ may be non-monotonic in the re-use factor ν. Indeed, while re-use relaxes the collateral constraint, it also increases the amount pledgeable in states where agents are constrained. This last effect makes the asset more valuable and can increase the liquidity premium[20] .

4.2 Intermediation with repo

Large dealer banks make for a significant share of the repo market where they play a role as intermediaries between natural borrowers and lenders. Consider for instance a hedge funds in need for cash. He would typically borrow from a dealer bank through a repo. The dealer bank would then tap in a Money Market Fund (MMF) cash pool through another repo to finance the transaction. Figure 5 illustrates this pattern of repo intermediation. In this section, we show that an intermediation equilibrium may exist when the dealer bank is more trustworthy than the hedge fund. A remarkable feature of our analysis is that intermediation arises endogenously although in our example, the Hedge Fund would be free to

[20] With re-use the liquidity premium is simply:

$$\mathcal{L} = \gamma_1^1 = \frac{1}{1-\nu} E\left[\left(\bar{p}(s,\nu) - \nu s\right) \left(u'(c_2^2(s)) - \delta\right) \right]$$
$$= \frac{1}{1-\nu} \int_{\underline{s}}^{s^*(\nu)} \frac{1-\nu(1-\theta)}{1-\theta} \left(u'\left[\omega + \frac{a}{1-\nu} \left(\frac{1}{1-\theta} - \nu \right) s \right] - \delta \right) s \mathrm{d}F(s)$$

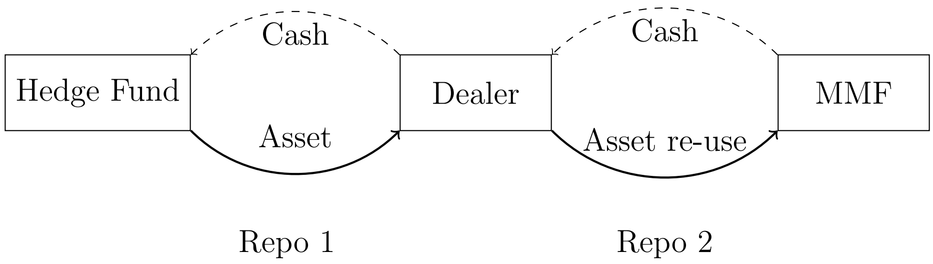

Figure 5: Intermediation with Repo

trade with the MMF[21]. We show indeed that the former prefers to deal with the dealer bank if it is a safer counterparty[22].

We modify the economy slightly. There are now two agents of type 1, $1L$ and $1H$, who differ with respect to their trustworthiness $\theta_L \leq \theta_H$ and possibly their marginal utility for consumption in period 2 and 3, δ_H and $\delta_L \leq \delta_H$ respectively[23]. Agent L initially owns all the asset, that is $a_0^L = a$ while $a_0^H = a_0^2 = a$. We say that there is intermediation when agents $1L$ does not trade with 2 meaning that he does not use a repo to borrow from agent 2. Intuitively, $1H$ can play a role as an intermediary since $\theta_H \geq \theta_L$ implies he is a more reliable counterparty for 2 than agent $1L$. This section derives conditions under which intermediation arises in equilibrium when agents are free to choose contracts.

4.2.1 Feasible Contracts

We build on our analysis of re-use to describe feasible contracts. As in the previous section a feasible repo contract $F_{ij} = (\bar{p}_{ij}, \nu_{ij})$ is a promise by borrower i to deliver

[21] Our analysis thus extends Infante (2015) and Muley (2015) which assume intermediation exogenously.

[22] There can also be institutional differences between the two trades involved. Indeed, repos between hedge funds and dealer banks are typically bilateral whereas dealer banks and hedge funds often trade via a tri-party agent (in the US, these are JP Morgan and BNY Mellon) which acts as a collateral custodian. See Federal Reserve Bank of New York (2010) for a discussion of Tri-Party repo. Although we do not model the distinction between bilateral and Tri-Party repo, our analysis still provides a fundamental explaination for this market segmentation.

[23] Although we do not need the difference indiscount factors to have intermediation in the sense explained below, it is necessary to obtain a chain of repo. A balance sheet cost τ for agent $1H$ to own the asset would play a similar role as $1H$ would prefer to lend in a repo to $1L$ rather than buying the asset spot.

$\bar{p}_{ij}(s)$ in state s together with a right to re-use a fraction ν_{ij} of the collateral for lender j, that verifies the no default constraints (5)-(6). We have shown in the previous section that unlimited re-use, that is $\nu_{ij} = 1$, can deliver the first best allocation. In this section, we assume re-use is limited, that is agent i can re-use at most a fraction ν_i of the collateral he receives. In order to focus on the intermediation role of repo, we set the following exogenous limits: $\nu_L = 0$, $\nu_H < 1$ and $\nu_2 < 1$. Agent L is a natural borrower and, as he will not lend, there is no loss in generality in preventing him to re-use collateral.

4.2.2 Intermediation Equilibrium

We are interested in an equilibrium with the following features. First, agent $1L$ and agent $1H$ trade using a spot transaction or/and a repo $F_{LH} = (\bar{p}_{LH}, \nu_H)$. Second, agent H borrows from agent 2 using another repo $F_{H2} = (\bar{p}_{H2}, \nu_2)$. As in the previous sections, we then derive the price $\tilde{p}_F$ for non-traded repo contracts $\tilde{F}$ to support the equilibrium. In particular, there should not be any profitable repo contract $\tilde{F}_{L2}$ for agents $1L$ and 2 to trade.

Observe that if the asset available for agent $1H$ to borrow from agent 2 is exogenously given, the previous analysis applies to the relationship between these two agents. However, the amount of asset available or free collateral depends on the transaction between agent $1L$ and $1H$. Consider indeed the collateral constraint of agent $1H$.

$$a_H^1 + \nu_H l^{HL} \geq b^{H2}$$

Agents $1L$ and $1H$ face a trade-off to transfer the asset from agent $1L$ on to agent $1H$. With a spot sale of 1 unit, 1 unit is available for $1H$ to borrow from agent 2 compared to only ν_H units if they trade in a repo, due to collateral segregation. On the upside, the repo increases the amount borrowed by $1L$ from $1H$ which is valuable when discount factors δ_L and δ_H differ. The solution to this trade-off will thus pin down the amount of free collateral available to $1H$ and ultimately the amount b^{H2} pledged to agent 2.

Given that this quantity is endogenous, it will be useful to extend our notation as follows. Define, $s^*(b, \theta_i, \nu)$ for $i = H, L$, as the solution in s^* to:

$$u'\left(\omega + \frac{bs^*}{1-\nu}\left[\frac{1}{1-\theta_i} - \nu\right]\right) = \delta_i,$$

and for $i = H, L$, define the repo contract $\bar{F}_{i2}(b, \nu_2) := (\bar{p}_{i2}(b, \nu_2), \nu_2)$ implictly as a function of the amount borrowed b:

$$\bar{p}_{i2}(b, \nu, s) = \begin{cases} \frac{s}{1-\theta_i} & \text{if } s < s^*(b, \theta_i, \nu_2) \\ \frac{s^*(b,\theta_i,\nu_2)}{1-\theta_i} + \nu_2(s - s^*(b, \theta_i, \nu_2)) & \text{if } s \geq s^*(b, \theta_i, \nu_2) \end{cases} \tag{23}$$

As before, $s^*(b, \theta_i, \nu)$ is the threshold in s above which marginal rates of subsitution between a type $1i$ and 2 agents can be equalized, given, in particular, the amount of asset b available. Building on our previous analysis, the repo contract descrived in (23) is then the contract that should emerge in equilibrium between those 2 agents. We now present the conditions for an intermediation equilibrium.

Proposition 7. ***Intermediation equilibrium****.*

Let $F_{LH} := (\bar{p}_{LH}, \nu_H)$ be given by

$$\bar{p}_{LH}(s) = \frac{s}{1-\theta_L} \qquad \forall s \in [\underline{s}, \bar{s}] \tag{24}$$

and $\hat{b}$ the solution to:

$$\int_{\underline{s}}^{s^*(\hat{b},\theta_H,\nu_2)} \left[u'\left(\omega + \frac{\hat{b}s}{1-\nu_2}\left[\frac{1}{1-\theta_H} - \nu_2\right]\right) - \delta_H\right] \mathrm{s}dF(s)$$
$$= \frac{(\delta_H - \delta_L)\theta_L}{(1-\nu_H)(1-\theta_L)} \frac{(1-\theta_H)(1-\nu_2)}{1-(1-\theta_H)\nu_2} \tag{25}$$

The equilibrium features intermediation iff $\hat{b} > 0$ and if:

$$\left[\frac{1-\nu_2(1-\theta_H)}{1-\nu_2}\right]\left[1 - \frac{1-\theta_H}{1-\theta_L}\right] \geq (1-\nu_H) \tag{26}$$

Three cases are then possible:

1. *$\hat{b} > a$. Agent $1L$ sells the asset spot to $1H$ who borrows $b_*^{H2} = a/(1-\nu_2)$*

from agent 2 *using repo* $F_{i2}(a, \nu_2)$.

2. $\hat{b} \in [\nu_H a, a]$. *Agent 1L uses a combination of a sale and a repo* F_{LH} *with* $1H$ *who borrows* $b_*^{H2} = \hat{b}/(1-\nu_2)$ *from agent* 2 *using repo* $F_{i2}(b^*, \nu_2)$

3. $\hat{b} \in [0, \nu_H a]$. *Agent 1L borrows from* $1H$ *using repo* F_{LH} *and* $1H$ *borrows* $b_*^{H2} = \hat{b}/(1-\nu_2)$ *from agent* 2 *using repo* $F_{i2}(\hat{b}, \nu_2)$

Condition (26) *is necessary in cases* 2) *and* 3).

In all three cases the amount b_*^{LH} *borrowed by agent* $1L$ *from* $1H$ *is given by:*

$$b_*^{LH} = \frac{a - (1-\nu_2)b_*^{H2}}{1-\nu_H} \tag{27}$$

Equation (25) is key to understand equilibrium intermediation. Observe indeed that $1H$ has two possible uses for each unit of asset or re-usable collateral he holds. The LHS measures the marginal gains from borrowing from agent 2. One unit of the asset used in repo F_{H2} increases agent 2 consumption in period 2 by $\frac{1-(1-\theta_H)\nu_2}{(1-\theta_H)(1-\nu_2)}s$ in states $s < s^*$ where the factor $(1-(1-\theta_H)\nu_2)/(1-\nu_2)$ is the collateral multiplier described in Section 4.1. The RHS measures the marginal gains from selling the asset to agent $1L$ who would then pledge it back to $1H$. These are equal to the corresponding net increase in borrowing $\theta_L/\left[(1-\theta_L)(1-\nu_H)\right]$ times the unit gains $\delta_H - \delta_L$ from doing so. The solution to this trade-off pins down the equilibrium amount of asset or free collateral b^* agent $1H$ uses to borrow from 2 when[24] $\hat{b} > 0$ We can then back up the nature of the trade(s) between $1L$ and $1H$ using equation (27) so that exactly b^* is available for $1H$ to trade with 2.

In case 1., gains from trade between agent $1H$ and 2 are so large that all the asset availble a is used in repo F_{H2}. This is only possible if agent $1L$ sells the asset to agent $1H$ as a fraction $1-\nu_H$ would be segregegated should they use a repo transaction. In case 3. however, gains from trade between $1L$ and $1H$ are large. These agents then use a repo F_{LH} because the extra borrowing capacity $\theta_L/\left[(1-\theta_L)(1-\nu_H)\right]$ over a mere spot trade overcomes the segregation cost. Observe that the repo contract F_{LH} between agents L and H does not reflect

[24] If the solution is $\hat{b} < 0$, agent 2 is not trading since agent prefers to re-use collateral to re-sell to $1L$ until all the asset is segregated.

any hedging motive as agent H is risk neutral. As a result of collateral segregation though, the amount of free asset for $1H$ to use with 2 reduces to at most $\nu_H a$. In the intermediate case 2., agents $1L$ and $1H$ are marginally indifferent between spot and repo trades and use a combination of both.

Condition (26) is sufficient for the intermediation equilibrium and necessary in cases 2. and 3. It states that trading through agent $1H$ dominates direct trading between agents $1L$ and 2. The LHS captures the benefits from intermediation which are increasing in $1/(1-\theta_H) - 1/(1-\theta_L)$, the extra amount agent $1H$ can pledge to 2 with respect to $1L$. Observe that as the re-use capacity of agent 2 ν_2 increases, this extra borrowing capacity matters less because agent 2 reaches the optimal level of consumption in more states. The RHS captures the cost of intermediation[25] proportional to the fraction of collateral segregated $1 - \nu_H$ when $1L$ borrows from $1H$. In case 2. and 3., we may see the chain of repo trades as a technology to combine the asset originally held by $1L$ with the higher trustworthiness of agent $1H$. The fraction of segregated collateral $1 - \nu_H$ appears as the cost to operate this intermediation technology. As shown by condition (26), the benefits may only materalize if $\theta_H > \theta_L$. In particular, if agents $1L$ and $1H$ were equally good counterparties, agent $1H$ would have no comparative advantage to borrow from agent 2.

Finally, equation (27) reflects the aggregate allocation of asset to support borrowing in the economy. Each unit of the asset can generate $1/(1 - \nu_H)$ units of repo-ed asset between $1L$ and $1H$ or $1/(1 - \nu_2)$ units between $1H$ and 2.

As we showed in the previous sections, trustworthiness θ increases one's borrowing capacity. In our economy, agent 2 is the natural lender. However, agent $1L$, the natural borrower, also has the lowest trustworthiness θ_L. Intermediation can be beneficial because it combines the asset from agent $1L$ with the trustworthiness $\theta_H \geq \theta_L$ of agent $1H$. Coming back to our motivating example, it seems natural to assume that a well known market player such as a dealer bank is a more reliable counterparty than a hedge fund.

[25] One may now understand why condition (26) is only sufficient in case 1. where no collateral is segregated since agent $1L$ and $1H$ trade spot rather than repo.

5 Conclusion

We analyzed a simple model of repurchase agreement. We derived the equilibrium lending contracts with limited commitment and showed that this contract shares many features of a repo contract: It is a spot sale combined with a forward contract. The forward contract contains a fixed repayment schedule that insures the lender against the risk that the price of the asset varies. We found that the repo haircut is an increasing function of counterparty risk as measured by the borrowers' ability to commit and a decreasing function of the asset inherent risk. Also, safe assets naturally command a higher liquidity premium than risky ones. We use the model to analyze the consequences of allowing re-use. We find that re-use are most useful when the asset is special, in the sense that its haircut is negative. Then re-use helps agents to achieve the first best allocation. Finally, we found conditions under which intermediation can arise endogenously in repo markets, thus giving us first elements to understand the endogenous formation of repo chains.

Obviously, we made simplifying assumptions. We considered a finite economy in which the limited commitment problem is especially binding. Extending the model to an infinite horizon is however not trivial, as the optimal contract will become dynamic, unless we resort to the usual subtleties. Also, we considered a real economy and abstracted from money on purpose. While traders active in the repo market trade cash, we felt that it would be too difficult in a first pass to model repo and money seriously. However, given the results and the recent progress of the monetary literature we do not doubt that embedding money is feasible. In the end, the model is rather tractable and could be used to understand current policy issues such as minimum haircuts, collateral transformation, or the effect of a large asset purchase by the monetary authority on the repo market. We leave these important topics for future research.

Appendices

A Equilibrium analysis of spot trade only

We now discuss the equilibrium allocation when only spot market trades are feasible. Since there is no market for forward contracts, the equilibrium conditions become

$$
\begin{aligned}
p_2(s) &= s \\
-p_1(1+\lambda_1^1) + \delta E[s] + \xi_1^1 &= 0, \\
-p_1(1+\lambda_1^2) + E\left[su'(\omega + sa_1^2)\right] + \xi_1^2 &= 0, \\
\lambda_1^1 \lambda_1^2 &= 0 \\
\xi_1^1 \xi_1^2 &= 0 \\
c_2^1(s) &= \omega + s(a_1^1 - a) \\
c_3^1(s) &= 2\omega + as
\end{aligned}
$$

As we have seen, the complete market trade amounts to a loan from agent 2 to agent 1. When only spot trades are available, agent 1 should sell his asset in period 1 and buy it back in period 1 to borrow implicitly from agent 2. A simple argument by contradiction shows that $\xi_1^2 = 0$ (agent 2 saves asset from date 1 to date 2). Also we guess and verify later that $\lambda_1^i = 0$. Then the equilibrium conditions are

$$
\begin{aligned}
-p_1 + \delta E[s] + \xi_1^1 &= 0, \\
-p_1 + E\left[su'(\omega + sa_1^2)\right] &= 0, \\
c_2^1(s) &= \omega - sa_1^2 \\
c_3^1(s) &= 2\omega + as
\end{aligned}
$$

so

$$
\xi_1^1 = E\left\{s\left[u'(\omega + sa_1^2) - \delta\right]\right\} \tag{28}
$$

Hence, either $\xi_1^1 = 0$ and a_1^2 solves (28), or $\xi_1^1 > 0$ and $a_1^2 = a$ where $E\left\{s\left[u'(\omega + sa) - \delta\right]\right\} > 0$. We now verify that consumption is non-negative for both agents at date 1. Agent 1 sells assets at date 1 and so trivially his consumption is positive. Agent 2 purchases a_1^2 at price p_1 and consumes

$c_1^2 = \omega - p_1 a_1^2$. When $\xi_1^1 = 0$, $p_1 = \delta E[s]$ and $c_1^2 = \omega - \delta E[s] a_1^2(\omega)$. Notice that the solution $a_1^2(\omega)$ to (28) is decreasing in ω. When $\xi_1^1 > 0$, $p_1 = E\left[su'(\omega + sa)\right]$ and so $c_1^2 = \omega - aE\left[su'(\omega + sa)\right]$. So in both case $c_1^2 > 0$ whenever ω is large enough.

Finally, notice that the liquidity premium is given by (28).

B Proof of Proposition 3

As we observed before, preferences are such that $p_2(s) = s$ and agent 1 holds all the asset at date 3, that is $a_2^1(s) = a$ for all s. The set of feasible repo contracts for agent $i \in \{1, 2\}$ is:

$$\mathcal{F}_i = \left\{ \bar{p} \in \mathcal{C}^0([\underline{s}, \bar{s}]) \mid 0 \leq \bar{p}(s) \leq \frac{s}{1 - \theta_i} \right\}$$

that is agent i can promise to repurchase one unit of the asset at a price equal at most to $s/(1 - \theta_i)$ in state s. The proof is in two steps. First, we argue that any equilibrium allocation can be obtained with only one repo contract sold by agent 1. Next we characterize this contract, given that agents must not be willing to deviate to any other feasible contract in$\mathcal{F}_1$.

Step 1

First observe that $\mathcal{F}_i$ is stable under linear combinations with positive coefficient (convex cone). Hence a combination of contracts shorted by i can be replicated by a single contract. Let F_{ij} be the repo contract in which agent i borrows $b^{ij} \geq 0$ from agent j. Without loss of generality agent 2 does not use a repo $F_{21}$to borrow but only lends in a repo F_{12} shorted by agent 1. Indeed, with these trades, the transfer between 1 and 2 in state s is given by :

$$T_{1 \to 2}(s) = (a - a_1^1)s + b^{12}\bar{p}_1(s) - b^{21}\bar{p}_2(s)$$

For agent 2 collateral constraint to be satisfied it must be that:

$$a_1^2 = a - a_1^1 \geq b^{21}$$

Using that $\bar{p}_2(s) \leq s$, we obtain that $T_{1 \to 2}(s) \geq 0$. Hence, a single contract F_1 sold by agent 1 is enough to finance transfer $T_{1 \to 2}$. For simplicity we call this contract $\bar{p}$. Agents need not use spot transaction since the spot payment schedule $p_2(s) = s$ belongs to $\mathcal{F}_1$.

We write the equilibrium conditions when agents trade repo $\bar{p}$:

$$
\begin{aligned}
-p_1 + \delta E[s] + \gamma_1^1 &= 0, \\
-p_F + \delta E\left[\bar{p}(s)\right] + \gamma_1^1 &= 0. \\
-p_1 + E\left[s u'(c_2^2(s))\right] + \gamma_1^2 &= 0, \\
-p_F + E\left[\bar{p}(s) u'(c_2^2(s))\right] &= 0. \\
\xi_1^1 \xi_1^2 &= 0 \\
c_2^1(s) &= \omega - \bar{p}(s) b^{12} \\
c_2^2(s) &= \omega + \bar{p}(s) b^{12}
\end{aligned}
$$

We can use the equilibrium conditions to derive the marginal willingness to trade any contract $\tilde{F} \in \mathcal{F}_1$ with schedule $\tilde{p}$ for both agents. In other words, we derive the prices $\tilde{p}^1_{F,s}$ and $\tilde{p}^2_{F,b}$ at which agent 1 (resp. agent 2) is ready to sell (resp. to buy) an infinitesimal amount of contract $\tilde{F}$.

$$
\begin{aligned}
\tilde{p}^1_{F,s} &= \delta E\left[\tilde{p}(s)\right] + \gamma_1^1 \\
\tilde{p}^2_{F,b} &= E\left[\tilde{p}(s) u'(\omega + s a_1^2 + \bar{p}(s)\ell^2)\right]
\end{aligned}
$$

For agents not to trade contract $\tilde{p}$ in equilibrium, the following inequality must hold:

$$
\tilde{p}^2_{F,b} \leq \tilde{p}^1_{F,s} \tag{29}
$$

Step 2

We now use inequality together with the equilibrium conditions to show that the equilibrium $\bar{p}$ is the contract characterized in Proposition 1. We distinguish two cases.

i) $\gamma_1^1 = 0$: agent 1 is unconstrained.

Then agents 1 and 2's (marginal) valuation for any contract $\tilde{p} \in \mathcal{F}_1$ must coincide, that is:

$$
E\left[\tilde{p}(s) u'(c_2^2(s))\right] = \delta E\left[\tilde{p}(s)\right] \tag{30}
$$

where $c_2^2(s) = \omega + \bar{p}(s) b^{12}$. Suppose there is an open interval $(s_1, s_2) \in \mathcal{S}$ such that for all $s \in (s_1, s_2)$, $u'(c_2^2(s)) - \delta \neq 0$ and has a constant sign. Let us then consider the piece-wise linear schedule $\tilde{p}$ such that $\tilde{p}(\underline{s}) = \tilde{p}(s_1) = \tilde{p}(s_2) = \tilde{p}(\bar{s}) = 0$ and $\tilde{p}(s_1/2 + s_2/2) = s_1$. The schedule $\tilde{p} \in \mathcal{F}_1$ would violate equality 30. It means that there cannot be an open interval on which $u'(c_2^2(s)) - \delta \neq 0$. Hence, by continuity, we must have for all $s \in \mathcal{S}$, $u'(c_2^2(s)) = \delta$, that is $c_2^2(s) = c^2_{2,*}$. This means that $\bar{p}$ is constant and in particular that agent 2 can finance $c^2_{2,*}$ in the lowest state $\underline{s}$:

$$c_2^2(\underline{s}) = \omega + \bar{p}(\underline{s})b^{12} = c_{2,*}^2$$

By definition of $\mathcal{F}_1$, and taking $b^{12} = a$, this is possible only if $s^* \leq \underline{s}$. In that case, although the equilibrium allocation is unique, the contracts traded are not. The expression of $c_2^2(s)$ only pins down[26] the product $b^{12}\bar{p}$, and the repurchase price $\bar{p}$ may lie anywhere in the interval $[\frac{s^*}{1-\theta_1}, \frac{s}{1-\theta_1}]$.

ii) $\gamma_1^1 > 0$: agent 1 is constrained.

This means that $b^{12} = a$. Agent 2 is unconstrained. Hence, his marginal valuation for a non-traded contract $\tilde{p} \in \mathcal{F}_1$ is equal to

$$\tilde{p}_F^1 = E[\tilde{p}(s)u'(c_2^2(s))]$$

Agent 1 marginal willingness to sell that contract is equal to:

$$\tilde{p}_{F,s}^2 = \delta E[\tilde{p}(s)] + \eta_1^1$$

Agents find it weakly optimal not to trade contract $\tilde{p}$ if and only if$\tilde{p}_{F,s}^2 \geq \tilde{p}_F^1$, that is

$$E\left[(\bar{p}(s) - \tilde{p}(s))\left(u'(\omega + \bar{p}(s)a_F) - \delta\right)\right] \geq 0 \tag{31}$$

Let us now define

$$\mathcal{S}^* = \left\{s \in \mathcal{S} \mid c_2^2(s) = c_{2,*}^2\right\}$$

If $\mathcal{S}^* = \emptyset$. Then, since $s^* \leq \underline{s}$, that is $c_2^2(s) < c_{2,*}^2$, by continuity of $\bar{p}$, we have $c_2^2(s) < c_{2,*}^2$ for all $s \in \mathcal{S}$. Then it is clear that the inequality above will be violated unless $\bar{p}(s) = s/(1-\theta_1)$. Hence the only possible value for $\bar{p}$ is $\bar{p}(s) = s/(1-\theta_1)$. Then it can be true that $\mathcal{S}^* = \emptyset$ if and only if $s^* \geq \bar{s}$. Then the equilibrium contract is unique since $\bar{p}$ cannot be replicated by any combination of contracts in $\mathcal{F}_1$.

Suppose now $\mathcal{S}^*$ is not empty (which implies $\underline{s} \leq s^* \leq \bar{s}$) and let $s_1 \geq s^* > \underline{s}$ be its minimal element. We have by continuity $c_2^2(s) < c_{2,*}^2$ for all $s \in (\underline{s}, s_1)$. Consider then $\tilde{p}$ defined as follows:

$$\tilde{p}(s) = \begin{cases} \frac{s}{1-\theta_1} & \text{if } s \leq s^* \\ \frac{s^*}{1-\theta_1} & \text{if } s \in [s^*, s_1] \\ \bar{p}(s) & \text{otherwise} \end{cases}$$

This would violate 31 unless $\bar{p} = \tilde{p}$ and proves that $\bar{p}(s) = s/(1-\theta_1)$ for $s \in [\underline{s}, s^*]$. We must now prove that $\bar{p}(s) = s^*/(1-\theta_1)$ for all $s \geq s^*$. Suppose it is not and let $s_2 \in [s^*, \bar{s})$ be then the

[26]In addition, agent 2 could also buy the asset spot to sell it in a repo F_2. In any case, having agent 1 sell a units of contract $\bar{p} = s^*/(1-\theta)$ is an equilibrium since agents do not (strictly) want to trade another contract.

greatest value such that $c_2^2(s) = c_{2,*}^2$ for all $s \in [s^*, s_2]$. Then let $s_3 = \min\{s \in \mathcal{S}^* | \, s > s_2\}$ if any. If there does not exist such s_3, by continuity $\forall s \in (s_2, \bar{s}]$, $c_2^2(s) > c_{2,*}^2$ so that $\bar{p}(s) > s^*/(1-\theta_1)$ - the case with the reserve inequality can be treated symmetrically. But then, inequality 31 would not hold with the following schedule:

$$\tilde{p}(s) = \begin{cases} \bar{p}(s) & \text{if } s \leq s_2 \\ \frac{s^*}{1-\theta_1} & \text{otherwise} \end{cases}$$

If s_3 exists, we can use a similar argument with a schedule $\tilde{p}$ equal to $s^*/(1-\theta_1)$ over (s_2, s_3) and equal to $\bar{p}(s)$ otherwise. This concludes the proof and shows that the equilibrium schedule $\bar{p}$ is as follows:

$$\bar{p}(s) = \begin{cases} \frac{s}{1-\theta_1} & s \leq s^* \\ \frac{s^*}{1-\theta_1} & s > s^* \end{cases}$$

By the same logic as in the previous case, since $b^{12} = a$, $\bar{p}$ is the unique contract such that $c_2^2(s) = \omega + \bar{p}(s)a$.

C Proof of Proposition 5

Proof. To show that the liquidity premium that the safer asset is higher than the one of the riskier asset, we take the difference in the liquidity premium. The liquidity premium for asset $i = A, B$ is

$$\mathcal{L}_i(\alpha) = \delta \int_{\underline{s}}^{s^{**}} \frac{\rho_i(s)}{1-\theta} \left[u' \left(\omega + \frac{a\rho_{\alpha_A}(s) + b\rho_{\alpha_B}(s)}{1-\theta} \right) - \delta \right] dF(s)$$

and so

$$\begin{aligned} \mathcal{L}_{a,b}(\alpha) &= \mathcal{L}_a(\alpha) - \mathcal{L}_b(\alpha) \\ &= \delta \int_{\underline{s}}^{s^{**}} \frac{\rho_{\alpha_a}(s) - \rho_{\alpha_b}(s)}{1-\theta} \left[u' \left(\omega + \frac{a\rho_{\alpha_A}(s) + b\rho_{\alpha_B}(s)}{1-\theta} \right) - \delta \right] dF(s) \\ &= \frac{\delta(\alpha_a - \alpha_b)}{1-\theta} \int_{\underline{s}}^{s^*} (s - \mathbb{E}[s]) \left[u' \left(\omega + \frac{a\rho_{\alpha_A}(s) + b\rho_{\alpha_B}(s)}{1-\theta} \right) - \delta \right] dF(s) \\ &> 0 \end{aligned}$$

where the inequality follows from the fact that $\alpha_a < \alpha_b$, and the integral is negative over the integration range.[27]

[27] Notice that we can expand this analysis for many more assets where s^{**} would then be given

The haircut as a function of α then is (using linearity of ρ_i in terms of s):

$$
\begin{aligned}
\mathcal{H}_i(\alpha) &= p_{1,i} - p_{F,i} \\
&= \delta \mathbb{E}\left[\rho_i(s) - \bar{p}_i(s)\right] \\
\frac{\mathcal{H}_i(\alpha)}{\delta} &= \mathbb{E}[s] - \int_{\underline{s}}^{s^{**}} \frac{\rho_i(s)}{1-\theta} dF(s) - \int_{s^{**}}^{\bar{s}} \frac{\rho_i(s^{**})}{1-\theta} dF(s) \\
&= \mathbb{E}[s] - \int_{\underline{s}}^{s^{**}} \frac{(1+\alpha_i)s - \alpha_i \mu}{1-\theta} dF(s) - \int_{s^{**}}^{\bar{s}} \frac{(1+\alpha_i)s^{**} - \alpha_i \mu}{1-\theta} dF(s) \\
&= \mathbb{E}[s] + \frac{\alpha_i \mu}{1-\theta} - \frac{(1+\alpha_i)}{1-\theta} \left[\int_{\underline{s}}^{s^{**}} s dF(s) + \int_{s^{**}}^{\bar{s}} s^{**} dF(s) \right]
\end{aligned}
$$

The term in brackets is less than $\mathbb{E}[s]$ therefore, for all assets A and B such that $\alpha_A < \alpha_B$ we obtain

$$\mathcal{H}_A(\alpha_A, \alpha_B) < \mathcal{H}_B(\alpha_A, \alpha_B)$$

i.e. the safe asset always commands a lower haircut than the risky asset.

□

D Mean Preserving Spread

Obviously, a higher $\bar{s}$ means that the asset is riskier. Given the distribution for s we obtain

$$
\begin{aligned}
\mathbb{E}[\bar{p}(s)] &= \frac{1}{\bar{s} - \underline{s}} \left[\int_{\underline{s}}^{s^*} \frac{s}{1-\theta} ds + \int_{s^*}^{\bar{s}} \frac{s^*}{1-\theta} ds \right] \\
&= \frac{1}{2(\bar{s}-1)(1-\theta)} \left[2(\bar{s}-1) - \frac{1}{2} (\bar{s} - s^*)^2 \right] \\
&= \frac{1}{1-\theta} - \frac{(\bar{s} - s^*)^2}{4(\bar{s}-1)(1-\theta)}
\end{aligned}
$$

Then the haircut $\mathcal{H}$ is always increasing in $\bar{s}$. First, notice that when $s^* \geq \bar{s}$, $\mathcal{H}$ depends only on $\mathbb{E}[s]$ but not on the distribution of s. Hence the haircut is not affected marginally by a mean-preserving spread. Secondly, when $s^* \leq \underline{s}$, it is clear from the expression above that the the lower bound for admissible haircut $\delta\mathbb{E}[s] - \delta\frac{2-\bar{s}}{1-\theta}$ increases with $\bar{s}$. Finally, when $s^* \in [\underline{s}, \bar{s}]$,

by

$$u'\left(\omega + \frac{\sum_{i=1}^{n} \rho_i(s^{**}) a_i}{1-\theta}\right) = \delta$$

where a_i would be the number of asset of type i pledged as collateral. The liquidity premium could be defined in the same way.

we obtain the following expression for the haircut

$$\begin{aligned}\mathcal{H} &= \delta(1-\mathbb{E}[\bar{p}(s)]) \\ &= \frac{\delta\theta}{1-\theta} + \frac{\delta\left(\bar{s}-s^*\right)^2}{4(\bar{s}-1)(1-\theta)}\end{aligned}$$

The derivative with respect to $\bar{s}$ is

$$\frac{\partial \mathcal{H}}{\partial \bar{s}} = \frac{\bar{s}-s^*}{4(\bar{s}-1)^2(1-\theta)}\left[2(\bar{s}-1)-(\bar{s}-s^*)\right] = \frac{(\bar{s}-s^*)(s^*-\underline{s})}{4(\bar{s}-1)^2(1-\theta)} \geq 0$$

Intuitively, with a mean-preserving spread, the collateral quality deteriorates and hence commands higher haircut to compensate the lender. Haircut increases with asset riskiness only because of the hedging motive of the lender.

The impact of an increase in risk on the liquidity premium $\mathcal{L}$ is however not clear. Indeed, in the case where $s^* \in [\underline{s}, \bar{s}]$, we have

$$\mathcal{L} = \frac{1}{2(\bar{s}-1)(1-\theta)} \int_{\underline{s}}^{s^*} s\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right) ds$$

Hence, we obtain

$$\frac{\partial \mathcal{L}}{\partial \bar{s}} = \frac{1}{2(\bar{s}-1)(1-\theta)}\left[-\frac{1}{\bar{s}-1}\int_{\underline{s}}^{s^*} s\left(u'(\omega + \frac{s}{1-\theta}a) - \delta\right) ds + \underline{s}\left(u'(\omega + \frac{\underline{s}}{1-\theta}a) - \delta\right)\right]$$

and the sign of this expression depends in particular on the coefficient of relative risk aversion of u.

Another variable of interest is the (gross) repo rate that can be defined as

$$r := \frac{\mathbb{E}[\bar{p}(s)]}{p_F} = \frac{1}{\delta}\frac{-\mathcal{H}+\delta\mathbb{E}[s]}{-\mathcal{H}+\delta\mathbb{E}[s]+\mathcal{L}}$$

Again, the effect of the mean preserving spread on the repo rate are not clear. However, numerical computations seem to indicate that the rate increase with riskiness.

Actually, we do not need to make distributional assumptions on s to get the result. As before, $s \sim F[\underline{s}, \bar{s}]$ but the asset payoff is now:

$$\rho_\alpha(s) = s + \alpha(s - \mathbb{E}[s]), \qquad \alpha \geq -1$$

When α=-1, the asset is perfectly safe and pays off $\mathbb{E}[s]$. An asset with a higher α has the same mean but a higher variance than an asset with a lower α. Indeed $Var[\rho_\alpha] = (1+\alpha)Var[s]$.

Let us now call $s^*(\alpha)$ the threshold such that $\rho_\alpha(s^*(\alpha)) = s^*$. Exactly as before, the optimal repo contract for asset α is

$$\bar{p}_\alpha(s) = \begin{cases} \frac{\rho_\alpha(s)}{1-\theta} & \text{for } s \leq s^*(\alpha) \\ \frac{\rho_\alpha(s^*)}{1-\theta} & \text{for } s \geq s^*(\alpha) \end{cases}$$

Remember that $\mathcal{H}_\alpha = \delta(\mathbb{E}[\rho_\alpha(s)] - \mathbb{E}[\bar{p}_\alpha(s)]) = \delta E[s] - \delta\mathbb{E}[\bar{p}_\alpha(s)]$. We have (assuming that $s^*(\alpha) \in [\underline{s}, \bar{s}]$)

$$\mathbb{E}[\bar{p}_\alpha(s)] = \frac{1}{1-\theta}\left[\int_{\underline{s}}^{s^*(\alpha)} \rho_\alpha(s)dF(s) + \rho_\alpha(s^*(\alpha))\left(1 - F\left(s^*(\alpha)\right)\right)\right]$$

Hence, we obtain

$$\begin{aligned}\frac{\partial\mathcal{H}_\alpha}{\partial\alpha} &= -\frac{\delta}{1-\theta}\left[\int_{\underline{s}}^{s^*(\alpha)}(s - \mathbb{E}[s])dF(s) + \frac{\partial s^*(\alpha)}{\partial\alpha}\rho_\alpha\left(s^*(\alpha)\right) f\left(s^*(\alpha)\right) - \rho_\alpha\left(s^*(\alpha)\right)\frac{\partial s^*(\alpha)}{\partial\alpha} f\left(s^*(\alpha)\right)\right] \\ &= -\frac{\delta}{1-\theta}\int_{\underline{s}}^{s^*(\alpha)}(s - \mathbb{E}[s])dF(s) \geq 0\end{aligned}$$

Here the asset risk is also the fundamental risk in the economy. With two assets (see below), we can compare haircuts for assets with different risk while keeping the economy unchanged. Actually this result together with that for two assets suggests that when fundamental risk increases, the difference in haircuts between risky and safe assets increase as well (a bit mechanical maybe).

E Proof of Proposition 6

Proof. If agents are constrained, the collateral constraint binds. We thus have

$$a_1^1 = b^{12} \tag{32}$$

$$a_1^2 = -\nu\ell^{21} \tag{33}$$

Even with re-use, there is a constraint on the amount of physical asset available. Precisely we have $a_1^1 + a_1^2 = a$. Also market clearing requires $b^{12} = \ell^{21}$. Summing (32) and (33) we obtain

$$\begin{aligned}a_1^1 &= \frac{a}{1-\nu} = b^{12} \\ a_1^2 &= -\frac{\nu}{1-\nu}a = -\nu b^{12}\end{aligned}$$

Using agent 2's budget constraint in period 2, we can write his consumption as

$$c_2^2(s) = \omega + \frac{a}{1-\nu}(\bar{p}(s,\nu) - \nu s)$$

Notice that while $\bar{p}(s,\nu)$ is of the general form (14), insurance requires that $\bar{p}(s,\nu)$ corrects for the cost agent 2 incurs from purchasing the asset to cover his short position in period 2. So following our earlier argument $\bar{p}(s,\nu)$ is

$$\bar{p}(s,\nu) = \begin{cases} \frac{s}{1-\theta} & \text{if } s < s^*(\nu) \\ \frac{s^*(\nu)}{1-\theta} + \nu(s - s^*(\nu)) & \text{if } s \geq s^*(\nu) \end{cases}$$

where $s^*(\nu)$ is implicty defined as follows

$$u'\left(\omega + \frac{as^*(\nu)}{1-\nu}\left[\frac{1}{1-\theta} - \nu\right]\right) = \delta.$$

Since $v \rightarrow \frac{1-(1-\theta)v}{1-v}$ is increasing in v, $s^*(\nu)$ is decreasing in v.

□

F Proof of Proposition 7

Proof. Under our conjecture, agents 1L and 1H may trade in a repo F_{LH} and agents $1H$ and 2 can trade in a repo F_{H2}. As usual, agents may also trade in the spot market. We then derive the conditions and characterize the repo contracts F_{LH} and F_{H2} for this conjecture is indeed an equilibrium.

Step 1: Agents problem and first order conditions

The problem of agent L writes :

$$\max_{a_1^L, b^{HL}} \omega + p_1(a - a_1^L) + p_{F,LH}b^{LH} + \delta_L E\left[2\omega + sa_1^L - \bar{p}_{LH}(s)b^{LH}\right]$$

$$\text{s.to} \quad a_1^L \geq b^{LH} \quad (\gamma_1^L)$$
$$b^{LH} \geq 0 \quad (\xi_{LH})$$

While the problem of agent H is:

$$\begin{aligned}
&\max_{a_H^1, l^{HL}, b^{H2}} \omega - p_1 a_H^1 - p_{F,LH}\ell^{HL} + p_{F,H2}b^{H2} \\
&+\delta_H E\left[2\omega + s a_H^1 + \bar{p}_{LH}(s)\ell^{HL} - \bar{p}_{H2}(s)b^{H2}\right] \\
&\qquad\qquad \text{s.to} \quad a_1^H + \nu_H \ell^{HL} \geq \; b^{H2} \quad (\gamma_1^H) \\
&\qquad\qquad\qquad\qquad \ell^{HL} \geq \; 0 \quad (\xi_{HL}) \\
&\qquad\qquad\qquad\qquad b^{H2} \geq \; 0 \quad (\xi_{H2})
\end{aligned}$$

Recall that a_1^H is the spot market trade of agent H. The variable ℓ^{HL} is the amount agent H lends to L and every unit of loans yields agent H a fraction ν_H of re-usable asset. These units can be re-sold spot, which decreases a_1^H, or re-pledged to agent 2, which increases b^{H2}. Finally, agent 2 solves

$$\begin{aligned}
&\max_{a_1^L, l^{2H}} \omega - p_1 a_1^2 - p_{F,H2}\ell^{2H} + E\left[u\left(\omega + s a_1^2 + \bar{p}_{H2}(s)\ell^{2H}\right)\right] \\
&\qquad\qquad \text{s.t. } a_1^2 \quad + \nu_2 \ell^{2H} \geq 0 \quad (\gamma_1^2) \\
&\qquad\qquad \ell^{2H} \qquad\quad \geq 0 \qquad (\xi_{2H})
\end{aligned}$$

where ℓ^{2H} is the loan of agent 2 to agent H.

Let us now write down the first order conditions for our 3 agents:

$$-p_1 + \delta_L E[s] + \gamma_1^L = 0 \tag{34}$$

$$p_{F,LH} - \delta_L E[\bar{p}_{LH}(s)] - \gamma_1^L + \xi_{LH} = 0 \tag{35}$$

$$-p_1 + \delta_H E[s] + \gamma_1^H = 0 \tag{36}$$

$$-p_{F,LH} + \delta_H E[\bar{p}_{LH}(s)] + \nu_H \gamma_1^H + \xi_{HL} = 0 \tag{37}$$

$$+p_{F,H2} - \delta_H E\left[\bar{p}_{H2}(s)\right] - \gamma_1^H = 0 \tag{38}$$

$$-p_1 + E\left[s u'(c_2^2(s))\right] + \gamma_1^2 = 0 \tag{39}$$

$$-p_{F,H2} + E\left[\bar{p}_{H2}(s) u'(c_2^2(s))\right] + \nu_2 \gamma_1^2 = 0 \tag{40}$$

Market clearing implies that $b^{ij} = \ell^{ji}$ for each pair of agents (i,j). Hence, we only use the notation b in the following. Observe that we introduced the positivity constraint on the amount borrowed by $1L$ to $1H$ as these agents may not use a repo transaction but a spot trade esclusively. A quick examination shows that all three collateral constraints must bind, that is $\gamma_1^L > 0$, $\gamma_1^H > 0$ and $\gamma_1^2 > 0$. This implies that:

$$\begin{aligned}
a_1^L &= b^{LH} \\
a_1^H + \nu_H b^{HL} &= b^{H2} \\
a_1^2 + \nu_2 b^{H2} &= 0
\end{aligned}$$

while market clearing for the asset yields:

$$a_1^L + a_1^H + a_1^2 = a$$

Using this last equations together with the saturated collateral constraints above, we obtain equation (27), that is

$$a = (1-\nu_H)b^{LH} + (1-\nu_2)b^{L2}$$

Quick manipulations of equations (34) to (40) give the following expressions for the Lagrange multipliers associated to the collateral constraints:

$$\begin{aligned}
\gamma_1^2 &= \frac{1}{(1-\nu_2)} E\left[(\bar{p}_{H2}(s) - s)\left(u'(c_2^2(s)) - \delta_H\right)\right] \\
\gamma_1^H &= \frac{1}{(1-\nu_2)} E\left[(\bar{p}_{H2}(s) - \nu_2 s)\left(u'(c_2^2(s)) - \delta_H\right)\right] \\
\gamma_1^L &= \gamma_1^H + (\delta_H - \delta_L)
\end{aligned}$$

Let b^* be the amount of asset available for the repo between $1H$ and 2 so that $b_*^{H2} = b^*/(1-\nu_2)$

$$c_2^2(s) = \omega + b_*^{H2}(\bar{p}_{H2}(s, b^*, \nu_2) - \nu_2 s)$$

Step 2 : Equilibrium Repo contracts

i) Equilibrium repo contract F_{LH} between $1L$ and $1H$

With usual equilibrium selection argument, agents $1L$ and $1H$ are not willing to trade contract $\tilde{F}_{LH}$ if and only if

$$\delta_L E[\tilde{p}_{LH}(s)] + \gamma_1^L + \xi_{LH} \geq \delta_H E[\tilde{p}_{LH}(s)] + \nu_H \gamma_1^H + \nu_H \xi_{HL}$$

Using equations (35) and (37), we obtain

$$(\delta_H - \delta_L) E\left[\bar{p}_{LH}(s) - \tilde{p}_{LH}(s)\right] \geq 0$$

which may only hold if

$$\bar{p}_{LH}(s) = \frac{s}{1-\theta_L}.$$

ii) Equilibrium repo contract F_{H2} between $1H$ and 2.

For a given amount b^* of asset available, agents H and 2 trade as in the previous section replacing a by b^*. Using our previous results, the equilibrium repo contract is $\bar{F}_{i2}(b^*, \nu_2) := (\bar{p}_{i2}(b^*\nu_2), \nu_2)$ where $\bar{p}(b^*, \nu_2)$ is defined in (23).

iii) No profitable contract between $1L$ and 2

We need to check that intermediation is optimal, that is agents L and 2 do not want to trade directly a contract $\tilde{F}_{L2}$. This requires

$$\delta_L E[\tilde{p}_{L2}(s)] + \gamma_1^L \geq E[\tilde{p}_{L2}(s)u'(c_2^2(s))] + \nu_2\gamma_1^2$$

We can rewrite the condition as

$$(1-\nu_2)\gamma_1^2 \geq E\left[(\tilde{p}_{L2}(s) - s)\left(u'(c_2^2(s)) - \delta_L\right)\right]$$

Using the expression of γ_1^2, we obtain:

$$E\left[(\bar{p}_{H2}(s) - s)\left(u'(c_2^2(s)) - \delta_H\right)\right] \geq E\left[(\tilde{p}_{L2}(s) - s)\left(u'(c_2^2(s)) - \delta_L\right)\right]$$

or,

$$E\left[(\bar{p}_{H2}(s) - \tilde{p}_{L2}(s))\left(u'(c_2^2(s)) - \delta_H\right)\right] \geq \frac{\theta_L(\delta_H - \delta_L)}{1-\theta_L}$$

In the LHS, we use $\tilde{p}_{L2} = s/(1-\theta_L)$ to find the tightest bound:

$$\left(\frac{1}{1-\theta_H} - \frac{1}{1-\theta_L}\right)\int_{\underline{s}}^{s^*(b^*,,\theta,\nu_2)} s\left(u'\left(c_2^2(s)\right) - \delta_H\right) \geq \frac{\theta_L(\delta_H - \delta_L)}{1-\theta_L} \geq (1-\nu_H)\gamma_1^H$$

\

where the last inequality derives from equations (34) -(37) and holds as an inequality when $b_{LH} > 0$. In this latter case, we can rewrite the necessary conditon above as

$$\left(1 - \frac{1-\theta_H}{1-\theta_L}\right)\frac{1}{1-\nu_2(1-\theta_H)} \geq \frac{1-\nu_H}{1-\nu_2}$$

which is sufficient condition (26). Observe that the condition will also be necessary whenever $b^{LH} > 0$.

Step 3 : Determination of b^*

We are left with determining the endogenous amount b^* available for the repo trade between $1H$ and 2 in order to describe the equilibrium completely.

i) $\xi_{LH} > 0$ or $\xi_{HL} > 0$ or $b^{LH} = 0$.

Since agent $1L$ collateral constrain binds, it must be that $a_L^1 = 0$ and $a_H^1 = a$. This implies that $b^* = a$ and $b^{H2} = \frac{a}{1-\nu_2}$. Agent 2 consumption is

$$c_2^2(s,a) = \begin{cases} \omega + \frac{as}{1-\nu_2}\left[\frac{1}{1-\theta_H} - \nu_2\right] & \text{if } s < s^*(a,\nu_2,\theta_{\mathrm{H}}) \\ c_{2,*}^2 & \text{if } s \geq s^*(a,\nu_2,\theta_{\mathrm{H}}) \end{cases}$$

This can be an equilibrium if $\xi_{LH} + \xi_{HL} \geq 0$. Using equations (35) and (37), the condition writes

$$\gamma_1^L \geq \nu_H \gamma_1^H + \frac{\delta_H - \delta_L}{1 - \theta_L}$$

which, using our derivations above, can be rewritten:

$$\frac{1 - (1 - \theta_H)\nu_2}{(1 - \theta_H)(1 - \nu_2)} \int_{\underline{s}}^{s^*(a,\theta_H,\nu_2)} \left[u'\left(c_2^2(a,s)\right) - \delta_H\right] \mathrm{sd}F(s) \geq \frac{(\delta_H - \delta_L)\theta_L}{(1 - \nu_H)(1 - \theta_L)}$$

Since the mapping

$$a \to \int_{\underline{s}}^{s^*(a,\theta_H,\nu_2)} \left[u'\left(c_2^2(a,s)\right) - \delta_H\right] \mathrm{sd}F(s)$$

is decreasing in its argument, the condition above is equivalent to condition $\hat{b} > a$ of case 1.
ii) $\xi_{HL} = \xi_{LH} = 0$, or $b^{LH} > 0$
In this case, we obtain two expressions for γ_1^H which impose the following equality:

$$\frac{1 - (1 - \theta_H)\nu_2}{(1 - \theta_H)(1 - \nu_2)} \int_{\underline{s}}^{s^*(b^*,\theta_H,\nu_2)} \left[u'\left(c_2^2(b^*,s)\right) - \delta_H\right] \mathrm{sd}F(s) = \frac{(\delta_H - \delta_L)\theta_L}{(1 - \nu_H)(1 - \theta_L)}$$

and pins down the amount $b^* \in [0, a]$ availabe for $1H$ to use in repo F_{H2}. Suppose first that $b^* \in [\nu_H a, a]$. Then it must be that agent $1H$ buys a fraction of the asset from $1L$. If they only trade in a repo, the maximum amount of free collateral available to $1H$ is $\nu_H a$. Suppose now that $b^* \in [\nu_H a, a]$, then using equation (27) and agent $1L$ collateral constraint, we obtain $a_1^L = b^{LH} > a$. Since agent $1L$ initially owns $a_0^L = a$, it means that he is a net buyer of the asset (through $1H$ re-selling in the repo). Hence, agent $1L$ and $1H$ only use repo F_{LH}.

□

References

Aghion, P. and P. Bolton (1992). An incomplete contracts approach to financial contracting. *The Review of Economic Studies 59*(3), 473–494. 3

Andolfatto, D., F. Martin, and S. Zhang (2014). Rehypothecation. Technical report. 1

Antinolfi, G., F. Carapella, C. Kahn, A. Martin, D. Mills, and E. Nosal (2015). Repos, fire sales, and bankruptcy policy. *Review of Economic Dynamics 18*(1), 21 – 31. Money, Credit, and Financial Frictions. 1

Araújo, A., J. Orrillo, and M. R. Páscoa (2000). Equilibrium with default and endogenous collateral. *Mathematical Finance 10*(1), 1–21. 1

BIS (1999). Implications of repo markets for central banks. Cgfs publication. 1

Bottazzi, J.-M., J. Luque, and M. R. Páscoa (2012). Securities market theory: Possession, repo and rehypothecation. *Journal of Economic Theory 147*(2), 477–500. 1

Comotto, A. (2014). Repo, re-use and re-hypothecation. Technical report, ICMA. 2

Copeland, A., A. Martin, and M. Walker (2014). Repo runs: Evidence from the tri-party repo market. *The Journal of Finance 69*(6), 2343–2380. 1

Dubey, P., J. Geanakoplos, and M. Shubik (2005). Default and punishment in general equilibrium1. *Econometrica 73*(1), 1–37. 1

Duffie, D. (1996). Special repo rates. *The Journal of Finance 51*(2), 493–526. 7, 17

Farboodi, M. (2015). Intermediation and voluntary exposure to counterparty risk. Technical report. 1

Federal Reserve Bank of New York (2010). Tri-party repo infrastructure reform. White paper. 1, 22

Fleming, M. J. and K. D. Garbade (2005). Explaining settlement fails. *Current Issues in Economics and Finance 11*(Sep). 4

Fuhrer, L. M., B. Guggenheim, and S. Schumacher (2015). Re-use of collateral in the repo market. Working Papers 2015-02, Swiss National Bank. 8, 4.1

Geanakoplos, J. (1996). Promises Promises. Cowles Foundation Discussion Papers 1143, Cowles Foundation for Research in Economics, Yale University. 1, 2.1

Geanakoplos, J. and W. Zame (2014). Collateral equilibrium, I: a basic framework. *Economic Theory 56*(3), 443–492. 1

Gorton, G. and A. Metrick (2012). Securitized banking and the run on repo. *Journal of Financial Economics 104*(3), 425 – 451. Market Institutions, Financial Market Risks and Financial Crisis. 1

Gottardi, P. and F. Kubler (2015). Dynamic competitive economies with complete markets and collateral constraints. *The Review of Economic Studies 82*(3), 1119–1153. 1

ICMA (2013). Frequently Asked Questions on repo. Technical report, International Capital Market Association. 17

Infante, S. (2013). Repo collateral fire sales: the effects of exemption from automatic stay. Technical report. 1

Infante, S. (2015). Liquidity Windfalls: The Consequences of Repo Rehypothecation. Finance and Economics Discussion Series 2015-22, Board of Governors of the Federal Reserve System (U.S.). 5, 1, 21

Krishnamurty, A., S. Nagel, and D. Orlov (2014). Sizing up repo. *The Journal of Finance 69*(6), 2381–2417. 1

Martin, A., D. Skeie, and E.-L. v. Thadden (2014). Repo runs. *Review of Financial Studies 27*(4), 957–989. 1

Maurin, V. (2015). Re-using collateral, a general equilibrium model of rehypothecation. Technical report, European University Institute. 1

Monnet, C. (2011). Rehypothecation. *Business Review* (Q4), 18–25. 1

Muley, A. (2015). Collateral reuse in shadow banking and monetary policy. Technical report, MIT. 5, 1, 21

Narajabad, B. N. and C. Monnet (2012). Why Rent When You Can Buy? A Theory of Repurchase Agreements. Technical report. 1

Parlatore, C. (2015). Collateralizing liquidity. Working paper, New York University, Stern School of Business. 1, 6

Singh, M. (2011). Velocity of pledged collateral: Analysis and implications. IMF Working Papers 11/256, International Monetary Fund. 3, 1

Singh, M. and J. Aitken (2010). The (sizable) role of rehypothecation in the shadow banking system. IMF Working Papers 10/172, International Monetary Fund. 1, 4.1

Tomura, H. (2013). Investment Horizon and Repo in the Over-the-Counter Market. UTokyo Price Project Working Paper Series 013, University of Tokyo, Graduate School of Economics. 1

Vayanos, D. and P.-O. Weill (2008). A search-based theory of the on-the-run phenomenon. *Journal of Finance 63*(3), 1361–1398. 17

Zhang, S. (2014). Collateral risk, repo rollover and shadow banking. Working paper series, London School of Economics. 1