Over-the-Counter Markets with Bargaining Delays

Anton Tsoy*

March 15, 2016

Abstract

In many over-the-counter (OTC) markets, price is negotiated bilaterally and the bargaining over price takes time. This paper develops a dynamic equilibrium model of OTC asset markets with both search delays and endogenous bargaining delays. We first show that trade delays arise in bargaining even when the information about the asset quality is almost public. This type of delays has several implications for the OTC liquidity different from the models with public or asymmetric information about the asset quality. First, conditional on the public information, the liquidity is U-shaped in the quality and assets in the middle of the quality range may not be traded. Second, search and bargaining delays have opposite effects on the market liquidity showing that the reduction in such delays through greater transparency, while welfare improving, may also hurt the market liquidity. Third, the substitutability of different asset classes leads to flights-to-liquidity during periods of market uncertainty and reveals adverse effects of gradual transparency policies. Finally, the paper derives the effect of asset liquidity, market liquidity and market tightness on asset prices.

Keywords: *search friction, trade delay, liquidity, asset prices, over-the-counter markets, transparency, flight-to-liquidity, private information*

*Einaudi Institute for Economics and Finance. E-mail: tsoianton.ru@gmail.com. I am grateful to my advisors, Glenn Ellison, Andrey Malenko, Juuso Toikka and Muhamet Yildiz, for invaluable guidance and numerous suggestions that significantly improved the paper. I also thank Fernando Alvarez, Hui Chen, Piero Gottardi, Veronica Guerrieri, Andrei Kirilenko, Claudio Michelacci, Konstantin Milbradt, Facundo Piguillem, Alp Simsek, Ivan Werning, and seminar participants at EIEF, EUI, Kellogg, UNC Kenan-Flagler, Econometric Society World Congress (Montreal), Systemic Risk Center at LSE, Market Design Workshop at NES. All remaining errors are mine.

1 Introduction

Many important asset markets are decentralized. Examples include over-the-counter (OTC) markets for commercial and residential real estate, asset-backed securities, derivatives, corporate and municipal bonds, credit-default swaps, private equity, sovereign debt, bank loans, etc. In such markets, asset prices are negotiated bilaterally and it takes parties time to agree on the price. These bargaining delays can range from months as in the real-estate or private equity markets to hours or even minutes as in the most liquid parts of the bond market. The existing literature abstracts from bargaining delays by adopting the Nash bargaining solution and instead focuses on search delays. In fact, search delays are thought of as a reduced form for all types of trade delays.[1] The goal of this paper is to understand how justified is this approach and to disentangle the effect of the bargaining delays on the asset liquidity and pricing.

The novelty of our approach is the departure from the Nash bargaining solution in the bargaining stage. We first show that in the standard alternating-offer bargaining game, bargaining delays arise even when parties have very precise information about the asset quality, as long as the public information is crude. This gap between the public and private information is relevant in many OTC markets. In such markets only a limited amount of public information about assets is available in the form of credit ratings, past quotes, etc., while agents are sophisticated in evaluating assets and have more precise information.[2,3] We consider the bargaining delays arising in the almost public information limit, as the precision of the private information goes to infinity, while holding fixed the precision of the public information.[4]

We then incorporate the bargaining delays arising in the limit of almost public information into otherwise standard dynamic equilibrium model of OTC markets à la Duffie

[1]Duffie (2012) summarizes the current approach as follows "[s]earch delays ... proxy for delays associated with reaching an awareness of trading opportunities, arranging financing and meeting suitable legal restrictions, negotiating trades, executing trades, and so on."

[2]The Committee on the Global Financial System (2005) gives the following account of the OTC trade: "Interviews with large institutional investors in structured finance instruments suggest that they do not rely on ratings as the sole source of information for their investment decisions ... Indeed, the relatively coarse filter a summary rating provides is seen, by some, as an opportunity to trade finer distinctions of risk within a given rating band. Nevertheless, rating agency 'approval' still appears to determine the marketability of a given structure to a wider market."

[3]**?** documents that even in primary markets, asymmetric information between the originator of the MBS and the investor is both present and statistically significant, however, the absolute magnitude of its effect on transactions costs and prices is small.

[4]This terminology comes from the epistemic literature (Aumann (1999)) where the public information establishes the common knowledge among agents. As the private information of parties becomes more precise, values become almost common knowledge, thus, the term almost public information limit.

et al. (2005). On the one hand, this limit approach captures positive bargaining delays that are determined by the precision of the public information about assets. On the other hand, it allows for the tractability in the analysis of the OTC model, as it abstracts from learning about the quality of the asset. Formally, we solve for the unique steady-state equilibrium of the economy, in which agents are occasionally hit by liquidity shocks, and to share risks, they can trade a continuum of assets in the market with search delays and endogenous bargaining delays. The bargaining delays are endogenous and determined in equilibrium by agents' valuations which in turn depend on the ability to sell quickly the asset in the future as well as the steady-state distribution of assets among agents in the economy.

The analysis provides several new predictions and policy implications. In equilibrium, not all assets are necessarily traded which allows the distinction between two trade margins: extensive (whether the asset is traded or not) and intensive (asset-specific negotiation delays). On the intensive margin, the novel testable implication is that the asset liquidity, captured by the real costs of the bargaining delay, is U-shaped in quality conditional on the public information about the asset. This pattern arises from the dynamics of the negotiation in the almost public information limit: the buyer continuously increases his price offers starting from the ask price and the seller continuously decreases her price offers starting from the bid price, until one of the sides accepts the offer of the opponent. Thus, the buyer of a high quality asset and the seller of a low quality asset are willing to accept early on an offer close to the bid and ask price, respectively, rather than wait for more favorable offers. At the same time, owners and buyers of assets in the middle of the quality range have incentives to delay trade to hold out for a more favorable price offer. This prediction is in contrast to the adverse selection models of asset trade, in which lower asset qualities are more liquid.

The extensive trade margin arises because of the buyers' option to continue the search for a different asset. In equilibrium, they follow a simple threshold strategy and "shop" for assets with the shortest negotiation times. In conjunction with the U-shaped liquidity pattern, this implies that a range of asset qualities in the middle may not be traded at all. For such assets, it takes parties too long to agree on the price, and buyers prefer to reject such assets and continue their search for an asset whose price takes less time to negotiate. The presence of the extensive trade margin shows that trade delays are relevant even in markets where search and bargaining delays are normally short, e.g. corporate bonds market, and hence, seemingly should not have a significant effect on liquidity. In such markets, short observed negotiation delays can imply that a range of assets is rejected by buyers, as they take too long to negotiate, which essentially makes them illiquid.

The analysis of both forms of delay allows the distinction between two trade frictions in decentralized markets: the *search friction* which is reflected in the ability of market participants to find a trading opportunity and the *bargaining friction* which is reflected in the ability of market participants to promptly negotiate the price once an opportunity is identified. The standard approach abstracts from the bargaining friction and focuses only on the search friction.

We show that this view is only partially justified. First, the limit trade delays generated by only search or only bargaining frictions are quite different. Conditional on the trade taking place, the former leads to stochastic delays for both sides (as in Duffie et al. (2005)), while the latter leads to deterministic delays for sellers of actively traded assets and stochastic delays for buyers. Second, on the intensive margin, the two frictions are indeed similar: an increase in the bargaining friction leads to an increase in the average bargaining delay. However, on the extensive margin, the two frictions have opposite effects on the *market liquidity* captured by the range of asset qualities always accepted by buyers.

The bargaining friction in our model is determined by the quality of the public information about assets. For example, during periods of heightened market uncertainty, the infrequently updated credit ratings become less reliable in assessing the risks associated with the asset, and hence, the bargaining friction increases. An increase in the bargaining friction leads to a larger bid-ask spread and longer negotiation delays. This results in a decrease in the market liquidity, as agents prefer to trade fewer assets for which the negotiation times do not increase significantly. In the recent financial crisis, the significant increase in downgrades of financial products (see Benmelech and Dlugosz (2010), Ashcraft et al. (2010)) indicates an increased quality heterogeneity of assets conditional on the public information. In line with my model, it was accompanied by the dried-up liquidity.

On the contrary, an increase in the search friction lowers the market liquidity. As it becomes easier to search for assets in the market, buyers prefer to reject more assets and accept for trade only the most liquid assets. The opposing effect of search and bargaining frictions on the market liquidity shows that transparency, although welfare improving, does not always improve the market liquidity. Many transparency policies are associated with both more efficient search and higher quality of public information. For example, promoting post-trade transparency through the prompt disclosure of past quotes improves the public information about assets, and thus, decreases the bargaining friction. At the same time, if agents are willing to hold only assets about which they have sufficient information, post-trade transparency expands the consideration sets of buyers, hence, shortening search delays and reducing the search friction. As a result, the overall effect of post-trade transparency on the market liquidity is ambiguous. This is consistent with the existing mixed evidence on the effect on liquidity of the post-trade transparency in the corporate bonds market (see Bessembinder et al. (2006), Edwards et al. (2007), Asquith et al. (2013), Goldstein et al. (2007)).

In the analysis of liquidity, different assets act as substitutes for risk-sharing. In the recent financial crisis of 2007-2008, traders reacted to the increase in market uncertainty by a shift in their preferences towards safer and more liquid assets, a phenomenon known as flight-to-liquidity (Dick-Nielsen et al. (2012), Friewald et al. (2012)). Similarly, opponents of greater transparency in OTC markets point out that it can result in the migration of trade to certain asset classes hurting the liquidity of the market as a whole.

We extend the baseline model to take into account the substitutability between asset classes. An increase in the bargaining friction for one asset class can result in flight-to-liquidity episodes wherein agents migrate to trading assets with lower bargaining friction, which exacerbates the negative effect of the increased market uncertainty on the liquidity. Interestingly, once we take into account the asset substitutability, even the reduction in the bargaining friction can have adverse effects. If the reduction is uneven across asset classes, and as a result, there is an asset class that is significantly more liquid than the rest of the market, then agents will migrate to trading assets in this class. This adversely affects the liquidity of the rest of the market and can result in an overall decrease in the market liquidity and welfare. This reveals the negative effects of gradual transparency policies. For example, the introduction of mandatory trade reporting in corporate bonds market was introduced in several stages. Asquith et al. (2013) shows that this hurt the liquidity of high-yield bonds for which the post-trade transparency was introduced later than for the investment grade bonds.

Finally, we derive an intuitive decomposition of asset prices into three components: fundamental value component, liquidity premium component and average-liquidity component. This decomposition is consistent with the empirical evidence that there is a significant non-default component in corporate spreads which depends both on the liquidity of bond and marketwide liquidity. (see, e.g., Longstaff et al. (2005), Bao et al. (2011)). The effect of different components on asset prices is unambiguous and depends on how they affect agents' outside options of continuing the search. Factors that improve the outside option of the seller, such as the fundamental value, asset liquidity, seller's match intensity, increase the price, while factors that improve the outside option of the buyer, such as marketwide liquidity and buyer's match intensity, decrease price.

Related literature This paper is related to several strands of literature. First, the paper builds on the search and bargaining model of OTC markets introduced in Duffie et al. (2005) and further developed to account for risk-aversion (Duffie et al. (2007)), unrestricted asset holdings (Lagos and Rocheteau (2007, 2009)), asset heterogeneity (Vayanos and Weill (2008), Weill (2008)), agent heterogeneity (Vayanos and Wang (2007), Shen et al. (2015), Hugonnier et al. (2014)).[5] These models use the Nash bargaining solution, and hence, implicitly assume that the asset quality is public information which implies that the trade happens immediately after agents meet. This paper contributes to this literature by considering the *almost* publicly information about asset quality which leads to positive negotiation delays and allows the study of the effect of endogenous negotiation delays on prices and liquidity. Importantly, we show that while on the intensive margin, negotiation delays are similar to search delays, on the extensive margin, they operate quite differently.

Second, the paper contributes to the literature exploring the relationship between the liquidity and asset quality. Dynamic asset trading models with adverse selection (Guerrieri and Shimer (2014), Kurlat (2013), Chang (2014)) predict a decreasing relationship: in order to provide incentives for sellers of lower-quality assets to reveal their quality, such assets should be more liquid.[6] While this literature focuses on the asymmetric information, the screening bargaining solution used in this paper is derived as the limit of the bargaining model with two-sided correlated private information. This results into a different U-shaped dependence of the liquidity on the asset quality.[7]

Third, the paper contributes to the theoretical literature that studies the effect of transparency on the efficiency and liquidity of OTC markets (Duffie et al. (2015), Asriyan

[5]In this respect, the paper is also related to the literature on asset pricing with transaction costs which explored exogenous proportional transaction costs (Constantinides (1986), Heaton and Lucas (1996), Huang (2003)), fixed trading costs (Lo et al. (2004)) and exogenous bid-ask spreads (Amihud and Mendelson (1986)). Like Duffie et al. (2005), this paper focuses on a different type of costs, the opportunity costs of delayed trade, however, in our model the delay, rather than being exogenously given, is endogenously determined.

[6]There is also a growing literature that introduces the adverse selection into the Walrasian competitive equilibrium (e.g. see Guerrieri et al. (2010), Kurlat (n.d.)) and imperfectly competitive equilibrium (e.g. see Lester et al. (2015)).

[7]He and Milbradt (2014), Chen et al. (2014) analyze the feedback loop between the liquidity and default, and show that assets closer to default are associated with higher bid-ask spreads. Both the channel and the prediction is different from this paper.

et al. (2015)). This literature shows that higher transparency reduces the information asymmetry between agents, and hence, may lead to more efficient risk sharing and higher liquidity. This paper shows that the effect of transparency on liquidity is ambiguous depending on whether it leads to the reduction in the bargaining or search friction. It also shows that adverse effects can arise because of the asset substitutability.

Forth, the paper is related to the theoretical literature on search-and-bargaining pioneered by Rubinstein and Wolinsky (1985) most of which focuses on the case of complete information and hence immediate agreement (see Osborne and Rubinstein (1990), Gale (2000) for an excellent survey). Exceptions include work by Satterthwaite and Shneyerov (2007) and Lauermann and Wolinsky (2014) who study the conditions for convergence to the Walrasian outcomes in search models with incomplete information where allocations are determined by static auction mechanisms. In contrast, my focus is on negotiation delays, and because of the bargaining friction, my model does not converge to the competitive outcome even as the search friction vanishes. Another paper that explicitly incorporates trade delays into a search model is Atakan and Ekmekci (2014). In their model, agents imitate exogenously given commitment types requesting a fixed share of the surplus, while in my model all agents are rational.

The structure of the paper is the following. Sections 2 and 4 present and solve the model. Section 5 provides the asset pricing and liquidity implications. Section 6 shows how the substitutability of asset classes leads to flights-to-liquidity and adverse effects of gradual transparency policies. Section 7 discusses the generality of results. Section 8 concludes and gives directions for future research. All proofs are relegated to the Appendix.

2 Model

This section describes the economy in which agents trade assets to share risks in a market with a random search. Subsection 2.1 introduces the novel screening bargaining solution. Subsection 2.2 and 2.3 define the (steady-state) equilibrium and central equilibrium quantities.

There is a continuum of agents of mass $a > 1$. Time $t \geq 0$ is continuous. There are two observable intrinsic types of agents which we call in anticipation of their equilibrium behavior buyers (b) and sellers (s).[8] The intrinsic type of each agent switches indepen-

[8] I will use female pronouns for sellers, and male pronouns for buyers.

dently from b to s with a Poisson intensity y_d, and from s to b with a Poisson intensity y_u. The initial distribution of types is stationary with a mass $\frac{y_u}{y_u+y_d}a$ of buyers and a mass $\frac{y_d}{y_u+y_d}a$ of sellers.

There is a continuum of asset qualities $\theta \in [0, 1]$ each in a unit supply. Agents are risk-neutral and discount the future at the common discount rate r. The flow payoff from asset θ is $k\theta$ for the buyer and $k\theta - \ell$ for the seller where k and ℓ are positive.[9,10] The interpretation is that assets are traded within an asset class defined by the public information. Examples of such classes are mortgage-backed securities rated AAA maturing in 10 years, investment grade zero-coupon bonds with short maturities, or renovated studios in downtown area. The quality θ is an index that aggregates various factors that affect asset payoffs and are not captured by the public information. Thus, k reflects the asset heterogeneity conditional on the public information. This interpretation of the asset quality comes from the type of bargaining delays analyzed in this paper (see the next subsection for more details). Higher asset qualities translate into higher flow payoffs for both types of agents. Sellers experience a transitory liquidity shock, and for them holding the asset is associated with additional *holding costs* $\ell > 0$. Thus, in a frictionless market, buyers would purchase assets from sellers.

Each agent is constrained to hold at most one asset. This way, we abstract from agents' portfolio decisions and focus on their risk-sharing motives. Assets are initially randomly distributed among agents. Since $a > 1$, not all agents own assets.

Agents can trade assets in a market with the search friction. There are two stages to the trading process: the search stage and the bargaining stage. Search is costless, and all unmatched agents participate in search. Searching agents are randomly matched to each other. The matching process is independent of the evolution of intrinsic types and is given by the quadratic matching technology commonly used in the search-and-bargaining literature (see e.g. Duffie et al. (2005)).[11] Buyers of mass m_b contact sellers of mass m_s with intensity $\frac{\lambda}{2}m_b m_s$, and so the total meeting rate of these two groups of agents is $\lambda m_b m_s$. The fact that the match is not instantaneous represents the search friction and the contact intensity parameter λ controls the severity of the search friction.

When a match is found, the agents involved choose whether to participate in the bargaining stage or continue the search. To rule out uninteresting equilibria where the

[9] I can normalize $\ell = 1$, as only the ratio k/ℓ matters. We prefer to keep the separate notation for the purpose of interpretation.

[10] The results on liquidity are shift invariant, so we can add an arbitrary constant $d > 0$ to flow payoffs to guarantee that all the prices and payoffs are positive.

[11] Duffie and Sun (2007) provides probabilistic foundations for this matching technology.

buyer rejects the trade because she anticipates that the seller will also reject the trade, we assume that sellers always choose to participate in the bargaining stage.[12] The buyer can proceed to the bargaining stage with his current stage or return to the search stage by saying "yes" or "no", respectively. We assume that the buyer can condition his strategy directly on the quality of the asset. As mentioned, the screening bargaining solution that we apply is a reduced form for bargaining between agents who get conditionally independent private signals about the quality which are infinitely more precise compared to the public information. Thus, the interpretation is that agents condition their strategies on these almost-perfect signals about the asset's quality.

The (mixed) strategy of the buyer $\sigma(\theta) \in [0, 1]$ specifies the probability with which the buyer matched with the seller of asset θ participates in the bargaining stage. Denote by Θ_L the set of assets such that $\sigma(\theta) = 1$, and by Θ_M the set of assets such that $\sigma(\theta) \in (0, 1)$. We call assets in Θ_L *unconditionally liquid* or simply *liquid*, assets in Θ_M *conditionally liquid*, and assets in $\Theta_I \equiv [0, 1]\backslash(\Theta_L \cup \Theta_M)$ *illiquid.*

Once agents proceed to the bargaining stage, they trade an asset θ with delay $t(\theta)$ at price $p(\theta)$. We assume that once the intrinsic type of one of the matched agents switches or agents complete the trade, the match is destroyed, and agents do not participate in search while matched. We next describe in details how $p(\theta)$ and $t(\theta)$ are determined though the screening bargaining solution.

2.1 Screening Bargaining Solution

Motivated by the Nash bargaining solution and its non-cooperative foundations, the literature on OTC markets commonly assumes that the surplus is split proportionately without delay once the match is found. In this subsection, we introduce an alternative screening bargaining solution applied in this paper. We first define the screening bargaining solution (SBS) for a general class of bargaining problems. In this section, we use the SBS as a reduced form and in the next section we provide microfoundations for it and show that it captures bargaining outcome when the information about the asset quality is almost public.

Consider the following general bargaining problem described by the tuple (ρ, v, c). There is a unit continuum of asset qualities $\theta \in [0, 1]$ and for each θ, the buyer's valuation is $v(\theta)$ and the seller's cost is $c(\theta)$. In equilibrium, v and c will correspond to endogenous buyer's gains from buying the asset and seller's losses from selling the asset, respectively.

[12] In equilibrium, the seller always gets a higher utility from bargaining than from continuing the search.

Assume that v and c are weakly increasing, almost everywhere continuously differentiable, and the trade surplus $\xi(\theta) \equiv v(\theta) - c(\theta)$ is positive for all θ. Time is continuous, and parties discount at rate ρ. If parties trade at time t at price p, then the payoff to the buyer is $e^{-\rho t}(v(\theta) - p)$ and the payoff to the seller is $e^{-\rho t}(p - c(\theta))$.

Before formally defining the SBS, let us first provide an intuitive description of how the SBS works out in terms of a related continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot)$. The seller continuously decreases her price offer p^s_t, and the buyer continuously increases his price offer p^b_t. Both sides take the paths of offers as given, but choose the time when they accept the offer of the opponent strategically, in particular, it is conditioned on the asset quality θ. The trade happens once one of the sides accepts the price offer of the opponent. Initial price offers p^s_0 and p^b_0 can be viewed as the bid and ask prices, respectively. These are the prices at which each side can trade immediately. However, generally agents prefer to wait for a more favorable price offer from the opponent.

The continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot)$ is a realistic description of the actual negotiations in OTC markets where parties start from extreme price offers and gradually moderate their offers until one of the sides accepts.[13] The next section provides the game theoretic foundations for this bargaining game. In particular, it addresses the question why both sides stick to price-offer paths $p^s_\cdot$ and $p^b_\cdot$ and do not condition their counter-offers on the asset quality.

In the pure-strategy Nash equilibrium of this bargaining game, for any asset quality θ corresponds the bargaining outcome consisting of the price $p(\theta)$ and the time $t(\theta)$ of trade. Of course, the outcome would depend on the choice of paths of price offers p^s_t and p^b_t. Let price offers be such that in the equilibrium outcome, the surplus is split proportionally. This uniquely pins down the trade delay. We call this equilibrium outcome the SBS. More formally, the SBS is defined as follows.

Definition 1. *The* screening bargaining solution (SBS) (p, t) *to the bargaining problem* (ρ, v, c) *with the surplus split* $\alpha \in (0, 1)$ *satisfies:*

[13] Lewis (2011)(pp. 212-213) describes the negotiation between Morgan Stanley and Deutsche Bank over the price of subprime CDOs:

> What do you mean seventy? Our model says they are worth ninety-five, said one of the Morgan Stanley people on the phone call.
>
> Our model says they are worth seventy, replied one of the Deutsche Bank people.
>
> Well, our model says they are worth ninety-five, repeated the Morgan Stanley person, and then went on about how the correlation among the thousands of triple-B-rated bonds in his CDOs was very low, ... he didn't want to take a loss, and insisted that his triple-A CDOs were still worth 95 cents on the dollar.

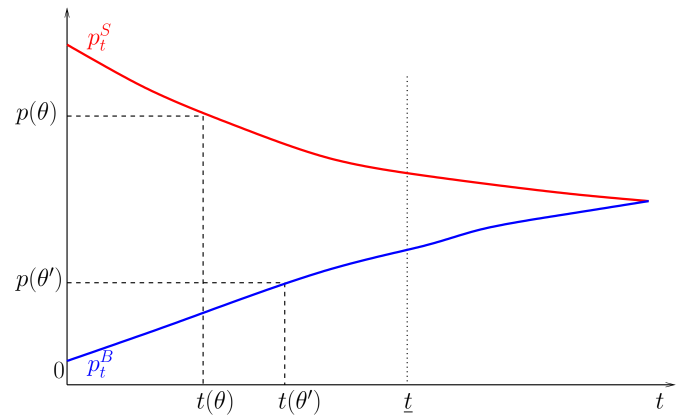

Figure 1: ***Illustration of the SBS.*** *For an asset quality $\theta > \theta^*$, the buyer accepts the seller's offer $p^s_{t(\theta)} = p(\theta)$ at time $t(\theta)$; for an asset quality $\theta' < \theta^*$, the seller accepts the buyer's offer $p^b_{t(\theta')} = p(\theta')$ at time $t(\theta')$.*

1. *for all $\theta \in [0, 1]$,*
$$p(\theta) = (1 - \alpha)v(\theta) + \alpha c(\theta); \tag{2.1}$$

2. $t(1) = t(0) = 0$ *and for some θ^*:*

$$\theta \in \operatorname{argmax}_{\theta' \in [\theta^*, 1]} e^{-\rho t(\theta')}(v(\theta) - p(\theta')), \text{ for } \theta \geq \theta^*, \tag{2.2}$$
$$\theta \in \operatorname{argmax}_{\theta' \in [0, \theta^*]} e^{-\rho t(\theta')}(p(\theta') - c(\theta)), \text{ for } \theta \leq \theta^*. \tag{2.3}$$

Condition (2.1) states that the price splits the surplus between the buyer and the seller in proportion α to $1 - \alpha$. Conditions (2.2) and (2.3) characterize the equilibrium delay in the bargaining game described above. For asset qualities above θ^*, the buyer gives in first and accepts the seller's offer at time $t(\theta)$. Condition (2.2) ensures that for such a buyer accepting at time $t(\theta)$ is indeed optimal. Symmetrically, for asset qualities below θ^*, the seller gives in first and accepts the buyer's offer at time $t(\theta)$ and condition (2.3) ensures optimality of time $t(\theta)$ (see Figure 1). We call this dynamics the *two-sided screening dynamics* motivated by the fact that $(t(\theta), p(\theta))$ is the screening contract for buyers of qualities $\theta > \theta^*$, and it is a screening contract for sellers of qualities below $\theta < \theta^*$.

The present paper applies the SBS as a reduced form for strategic bargaining. In equilibrium, there is the endogenous value of trade for the buyer $v(\theta)$ and the cost of trade for the seller $c(\theta)$ during the bargaining stage (see equations (4.7) and (4.8) below). In the bargaining stage, the match can be exogenously destroyed if the intrinsic type of one of the agents switches, so the *efficient discount factor* is $\rho \equiv r + y_u + y_d$. A tuple

(ρ, v, c) defines bargaining problem to which we apply the SBS to determine the price and bargaining delay in equilibrium. To guarantee that v and c are, indeed, weakly increasing, we assume that the buyer's share of the surplus is sufficiently large:

$$\alpha \geq \frac{y_d}{r + y_d}. \tag{2.4}$$

We additionally restrict attention to equilibria in which functions v and c are weakly increasing on $[0,1]$ and absolutely continuous on $\Theta_L \cup \Theta_M$. If some asset θ is expected to trade with a significant delay, this would lead to a discontinuity in v and c at θ, which in some cases can in turn justify the longer negotiation delay. The continuity requirement on v and c rules out this sort of self-sustained illiquidity.

Lastly, let us stress the assumption that agents do not make any price offers before they proceed to the bargaining stage, and agents agree to start the negotiation or reject and continue the search by simply saying "yes" or "no" to the current match. This assumption rules out conditional offers, e.g. when the buyer threatens to leave if his offer is not accepted or the seller promises to offer a low prime if the buyer agrees to start the negotiation. This assumption can be motivated by the limited commitment of agents before the bargaining stage stage. It allows the separation of the bargaining stage and justifies the application of the screening bargaining solution.

2.2 Equilibrium

Now, I describe the distribution of asset holdings among agents and define the steady-state equilibrium.

Each agent can be either matched (m) or unmatched (u). We refer to the intrinsic type of the agent and his match status as the *type* $\tau \in \{bu, su, bm, sm\}$ of the agent. The *asset position* of the agent $[0,1] \cup \{\phi\}$ is the quality of the asset that the agent owns or bargains over. We use notation ϕ for agents who do not own an asset and are not matched to a seller. The evolution of types and asset holdings is depicted in Figure 2. For example, consider a group of matched sellers, each of whom holds an asset of quality θ. Then the transition from this group could happen according to three possible scenarios. First, the bargaining stage is completed and the asset changes hands (bold arrows from block of matched agents in Figure 2). Second, a seller in this group recovers from liquidity shock and becomes a buyer (arrow indexed by intensity y_u). Finally, the buyer to whom the seller is matched switches intrinsic type and the match is destroyed (arrow indexed

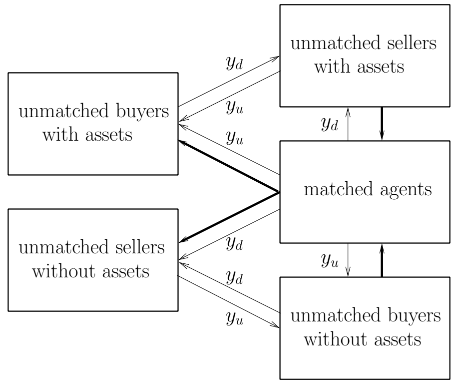

Figure 2: ***The evolution of types and asset holdings.*** *Bold arrows indicate transitions between types and changes in asset holding caused by bargaining, and thin arrows indicate transitions caused by the switching of the intrinsic types (intensities are written next to arrows).*

by intensity y_u).

The economy is in steady state. Denote the steady-state distribution of assets among different types of agents by $M = \{M_\tau \in \Delta([0,1]), \tau \in \{bm, bu, sm, su\}\}$. For example, for any measurable set $\Theta \subseteq [0,1]$, $M_{bu}(\Theta)$ gives the mass of unmatched sellers that own assets in Θ, and $M_{bm}(\Theta)$ gives the mass of matched buyers that bargains over some asset in Θ. We consider equilibria such that there exists the mass density function μ_τ of M_τ.

There are several balance conditions imposed on M. First, for any asset θ, the sum of agent positions is equal to the supply of the asset,

$$\mu_{su}(\theta) + \mu_{bu}(\theta) + \mu_{bm}(\theta) = 1. \tag{2.5}$$

Second, since all assets are in unit supply and the total mass of assets is a, the mass of agents that do not hold any asset is equal to $a - 1$,

$$M_{su}(\phi) + M_{bu}(\phi) + M_{bm}(\Theta_L \cup \Theta_M) = a - 1. \tag{2.6}$$

Third, the number of matched agents of each intrinsic type should coincide with the number of matches,

$$\mu_{sm}(\theta) = \mu_{bm}(\theta). \tag{2.7}$$

Finally, the steady-state assumption requires that there be no changes in the distribution M over time. We analyze the equilibrium of the model in steady state defined as follows.

Definition 2. *A tuple (σ, M) constitutes an* equilibrium *if the buyer's strategy σ is optimal given M, and M is the stead-state distribution of assets generated by σ.*

2.3 Market Thickness and Trade Margins

Before proceeding to the analysis, we introduce measures of market thickness and liquidity.

Let $\Lambda_s \equiv \lambda M_{bu}(\phi)$ be the contact intensity with unmatched buyers without an asset, and $\Lambda_b \equiv \lambda M_{su}(\Theta_L)$ be the contact intensity with sellers of liquid assets. Λ_s and Λ_b capture how easily each side of the market can find a trade partner. Both are measures of market thickness and as we will show below are closely related. By convention, we will only refer to Λ_s as the market thickness. Let $F_L \in \Delta(\Theta_L)$ be the steady-state probability distribution of asset qualities in the pool of unmatched sellers of liquid assets.[14]

We analyze two trade margins. First, the *extensive margin*, captured by $\sigma(\theta)$, reflects whether the asset is actively traded in the market, i.e. always accepted for trade, or can be rejected by some buyers. To capture the extensive margin of the whole market, let $L \equiv |\Theta_L|$ be the mass of assets in that are always accepted by buyers. We refer to L as the *market liquidity*; higher L means that the buyer accepts a broader range of assets for trade.

Second, the *intensive margin* reflects how quickly the asset gets negotiated which is captured by the quantity $x(\theta) \equiv e^{-\rho t(\theta)}$. Then $x(\theta)$ is the factor by which the surplus from trade of the asset θ is dissipated due to the negotiation delay. We refer to $x(\theta)$ as the *liquidity* of asset θ. The empirical counterpart of the liquidity $x(\theta)$ is the trade volume.[15] As we show in the Appendix, the trade volume is given by

$$\frac{\Lambda_s \sigma(\theta) y_d}{y_u + y_d + \Lambda_s \sigma(\theta)} x(\theta)^{\frac{y_u + y_d}{\rho}}, \tag{2.8}$$

which is an increasing function of the liquidity $x(\theta)$. The intensive margin for the whole market is captured by the average liquidity $\overline{x} \equiv \frac{1}{L} \int_{\theta \in \Theta_L} x(\theta) dF_L(\theta)$ and the aggregate liquidity $X \equiv \int_{\theta \in \Theta_L} x(\theta) d\theta$. It follows from (2.8) that when r is small compared to the intensity of shocks and recoveries $y_u + y_d$ and the set of conditionally liquid asset Θ_M is small, the average and the aggregate trading volumes are close to $\overline{x}$ and X, respectively.

[14] Alternatively, one could consider assets in $\Theta_L \cup \Theta_M$ to measure the market liquidity. Here and further, we focus on L, as it allows for clearer comparative statics and simulations indicate that the difference between two measures is often very small.

[15] The trade volume is also equal to the asset turnover, as each asset is in the unit supply.

3 Microfoundation for the Screening Bargaining Solution

This section provides game-theoretic foundations for the screening bargaining solution (SBS) introduced in the previous section. Readers more interested in the implications of the bargaining delays for the OTC liquidity may skip it on first reading.

The (generalized) Nash bargaining solution (NBS) commonly used in the literature is derived from the static axiomatic approach (Nash (1950), Roth (1979), Binmore (1987)). It predicts the proportional split of the surplus (according to the parties' bargaining power), but is silent about the delay required to reach this split. Rubinstein (1982) and Binmore et al. (1986) take the non-cooperative approach (by modelling explicitly the bargaining protocol) to show that the split in the NBS, which we refer to as the *Nash split*, is attained without delay when the information about values *is* public and offers are frequent. In this section, we relate the SBS outcome to the outcome of bargaining when the information about values is *almost* public and offers are frequent, and show that in such a model, the trade delay is necessary to attain the Nash split.

Consider the following discrete-time bargaining game $\mathcal{G}(F, \Delta)$. The seller's type θ^s and the buyer's type θ^b are jointly distributed on $[0,1]^2$ according to the CDF F with strictly positive, continuously differentiable density f. Types are affiliated, i.e. f is log-supermodular. We can think of types as noisy private signals about the underlying asset quality θ. The affiliation of signals captures the correlation of signals with the underlying asset quality: a buyer is more likely to receive a high signal θ^b when the asset quality θ is high and thus, the seller's signal θ^s is likely to be high as well. This signal structure is similar to that used in the global games literature (see, e.g., Morris and Shin (1998)).

Values are private and given by $v(\theta^b)$ for the buyer and $c(\theta^s)$ for the seller. We assume that functions v and c are strictly increasing, continuously differentiable,[16] and $v(0) > c(1)$. The latter is the no-adverse selection condition: the trade is always efficient. We additionally assume that $v(\theta) = c(\theta) + \xi$ which holds for the specification of v and c in the OTC model.[17] This assumption is not necessary for our results, but it simplifies certain steps in the proofs.

[16]In the equilibrium of our OTC model, functions v and c are only guaranteed to be weakly increasing and they may have discountinuous jumps. Such functions can be approximated (e.g., in L^1 norm) by strictly increasing, continuously differentiable functions, and this section provides microfoundations for this case in the sense that it describes the bargaining outcomes for arbitrarily close specifications of v and c.

[17]See (A.28) and (A.29) in Appendix.

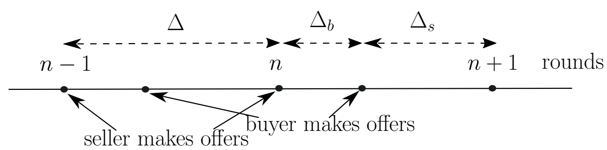

Figure 3: Timing of the discrete-time bargaining game.

Both sides discount time at a constant rate ρ. Bargaining happens in discrete rounds of length Δ. In the beginning of the round, the seller makes a price offer or accepts the last price offer of the buyer. After delay Δ_b, the buyer either accepts the last price offer of the seller or makes a counter-offer. After that, time $\Delta_s = \Delta - \Delta_b$ elapses and the new round starts. Figure 3 illustrates the bargaining protocol. The ratio $\frac{\Delta_b}{\Delta_b+\Delta_s} = \alpha$ captures the bargaining strength of the buyer.[18] The game stops when one of the parties accepts the price offer of the opponent with trading happening at the accepted price. Note that as $\Delta \to 0$, parties are able to make offers and respond almost continuously.

The solution concept is Perfect Bayesian equilibrium (PBE). We focus on PBEs in strategies that have the following simple *interval form*: after any history, the set of types that pool with each other and make the same counter-offer (or accept) is an interval. This requirement is stronger than pure strategies, as it rules out strategies in which two types pool with each other, but separate from some types in between. However, it still allows for rich signaling possibilities.[19,20]

We additionally introduce the following refinement. Call a party *informed* after a history if its posterior beliefs assign probability 1 to a single type of opponent. We require that the support of players' posterior beliefs about the opponent's type cannot expand over time, unless there is an informed party, in which case the beliefs of only the informed party are not allowed to expand. The requirement that the support of beliefs does not expand (the support restriction), is standard in the bargaining literature (see Grossman and Perry (1986), Rubinstein (1982), Bikhchandani (1992)). The existing PBE constructions in bargaining games with one-sided uncertainty and two-sided offers,

[18] This intepretation of α is standard in the bargaining literature (see Osborne and Rubinstein (1990)): the buyer's ability to commit to a longer delay before the counter-offer increases his surplus share in the complete information game. Similarly, Lemma 4 in Appendix shows that in our model, upper and lower bounds on the price of trade are decreasing in α.

[19] In the bargaining with one-sided private information and two-sided offers (e.g. Gul and Sonnenschein (1988), Ausubel and Deneckere (1989), **?**), the cheap-talk messages that are not accepted, but reveal information are normally ruled out by assumption. The restriction to interval strategies allows for such cheap-talk messages.

[20] As we discuss below in footnote 29 under a stronger notion of the correlation of types, the restriction to interval strategies itself does not prevent achieving the Nash split without delay.

however, do not always satisfy this requirement.[21] In order to guarentee the existence, we slightly weaken the support restriction for the case when one party fully revealed its type.[22]

The SBS is intended to capture bargaining when parties make offers almost continuously and the signals about the quality are very precise. We next formalize this idea. The (bargaining) outcome consists of a pair of functions (t, p) where $t(\theta^s, \theta^b)$ and $p(\theta^s, \theta^b)$ are the time and price, resp., at which types θ^s and θ^b trade. A PBE outcome is the outcome generated by PBE strategies. For a fixed distribution of types F, consider a sequence of PBE outcomes $(t_{F,\Delta}, p_{F,\Delta})$ indexed by $\Delta \to 0$, and say that $(t_{F,\Delta}, p_{F,\Delta})$ converges to the continuous-time limit (t_F, p_F) if $(t_{F,\Delta}, p_{F,\Delta}) \xrightarrow{p} (t_F, p_F)$.[23,24]

Let F^* be the uniform distribution on the diagonal $\theta^s = \theta^b$. Under F^*, $\theta = \theta^s = \theta^b$ and the quality of the asset is a public information. Consider a sequence of distributions $F \xrightarrow{p} F^*$ such that for any $\varepsilon > 0$,

$$\sup_{(\theta^s,\theta^b):|\theta^s-\theta^b|>\varepsilon} \max\{f(\theta^b|\theta^s), f(\theta^s|\theta^b)\} < \varepsilon$$

for all F sufficiently far in the sequence.[25] When F is close to F^*, the quality is almost public information in the following sense. For any $\varepsilon_0 > 0$, conditional on θ^s, the seller assings probability greater than $1-\varepsilon_0$ to the buyer's type being within ε_0 of θ^s, she assigns a probability greater than $1-\varepsilon_0$ that the buyer assigns probability greater than $1-\varepsilon_0$ that the seller's type is within ε_0 of θ^s and we can continue these statements up to perhaps very large (for F very close to F^*), but necessarily finite order (hence, the information about the quality is almost public, but not public).

The main result of this section is that the bargaining outcomes are quite different when θ is public information ($F = F^*$), and when it is almost public information ($F \approx F^*$). Let us start with the former. Denote the price of the Nash split by $\mathbf{p}(\theta^s, \theta^b) = (1-\alpha)v(\theta^b) + \alpha c(\theta^s)$. The following proposition due to Binmore et al. (1986) states that in this case trade happens without delay.

[21] See e.g. Grossman and Perry (1986).

[22] The first part of Theorem 1 below holds without this modification of the support restriction, and we only use it in the proof of the second part (see footnote 38 in the proof of Lemma 12).

[23] Here and further, $\xrightarrow{p}$ denotes convergence in probability, e.g. $(t_{F,\Delta}, p_{F,\Delta}) \xrightarrow{p} (t_F, p_F)$ as $\Delta \to 0$ if for all $\varepsilon > 0$, $\lim_{\Delta\to 0} \mathbb{P}_F \left(|t_{F,\Delta} - t_F| < \varepsilon \text{ and } |p_{F,\Delta} - p_F| < \varepsilon\right) = 1$.

[24] There are known technical issues in definining games in continuous time (see Simon and Stinchcombe (1989)). For this reason, it is standard in the bargaining literature to take a limit $\Delta \to 0$ in the discrete-time game to obtain predictions that do not depend on the protocol of bargaining (e.g. the order of offers).

[25] See Online Appendix for an example of such a sequence.

Proposition 1 (Binmore et al. (1986)). *(t_{F^*}, p_{F^*}) does not depend on the sequence $(t_{F^*,\Delta}, p_{F^*,\Delta})$ and $(t_{F^*}, p_{F^*}) = (0, \mathbf{p})$.*

Consider now the case of almost public information about the asset quality. Denote by $\mathbf{t}(\theta^s)$ the delay associated with quality θ^s in the SBS as given by (2.2) – (2.3).

Theorem 1. *1. Consider a sequence of PBE continuous-time limits (t_F, p_F) indexed by $F \xrightarrow{p} F^*$. If $p_F \xrightarrow{p} \mathbf{p}$ as $F \xrightarrow{p} F^*$, then there exist $0 < x_l < x_h < 1$ and $0 < \underline{\theta}^s < \overline{\theta}^s < 1$ such that*

$$x_l > \limsup_{F \xrightarrow{p} F^*} \mathbb{E}_F[e^{-\rho t_F}], \tag{3.1}$$

$$x_h > \liminf_{F \xrightarrow{p} F^*} \mathbb{E}_F[e^{-\rho t_F} | \theta^s < \underline{\theta}^s \text{ or } \theta^s > \overline{\theta}^s]. \tag{3.2}$$

2. There exists a sequence of PBE continuous-time limits (t_F, p_F) such that $(t_F, p_F) \xrightarrow{p} (\mathbf{t}, \mathbf{p})$ as $F \xrightarrow{p} F^*$.[26]

Theorem 1 shows that the bargaining outcome when information the asset quality is not public differs drastically from the case when the quality is a pubic information, and importantly, this difference does not vanish as players's signals become very precise (F arbitrarily close to F^*). First, the bargaining delay is necessary to attain the Nash split of the surplus (inequality (3.1)). Second, the bargaining delay is generally non-monotone: it is lower for qualities closer to extremes of the distribution (0 and 1) and higher in the middle (inequality (3.2)). Finally, the SBS can be approximated by the PBE outcomes as offers become frequent and the information about quality becomes almost public. As we will show in the next section, the SBS outcome captures positive and non-monotonic bargaining delays (U-shaped liquidity) that we obtain in Theorem 1.

Let us provide the intuition for these results. First, why the delay is necessary to attain the Nash split? Although the formal proof is quite involved, the underlying idea is simple. Suppose to contradiction, for any Δ and F arbitrarily close to 0 and F^*, resp., there were a PBE in which trade happens with high probability without a significant delay. Since sufficiently different asset qualities should be traded at sufficiently different prices in order to match the Nash split, it is necessary that at least one of the sides reveals quite precisely its signal. Then one side, say the buyer, can relatively quickly convince the seller that its value is relatively low. But this implies that high types of the buyer

[26]That is, for all $\varepsilon > 0$, $\lim_{F \xrightarrow{p} F^*} \mathbb{P}_F\left(|t_F - \mathbf{t}| < \varepsilon \text{ and } |p_F - \mathbf{p}| < \varepsilon\right) = 1$.

can mimic lower types and get a more favorable price by only slightly delaying the trade, which is a contradiction to the sequential rationality.[27]

The non-monotonicity of the bargaining delay is also quite intuitive. We show that the buyer has the option to trade immediately at price close to $\mathbf{p}(1)$ which is the complete-information price of trade when both players' types equal 1.[28] Since $p_F \xrightarrow{p} \mathbf{p}$, the buyer's types close to 1 expect to trade with a high probability at a price close to $\mathbf{p}(1)$, thus, for them the expected bargaining delay cannot be too long. Symmetric argument shows that for the seller's types close to 0, the expected bargaining delay is relatively show, as they have the option to trade immediately at a price close to $\mathbf{p}(0)$ (which is the complete-information price of trade when both players' types equal 0). Therefore, types close to the extremes of the range are guaranteed to trade relatively quickly which gives us inequality (3.2).

Now, let us turn to why the SBS can be approximated by the PBEs for $F \approx F^*$, but not when $F = F^*$? When the information about the quality is public, there is a unique split of the surplus sustainable in any continuation equilibrium and so, it is not possible to reward or punish players to sustain the delay. This is, however, possible when the information is noisy. We construct PBEs approximating the SBS in the grim trigger strategies. In particular, we specify that if e.g. the seller deviates from the equilibrium path, then the buyer infers that the seller's signal is very low and the seller is very desperate to trade (formally, the buyer beliefs that the seller's type is 0). After such an optimistic updating, in the continuation equilibrium the buyer almost immediately gets the maximal share of the surplus. By specifying such a punishment path, we can sustain the equilibrium path that involves delay. Despite the fact there is an efficiency loss due to the bargaining delay on the equilibrium path and both parties assign a high probability to it, nobody wants to seem desperate and deviate from the equilibrium path.

Theorem 1 highlight the crude public information, rather than parties' private information, as the source of the bargaining delay. The assumption of the coarse public information about the asset quality is relevant in many OTC markets. Credit ratings for financial assets put only crude bounds on the risks associated with the asset, and experienced traders rely on their private information sources to further refine these bounds. Likewise, in the real estate, an experienced realtor goes beyond the public profile of the

[27]Here, the assumption that the support cannot expand is crucial: once the buyer signals that his value is relatively low, this gives him a guarantee of relatively low price in any continuation play. Lemma 4 in Appendix shows how the bounds on the price of trade depend on the support of types remaining in the game.

[28]SeeLemma 4 in Appendix.

house and assesses various characteristics of the neighborhood, such as safety and demographics, to determine more precisely its value. In our model, the precision of the public information is captured by the slope of functions v and c (or referring to the primitives of the OTC model by the parameter k): the more homogenous the assets, the smaller the differences in the prices at which assets trade, and in turn, the smaller bargaining delay is required to attain the Nash split.

Finally, let us motivate our focus on the PBEs of the bargaining game approximating the SBS. First, the multiplicity of equilibria is the major concern in the literature on bargaining with two-sided private information about values and two-sided offers. In particular, in our model, along with PBEs described in Theorem 1, there is a continuum of PBEs in which trade happens immediately at some price in $(c(1), v(0))$. We choose to focus on PBEs that approximate the Nash split. On the one hand, this allows us to better contrast our results to that in the OTC literature which applies the Nash bargaining solution. On the other hand, the original paper by Nash (1950) gives axiomatic foundations for the Nash split of the surplus. Thus, it is reasonable to assume that such a split would be a natural focal point for equilibrium split when players have very precise information about the quality.

Second, Tsoy (2014) shows that the SBS outcome is robust to the assumptions about the distribution of types. Tsoy (2014) considers F with positive mass only on a band around the diagonal $\theta^s = \theta^b$ and approximate F^* by making this band very narrow. This correlation structure is much stronger that the one used in this paper in that players know, rather than assign very high probability, that their signals are close to each other. Under this stronger correlation structure, Tsoy (2014) constructs PBEs approximating the SBS.[29]

Third, technically, the sharp constast between the outcomes with public and almost public information about the asset quality stems from the order of limits. Rubinstein (1982)'s analysis first assumes that the idealized complete-information model $\mathcal{G}(F^*, \Delta)$ is a good approximation for $\mathcal{G}(F, \Delta)$, and then makes offers frequent. Our analysis first makes offers frequent in $\mathcal{G}(F, \Delta)$, and then takes limit $F \xrightarrow{p} F^*$. We beliefve that our approach better captures the negotiation in OTC markets where there are virtually no restrictions on the protocol of bargaining, while the private information of parties is

[29] Under this stronger correlation structure and dispensing with the assumption that the support of beliefs does not expand, Tsoy (2014) also contructs PBEs that attain the Nash split without delay. On the one hand, Theorem 1 shows that such an outcome is not robust to the model of the correlation used in this paper. On the other hand, the construction of the efficient outcome in Tsoy (2014) also uses interval strategies suggesting that such strategies are not very restrictive.

a relevant feature. Relatedly, the proof of Theorem 1 suggests that the limiting case ($F \to F^*$) sheds light also on the bargaining dynamics when the noise in the signals is not small. We consider the continuous-time bargaining game introduced in the previous section for any distribution F (not only for F^* as in the SBS) and show that the trade dynamics similar to that in the SBS arises in BNEs of this game.

4 Equilibrium Analysis

This section characterizes the unique equilibrium (Theorem 2). First, Lemma 1 derives the steady-state distribution M of assets among different agents. Next, Lemma 2 shows that the buyer's optimal strategy σ has a simple threshold form. Finally, Lemma 3 uses the SBS to pin down the liquidity profile x.

4.1 Steady-State Distribution

In this subsection, we show that for a given strategy profile σ and liquidity profile x, there exists a unique steady-state distribution M, and we describe its properties. The following lemma describes $\Lambda_s, \Lambda_b, F_L$ in the unique steady-state distribution corresponding to a given strategy profile and specification of delay.

Lemma 1. *For any σ, x, L and Λ_s, there exists a unique steady-state distribution M in which Λ_s is the unique solution to*

$$\frac{\Lambda_s}{\lambda} = \frac{y_u}{y_u + y_d}(a-1) - \frac{y_d}{y_u + y_d}\int_0^1 \frac{\Lambda_s \sigma(\theta)}{y_u + y_d + \Lambda_s \sigma(\theta)} d\theta, \tag{4.1}$$

Λ_b is given by

$$\Lambda_b = \frac{\lambda y_d L}{y_u + y_d + \Lambda_s}, \tag{4.2}$$

and F_L is uniform conditional on $\theta \in \Theta_L$.

Lemma 1 shows that Λ_s, Λ_b, and F_L depend only on σ, but not on $t(\cdot)$.[30] The result that the liquidity characteristic $t(\cdot)$ does not affect the distribution of assets F_L may be counter-intuitive at first sight, as one may expect that more liquid assets are traded more quickly and so are more abundant in the market. To see why, observe that the inflow into the group of sellers of asset θ is formed from matched sellers whose counter-party is hit

[30] Unlike Λ,Λ_b,F_L, steady-state distribution M derived explicitly in the Appendix depends on the delay profile $t(\cdot)$.

by a liquidity shock and from unmatched buyers owning asset θ who are hit by a liquidity shock (see Figure 2). Both these inflows have intensity y_d. At the same time, the outflow from this group of sellers happens because of the recovery from the shock of sellers and the formation of new matches. The former has intensity y_u and the latter has intensity Λ_s, and both are again independent of $t(\cdot)$. Therefore, $t(\cdot)$ only changes the distribution of agents between those who have already completed a trade and those still bargaining but does not affect the mass of sellers in the search stage.

Equation (4.1) has a natural interpretation. The left-hand side gives the mass of buyers without an asset, which in the absence of trade, equals $\frac{y_u}{y_u+y_d}(a-1)$. When agents are allowed to trade the mass of buyers without an asset decreases, which reflects the fact that ownership of assets becomes more efficient.

An interesting feature that follows from equation (4.1) is that if buyers accept a greater variety of assets this reduces the chances of the seller to be matched. In particular, if σ weakly increases for all θ, then it follows from equations (4.1) and (4.2) that Λ_s decreases and Λ_b increases. The more assets buyers accept, the more likely it is for the buyer to find a match, however, this implies more competition for sellers and for them the likelihood of forming a match decreases. Notice that this happens despite the fact that there are no search externalities, i.e. the fact that additional sellers are searching for buyers does not reduce the chances of others to be matched. The competition between sellers arises, however, for the following reason: the fact that buyers accept a wider variety of assets implies that more buyers find matches. These buyers are either busy in the bargaining stage or have already completed their trades. This reduces the number of buyers searching in the market and reduces the likelihood of a match for unmatched sellers.

4.2 Optimal Search Strategy

Now, given a steady-state distribution M and delay profile $t(\cdot)$, we compute the optimal strategy σ. For $\tau \in \{bu, su, bm, sm\}$, let $V_\tau(\theta)$ be the expected utility of an agent of type τ owning (or bargaining over) asset θ, and for $\tau \in \{bu, su\}$, let $V_\tau(\phi)$ be the expected utility of an agent of type τ owning no asset. Value functions during the search stage are determined by the following Bellman equations,

$$rV_{su}(\phi) = y_u(V_{bu}(\phi) - V_{su}(\phi)), \tag{4.3}$$

$$rV_{bu}(\theta) = \overline{v}(\theta) + y_d(V_{su}(\theta) - V_{bu}(\theta)), \tag{4.4}$$

$$rV_{bu}(\phi) = y_d(V_{su}(\phi) - V_{bu}(\phi)) + \Lambda_b\left(\mathbb{E}\left[V_{bm}(\theta)|\theta \in \Theta_L\right] - V_{bu}(\phi)\right) \tag{4.5}$$

$$rV_{su}(\theta) = \underline{v}(\theta) + y_u(V_{bu}(\theta) - V_{su}(\theta)) + \sigma(\theta)\Lambda_s\left(V_{sm}(\theta) - V_{su}(\theta)\right). \tag{4.6}$$

The depreciation of value functions in the left-hand side of equations (4.3) – (4.6) equals the sum of flow payoffs and changes in value functions due either to switches of intrinsic types or the formation of matches. For example, consider equation (4.5). The flow payoff of the searching buyer without an asset is zero. If the buyer is hit by a liquidity shock, his value function drops to $V_{su}(\phi)$, while if he is matched to a seller, then his value function increases to $\mathbb{E}\left[V_{bm}(\theta)|\theta \in \Theta_L\right]$. Notice that if a buyer is matched to a seller of an asset in Θ_M, then his continuation utility is $V_{bu}(\phi)$ irrespective of whether he starts to negotiate or continues to search. Therefore, in equation (4.5), it is sufficient to consider the case when the buyer is matched to sellers of assets Θ_L and the relevant distribution is F_L. This is however not the case in equation (4.6) which described the value function of the seller of asset quality θ. In equilibrium, such a seller strictly prefers to start the negotiation, and hence, for her the probability with which the asset that she offers for trade is accepted is important.

To determine the price of trade, we compute the benefits $v(\theta)$ from trade for the buyer of asset θ, and the costs of trade $c(\theta)$ for the seller of asset θ. Let $\hat{c}(\theta)$ be the value for the seller of asset θ from staying in the match but never selling the asset until the recovery from the liquidity shock happens, and $\hat{v}$ be the value for the buyer from staying in the match but not buying from the current seller. Then

$$c(\theta) = -(V_{su}(\phi) - \hat{c}(\theta)), \tag{4.7}$$

$$v(\theta) = V_{bu}(\theta) - \hat{v}. \tag{4.8}$$

In the Appendix we show that the trade surplus is constant and equal $\xi \equiv \frac{\ell}{\rho}$. This follows from the fact that holding costs do not depend on the asset quality translates. By the assumption of the proportional split of the surplus, the price of trade is given by (2.1). Given the Bellman equations (4.3) – (4.6), the price of trade (2.1) and the liquidity profile x, one can find value functions and determine optimal strategies. The following lemma states that the equilibrium strategy takes the simple threshold form.

Lemma 2. *The asset of quality θ is always accepted by buyers ($\theta \in \Theta_L$) whenever*

$$x(\theta) > \underline{x} \equiv \frac{\Lambda_b}{\rho + \Lambda_b}\overline{x}, \tag{4.9}$$

and is always rejected whenever the inequality in (4.9) is reversed. Moreover,

$$V_{bu}(\phi) = \alpha \frac{r + y_u}{r} \xi \underline{x}. \tag{4.10}$$

.

From Lemma 2, buyers search for sufficiently liquid assets in the market. In fact, if not all assets are accepted for trade in equilibrium, there is a non-trivial search process occurring in equilibrium. The buyer may reject several assets before he finds a sufficiently liquid asset for which he proceeds to the bargaining phase. The threshold of the buyer depends on the average liquidity and the ability to find liquid assets in the market. If the search is fast (Λ_b is large), then the buyer's threshold is close to the average liquidity, i.e. the outside option of the buyer is essentially to go back to the market and get a random draw from the pool Θ_L. If the search is slow (Λ_b is small), then the buyer accepts a wide range of assets, as finding another asset entails a significant delay.

4.3 Liquidity Profile

The last step in characterizing the equilibrium is to show how profile x is determined.

Lemma 3. *Either $\Theta_L = [0,1]$ or there exist $0 < \check{\theta} < \underline{\theta} \leq \theta^* \leq \hat{\theta} < 1$ such that $\Theta_L = [0, \check{\theta}] \cup [\hat{\theta}, 1]$ and $\Theta_M = (\check{\theta}, \underline{\theta})$. Moreover,*

$$x(\theta) = \begin{cases} 1 - \dfrac{v(1) - v(\theta)}{\alpha\xi}, & \text{for all } \theta > \hat{\theta}, \\ 1 - \dfrac{c(\theta) - c(0)}{(1-\alpha)\xi}, & \text{for all } \theta \leq \check{\theta}. \end{cases} \tag{4.11}$$

4.4 Equilibrium

The next theorem combines equilibrium conditions for M, σ, x derived in the previous subsections to characterize the unique equilibrium.

Theorem 2. *There exists a unique equilibrium characterized by* (Λ_s, L) *solving:*

$$\Lambda_s \geq \frac{\lambda y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L}-1\right)-L\right)-y_u-y_d, \text{ with equality iff } L<1, \tag{4.12}$$

$$L = \frac{y_u+y_d+\Lambda_s}{y_d\Lambda_s}\left(y_u(a-1)-(y_u+y_d)\frac{\Lambda_s}{\lambda}-h(L,\Lambda_s)\right); \tag{4.13}$$

$$\text{where } h(L,\Lambda_s) \equiv \int_0^{\min\left\{1,\frac{\rho+\Lambda_s}{\Lambda_s}\frac{(1-L)k}{(1-\alpha)y_d\xi}e^{\frac{k}{r\xi}L}\right\}} \frac{(1-s)y_d}{1+\frac{y_u+y_d}{\Lambda_s}-\left(1-\frac{y_u+y_d}{\rho}\right)s}ds.$$

The equilibrium is characterized by the market liquidity L and the market thickness Λ_s. Let me sketch the solution of the model. Lemma 1 shows that the distribution M is pinned down by the market thickness parameters Λ_s and L as well as strategy σ and liquidity profile x. We then use Lemma 2 and 3 as well as expressions for value functions to derive σ and x for given Λ_s and L. Therefore, the equilibrium is characterized by Λ_s and L that satisfy two requirements.

Equation (4.12) reflects the optimality of the buyer's strategy and it produces an increasing relationship between Λ_s and L. This captures the fact that when it is easier for the seller to find trade partner, it is harder to find a trade partner for the buyer conditional on the same range of actively traded assets (see equation (4.2)) and so, it is optimal for the buyer to extend the range of acceptable assets. Equation (4.13) reflects the steady-state requirement that produces a decreasing relationship between Λ_s and L. It states that when buyers accept more assets for trade, fewer buyers are searching in the market, as more of them have already traded or are in the process of negotiation. Thus, the market thickness Λ_s decreases and it is harder for the sellers to find a trade partner. The equilibrium determination is depicted in Figure 4.

5 Main Results

This section applies the equilibrium characterization in Theorem 2 to derive asset pricing and liquidity implications. The search and bargaining frictions, although similar on the intensive margin, differ on the extensive margin. This implies that transparency policies have an ambiguous effect on the liquidity of decentralized markets. Further, because of the extensive margin, despite the short observable trade delays, search and bargaining frictions can have an important effect on the asset and market liquidity.

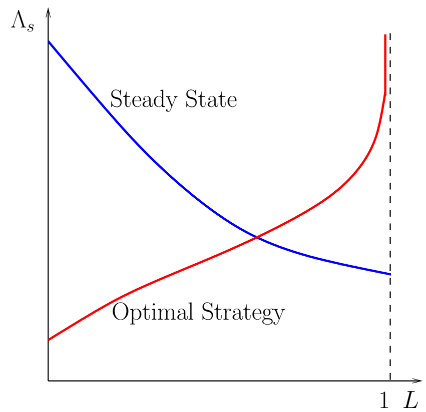

Figure 4: Equilibrium is determined as the intersection of the increasing curve given by equation (4.12) *reflecting the optimality of strategy* σ *and the decreasing curve given by equation* (4.13) *reflecting the stead-state distribution of assets in the economy.*

Asset Prices The next proposition gives the asset price decomposition into three components: the fundamental value, the liquidity premium and the average liquidity components.

Proposition 2. *Prices of assets are given by*

$$p(\theta) = \underbrace{\frac{1}{r}\left(k\theta - (r + y_d)\xi\right) + (1-\alpha)\xi}_{\text{fundamental value}} + \underbrace{(1-\alpha)\frac{y_d}{r}\frac{\sigma(\theta)\Lambda_s}{\rho + \sigma(\theta)\Lambda_s}\xi x(\theta)}_{\text{liquidity premium}} - \underbrace{\alpha\frac{y_u}{r}\frac{\Lambda_b}{\rho + \Lambda_b}\xi\overline{x}}_{\text{average liqudity}} . \tag{5.1}$$

Moreover, the bid-ask spread $p(1) - p(0) = \dfrac{k}{r}$.

The first component is the price if there were no opportunity to search for another asset in the market. This price captures the value of holding the asset for the seller plus his fraction of the trade surplus. The other two components reflect how the outside options created by the search market affect prices. The second component of the price depends on $x(\theta)$ and reflects the liquidity premium. The more liquid the asset is, the higher the price the buyer is willing to pay. This effect is driven by the outside option of the seller to search in the market for another buyer. For a more liquid asset, after the new match is formed, less surplus is dissipated due to delay, which increases the outside option of the seller and hence increases the price of asset. Observe that this outside option depends on the ability of the seller to find a buyer (Λ_s). The more unmatched buyers in the market, the more valuable the outside option of the seller and the higher the price sensitivity to the asset's liquidity. The fact that the sensitivity of the price to liquidity depends on aggregate market conditions was documented empirically (see Bao et al. (2011), Friewald et al. (2012)). The third component is the effect of average (across assets in Θ_L) liquidity in the market. This component accounts for the buyer's outside option of finding another seller. Naturally, the outside option of the buyer is increasing in the average liquidity and pushes the price down. Therefore, the third component has a negative sign. This effect is larger the easier it is for the buyer to find a trade partner (higher Λ_b).

Market tightness also affects prices although not directly but through their sensitivity to asset liquidity and average asset liquidity. When the mass of searching buyers in the market is higher, it is easier for a seller to find a counter-party. Hence, the gains for the seller from holding a more liquid asset are higher, which translates into the higher sensitivity of the asset price to asset liquidity and leads to an increase in asset prices. On the contrary, when the mass of searching sellers in the market is higher, the buyer can more easily find a seller in the market. Hence, the gains for the buyer from an increase in average asset liquidity are higher, which translates into the higher sensitivity of the asset price to average asset liquidity and a dampening of prices.

When $x(\theta) = \overline{x}$ for all θ, my model reduces to Duffie et al. (2007) which already allows us to distinguish between the default and non-default components of asset prices. Equation (5.1) further separates the liquidity premium component which varies in the cross-section of assets, and the average-liquidity component which will be shown in Section 6 to depend on the liquidity of other asset classes. Existing empirical evidence on the behavior corporate spreads suggests that the decomposition in the pricing equation (5.1) captures key components of asset prices in OTC markets. Longstaff et al. (2005) shows that the default component does not explain entirely corporate spreads. The non-default component varies with liquidity measures in the cross-section of assets and depends on the marketwide liquidity in the time series analysis. While Longstaff et al. (2005) does not provide a direct test of my theory, my model can be useful in explaining these effects on corporate spreads. The last two components in equation (5.1) correspond to the non-default component. While the liquidity premium component ensures the variation of the non-default component across assets, the average-liquidity component ensures the variation of the non-default component with respect to the marketwide liquidity.

Finally, notice that the bid-ask spread depends on the asset heterogeneity. When agents vary significantly in flow payoffs, the variation in prices is higher and in the SBS, the negotiation starts from offers that are farther apart.

U-shaped Liquidity Lemma 3 implies that the liquidity is U-shaped in quality. This follows from the dynamics behind the SBS. For the lowest qualities, the seller's value of the asset as low and so, she prefers to accept a larger discount earlier rather than wait longer for more favorable offers. Symmetrically, the buyer of the highest qualities has the highest value and so, he prefers to accept a high price early on rather than wait for the lower offers from the seller. It is qualities in the middle that trade with the longest delay and hence, are less liquid. In fact, Lemma 3 implies that they may not be traded at all. Since the buyer is looking for relatively liquid assets in the market, he may prefer to continue the search rather than starting the lengthy negotiation over the price of the asset quality in the middle of the quality range.

The U-shaped prediction is the novel empirical implication of the paper and it differs from the implication of Guerrieri and Shimer (2014).[31] They study the model with asymmetric information and discover an increasing relationship between the liquidity and default risk. This stems from the fact that in order to incentivize owners of assets to reveal their private information, assets of higher quality should be traded at higher prices but with lower probability compared to the lower-quality assets. The existing empirical literature gives a contradicting evidence about the relation between the asset quality and liquidity. Longstaff et al. (2005), Ericsson and Renault (2006) document a positive correlation between illiquidity and default risk for corporate bonds. Beber et al. (2009) shows that for Euro-bonds the correlation is reversed: more risky sovereign debt is also more liquid. The model in this paper reconciles this evidence within a single framework. While in general the dependence in my model is U-shaped, the shape can be significantly skewed to either side depending on the specification of the payoff function[32] and the split of the surplus α (see Online Appendix for the comparative statics with respect to α).

Search and Bargaining Frictions Recall that the search friction decreases with the contact intensity λ, and the bargaining friction increases with the asset heterogeneity k. The next proposition shows how the market liquidity and average delay react to both types of frictions.

Proposition 3. *Suppose the equilibrium before and after the change of parameters has $\Theta_I \neq \phi$. The following comparative statics hold:*

[31]The companion paper (Tsoy, 2015) derives a similar prediction in the bargaining model where v and c are exogenous. This paper in contrast shows that the U-shaped liquidity pattern is present even when v and c are endogenous, and in particular, they both determine and depend on the negotiation delays.

[32]See non-linear specifications in Section 7.

1. *An increase in k leads to a decrease in $L, \underline{x}, \overline{x}$, and an increase in Λ_s.*

2. *An increase in λ leads to a decrease in L and an increase in $\underline{x}$ and $\overline{x}$.*

To see the effect of the bargaining friction, let us first consider the case when k is close to 0. Then the differences in flow payoffs across assets are very small, and thus, various asset qualities are traded at similar prices. Therefore, there are little incentives to delay trade for a slightly more favorable offer, and negotiations are short. An increase in the asset heterogeneity k leads to the differences in prices at which assets are traded to increase. As a result, the negotiation starts from the offers that are farther apart and agents have higher incentives to delay trade and wait for a more favorable price offer. The increase in the bargaining delays makes fewer assets attractive for trade to buyers and so, the market liquidity L drops. However, if buyers reject too many assets the search delays increase. Hence, buyers are also willing to tolerate longer negotiation delays, i.e. $\underline{x}$ decreases. As a result, the average liquidity $\overline{x}$ decreases. Notice that the increase in bargaining friction increases the market thickness Λ_s. This is the effect of competition among sellers for buyers. When bargaining friction is greater, fewer assets are actively traded. Therefore, a larger fraction of unmatched buyers searches for more scarce trade opportunities, which improves the match intensity for sellers.

On the contrary, an increase in the search friction leads to a higher market liquidity. When it is harder for buyers to find another sellers, the outside option of searching further in the market depreciates. As a result, buyers are willing to accept a wider range of asset qualities for trade. Recall from Lemma 3, that the buyer's utility is proportional to $\underline{x}$. Proposition 3 shows that both frictions reduce the utility of buyers, while they have opposite effects on the market liquidity.

In the literature, the search friction is thought of as a reduced form for all types of frictions leading to trade delay. My analysis partially justifies this approach. On the intensive margin, the bargaining friction is similar to the search friction. An increase in the bargaining friction leads to higher average negotiation delays. However, on the extensive margin the two frictions operate quite differently. Therefore, only one type of friction cannot serve as a proxy for the other.

Vanishing Frictions In my model, we have both search and bargaining frictions and the interaction of the two produces interesting predictions for trade margins. One can see from the system characterizing the equilibrium in Theorem 2 that when the bargaining friction vanishes ($k \to 0$), the model reduces to that of Duffie et al. (2007). A natural question is whether the search friction is essential for negotiation delays. The next proposition shows that even when the search friction vanishes ($\lambda \to \infty$), in the limit there are non-trivial negotiation delays and some assets can be not traded. This contrasts with Duffie et al. (2007) where in the limit of vanishing search frictions the equilibrium is efficient.

Proposition 4. *The limit of equilibria as $\lambda \to \infty$ is characterized by* (Λ_b^*, L^*) solving:

$$L^* = \frac{\Lambda_b^*}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L^*} - 1\right) - L^*\right), \text{ with equality iff } L < 1,$$

$$L^* = \frac{y_u}{y_d}(a-1) - (y_u + y_d)\frac{L^*}{\Lambda_b^*} - \int_0^{\min\left\{1, \frac{k(1-L^*)}{(1-\alpha)y_d\xi}e^{\frac{k}{r\xi}L^*}\right\}} \frac{1-s}{1-s+\frac{y_u+y_d}{\rho}s}ds.$$

Moreover, $\Lambda_s \to \infty$, while $\frac{\Lambda_s}{\lambda} \to M_{bu}^*(\phi) \in (0, \infty)$.

While Proposition 4 follows simply from taking the limit of (4.12) and (4.13), it is not immediate why buyers do not search for the most liquid (with $x(\theta) = 1$) asset given that search delays are virtually zero. Proposition 4 shows that in the limit there is a shortage of sellers in the market[33]. When holders of liquid assets are hit by a liquidity shock, they almost immediately find a trade partner and start bargaining. Because of the shortage of sellers, it takes buyers some time to find an appropriate asset for trade (Λ_b is finite). Thus, the buyer accepts the range of asset qualities.

Transparency and News The analysis of two trade frictions adds to the debate about the effect of transparency on the liquidity of OTC markets. There is a tendency toward increasing transparency of OTC markets. In July 2002, the Transaction Reporting and Compliance Engine (TRACE) was introduced in the U.S. corporate bond market that required the public reporting of nearly all transactions with minimal delays. Recent financial crises increased the pressure for a greater transparency in other markets, such as credit derivatives and credit-default swaps. My analysis shows that the transparency has a bright and dark side.

On the one hand, such policies as more accurate and frequently updated credit ratings, introduction of benchmarks, and dissemination of past quotes, improve the quality of public information. As a result, conditional on better public information, the asset heterogeneity is reduced which decreases the bargaining friction and thus increases the

[33] Indeed, it follows from $\Lambda_b \to \Lambda_b^*$ that $M_{su}(\Theta_L) = \frac{\Lambda_b}{\lambda} \to 0$.

market liquidity. Bessembinder et al. (2006), Edwards et al. (2007) provide empirical evidence that the introduction of TRACE improved corporate bonds liquidity and led to a decrease in transaction costs.

On the other hand, such policies as better trading platform that allow for a more efficient search and greater post-trade transparency reduce the search friction and thus lead to lower market liquidity. This of course does not mean that such form of transparency is bad for welfare. On the contrary, the analysis of the welfare in Proposition 3 and numerical example show that the reduction in the search friction leads to a more efficient risk sharing. However, because of the reduction of the market liquidity, this is not a Pareto improvement and owners of assets that become illiquid are worse off from such policies.

The analysis of bargaining friction reveals that the shocks that affect the quality of public information decrease the market liquidity. In the recent financial crisis, because of the bad news from the housing market, credit ratings became less informative about the quality of mortgage-backed securities, collateralized debt obligations, and other assets.[34] Downgrades of structured products coincided with dried-up liquidity of structured finance products (see Brunnermeier (2008)). This is consistent with predictions of this paper: drop in the quality of ratings lead to an increase in the bargaining friction which in turn results in the drop in the market liquidity.

Short Delays and Selling Pressure A common criticism of the search and bargaining models is that in many OTC markets, such as the corporate bonds market, the perception is that trade delays are not significant, and hence, it may be questionable how important they are for prices and liquidity. The analysis of both trade margins shows that even small trade delays can significantly impact asset prices and liquidity. The effect comes through the extensive margin. When the search delays are short, by Proposition 3, few assets are actively traded, while the average bargaining delays are also short. Thus, short *observed* search and bargaining delays do not mean that assets can be quickly sold.

Another question is whether it is possible that a relatively wide range of assets is rejected by buyers. My interpretation is that such assets are more sensitive to market conditions. That is, in normal times all assets are actively traded (although with varying delay), however, in times of lower liquidity, the liquidity of certain assets in the middle is impaired. As a result, on a long horizon all assets are traded at least once, but the trading

[34] Benmelech and Dlugosz (2010) reports that for structured finance products, both the amount and the size of downgrades increased significantly during 2006-2008. Ashcraft, Goldsmith-Pinkham, and Vickery (2010) documents a spike in downgrades for subprime and Alt-A mortgage-backed securities.

activity varies because of varying sensitivity to market condition. The next proposition shows that the market liquidity drops during the times of selling pressure. We capture the selling pressure via a simultaneous offsetting increase in y_d and decrease in y_u, so that the long-run ratio of sellers in the population increases.

Proposition 5. *Suppose that y_d increases and y_u decreases so that $y_u + y_d$ stays constant. Then L decreases.*

Trading Activity In the next section, we study how the migration of agents between asset classes causes the market liquidity and trading activity to change. As a preliminary observation, the next proposition derives the comparative statics with respect to the number of agents a.

Proposition 6. *An increase in a leads to an increase in L and Λ_s and a decrease in $\underline{x}$.*

Expectedly, when more agents participate in trading, the market liquidity L increases. Interestingly, the buyer's strategy threshold and the buyers' utility is decreasing in a. When there are more agents, there are both more buyers and more sellers in the market. Thus, the competition among buyers for unmatched sellers of liquid assets increases, which in turn forces buyers to accept a wider range of assets. As a result, buyers' utility goes down together with their strategy threshold $\underline{x}$, and a wider range of assets is actively traded.

6 Transparency and Flights-to-Liquidity

This section shows that the substitutability between asset classes leads to flights-to-liquidity during periods of market uncertainty and adverse liquidity effects of the gradual transparency policies.

I first extend the baseline model to two asset classes. There are two asset classes indexed by $i = 1, 2$, each of mass 1 and a mass $a > 2$ of agents. Asset classes i are characterized by their asset heterogeneity k_i. The mass $a_i \geq 1$ of agents trading assets in each class i is determined in equilibrium so that $a_1 + a_2 = a$. Other than that, parameters of the search-and-bargaining model are as in the baseline model in Section 2. The equilibrium in the multi-class model is defined next. Subscripts indicate equilibrium quantities for the corresponding asset class.

Definition 3. *A tuple* $(\sigma_i, M_i, a_i)_{i=1,2}$ *is a* multi-class equilibrium *if* (σ_i, M_i) *is the equilibrium of the baseline model with mass of agents* a_i *and the following conditions hold*

$$\begin{cases} \underline{x}_1 = \underline{x}_2, & \text{if } a-1 > a_1 > 1, \\ \underline{x}_1 \leq \underline{x}_2, & \text{if } a_1 = 1, \\ \underline{x}_1 \geq \underline{x}_2, & \text{if } a_1 = a-1. \end{cases} \tag{6.1}$$

The interpretation of condition (6.1) is that if trading assets in one of the classes brings a higher utility to the buyer, buyers will migrate into trading this asset class. To see this, recall that the buyers' utility of trading each asset class is proportional to strategy thresholds $\underline{x}_1$ and $\underline{x}_2$ (cf. Lemma 2). If both are equal, then buyers are indifferent between the two classes. If one is greater, then all agents migrate to the more preferable (for buyers) class making the other class illiquid. The equilibrium of the two-class model always exists and is unique.

Theorem 3. *There exists a unique two-class equilibrium.*

I next show that a flight-to-liquidity occurs as a response to an increase in the market uncertainty about one of the asset classes.

Proposition 7. *Suppose that* k_1 *increases and/or* k_2 *decreases. Then the market liquidity* L_1*, the aggregate liquidity* X_1 *and the participation* a_1 *decrease, while* L_2, X_2 *and* a_2 *increase.*

When the bargaining friction increases for the first asset class, agents migrate to trading assets in class 2 for which the bargaining friction is lower. This flight-to-liquidity exacerbates the drop in liquidity. By Propositions 3 and 6, both an increase in k and a decrease in a lead to a decrease in L. As a result, as fewer agents are trading assets in class 1, the negative effect on the market liquidity from the increase in the bargaining friction is exacerbated.

OTC markets are known to be prone to flights-to-liquidity episodes when, due to increased market uncertainty, agents shift their portfolio preferences to safer and more liquid assets. These phenomena are associated with dried-up liquidity in markets for more risky assets. Friewald et al. (2012), Dick-Nielsen et al. (2012) show empirically that flight-to-quality episodes were observed during the recent liquidity crisis of 2007-2008.

An important observation is that the level of payoffs in each asset class does not affect the equilibrium characterization in Theorems 2 and 3, and only affects the levels of utilities and prices. Therefore, my model stresses that the flights are flights-to-liquidity rather than flights-to-quality. In particular, if asset class 1 experiences an increase in market uncertainty (k) but at the same time a decrease in the level of payoffs, e.g. a decrease in the aggregate default probability for corporate bonds, then the direction and the magnitude of the migration to trading assets in class 2 would not change. This is consistent with the empirical evidence that default risk plays a smaller role than liquidity in flights (see Beber et al. (2009)).[35]

Gradual Transparency Policies The dried-up liquidity during the recent financial crisis inspired regulators to shift trading of financial assets from OTC markets to more centralized platforms. E.g. Title VII of Dodd-Frank calls for a greater transparency of trading in credit-default swaps and credit derivatives. Before the crisis in 2002, the public dissemination of trades in the corporate bonds market was introduced through the Trade Reporting and Compliance Engine (TRACE). Interestingly, the TRACE was introduced in several phases with early phases requiring disclosure only for larger issues of investment grade bonds, and later phases expanding the requirement to high-yield bonds and other assets, such as agency-backed securities.

The analysis of the flights-to-liquidity shows that such gradualism in introducing transparency can hurt the market liquidity. More specifically, consider the following stylized model with two asset classes. We suppose that $k_1 > k_2$ and we interpret the first asset class as high-yield bonds, and the second asset class as investment grade bonds. Suppose that $k_1 >> 0$ so that $L_1 < 1$, while $k_2 \approx 0$ so that $L_2 = 1$. Consider the effect of the introduction of the post-trade transparency in the asset class 2. This leads to a decrease in the bargaining friction in the second asset class. Proposition 7 implies that the trading will shift into the second asset class hurting the liquidity of the first asset class. Thus, this measure will not increase the market liquidity of the second asset class (it is already 1). However, it will reduce the market liquidity of the first asset class, thus, reducing the overall market liquidity $L_1 + L_2$. Moreover, by Proposition 7 it will also lead to a decrease in the aggregate liquidity X_1 (which captures the trading volume) in the first asset class. Asquith et al. (2013) shows that the introduction of TRACE while decreasing the price dispersion, also decreased the trading activity in high-yield bonds which is consistent with the mechanism described above.

[35]The analysis of flights-to-quality in Section 6 reveals that in the corporate bond market, the latter component would decrease asset prices of all bonds with the improvement in the liquidity of the Treasure market, a regularity confirmed empirically in Longstaff et al. (2005).

7 Extensions

This paper focuses on the effect of endogenous bargaining delays on the liquidity and prices. To capture the endogenous bargaining delays in a tractable way, we apply the screening bargaining solution and restrict attention to linear flow payoffs, the size of the holding costs that does not depend on the asset quality, and equal supply of each asset quality. In this section, we discuss the generality of my results.

There are several reasons to consider more general payoff specification. Post-search delays can arise for a variety of other reasons including asymmetric information about the quality, pre-trade evaluation of assets, work-up procedures, gradual execution of the deal to maintain the privacy, time it takes to raise financing for the deal. Further, one can argue both that more risky assets are associated with higher gains from trade (e.g. agents holding toxic assets are especially eager to sell them) and that higher-quality assets are associated with higher benefits for the holder (e.g. such assets can be used as collateral for cheaper short-term borrowing).

Suppose that flow payoffs of the buyer and the seller are given by general functions $\overline{v}$ and $\underline{v}$, respectively. Functions are assumed to be strictly increasing and continuously differentiable such that $\overline{v}(\theta) > \underline{v}(\theta)$ for all θ. Denote by $\ell(\theta) \equiv \overline{v}(\theta) - \underline{v}(\theta)$ the holding costs that can vary with θ. Also suppose that in the bargaining stage trade happens with delay $t(\theta)$ which can be either exogenous or be determined in equilibrium by some mechanism. We also assume that the supply of an asset quality θ is given by $f(\theta)$ where f is continuous and positive for all θ, with c.d.f. F.

With this more general specification of payoffs, the key quantity is $z(\theta) = \frac{\ell(\theta)}{\rho}x(\theta)$, the *expected surplus* from trade. The interpretation is that with probability $1 - x(\theta)$, the match is destroyed because of one of the sides switches its types and the realized surplus in the match is zero, and with complementary probability $x(\theta)$, the surplus $\frac{\ell(\theta)}{\rho}$ is realized after agents negotiate for time $t(\theta)$. Let $L \equiv \int_{\theta\in\Theta_L} dF(\theta)$ be the mass of liquid assets and $\overline{z} \equiv \frac{1}{L}\int_{\theta\in\Theta_L} z(\theta)dF(\theta)$ be the average liquidity. Both the threshold form of the buyers' strategy and the asset price decomposition generalize.

Theorem 4. *The asset of quality θ is always accepted by buyers ($\theta \in \Theta_L$) whenever*

$$z(\theta) > \underline{z} \equiv \frac{\Lambda_b}{\rho + \Lambda_b}\overline{z}, \tag{7.1}$$

and is always rejected whenever the inequality in (7.1) *is reversed. Prices of assets are* *given by*

$$p(\theta) = \underbrace{\frac{1}{r}\left(\overline{v}(\theta) - (r+y_d)\frac{\ell(\theta)}{\rho}\right) + (1-\alpha)\frac{\ell(\theta)}{\rho}}_{\textit{fundamental value}} + \underbrace{(1-\alpha)\frac{y_d}{r}\frac{\sigma(\theta)\Lambda_s}{\rho+\sigma(\theta)\Lambda_s}z(\theta)}_{\textit{liquidity premium}} - \underbrace{\alpha\frac{y_u}{r}\frac{\Lambda_b}{\rho+\Lambda_b}\overline{z}}_{\textit{average liqudity}} . \tag{7.2}$$

Theorem 4 shows that under general payoffs, the buyers' preferences are not guided solely by the liquidity considerations. Instead, the buyer trades off the post-search trade delay and the surplus from trade. Even when the gains from trade are large, the buyer may reject the asset because of the high delay associated with it. In the working paper version of this paper, we prove Theorem 4 and analyze numerically the non-linear specifications of payoffs in my model as well as the model with exogenous post-search delays. Importantly, the conclusion of Theorem 4 does not depend on the particular specification of trade delay. This suggests that it can be used for the empirical test of different theories of asset-specific trade delays. For example, instead of the SBS one can use the bargaining model with interdependent values in Fuchs and Skrzypacz (2013) (or a more general version of it in Deneckere and Liang (2006)). In fact, heuristically the equilibrium of such a model can be obtained from my model by setting $\alpha = 0$ and specifying in Definition 1 that $\theta^* = 1$.[36]

8 Conclusion

This paper develops a tractable model of decentralized asset markets with both search and endogenous bargaining delays. The key to the tractability is the application of the novel screening bargaining solution that captures bargaining delays due to the gap between the coarse public and precise private information. The analysis allows for the separation between the intensive and extensive trade margins, as well as between the search and bargaining frictions. The liquidity of the asset is U-shaped in the quality and assets in the middle of the quality range may not be traded at all. While on the intensive margin, search and bargaining frictions operate similarly, on the extensive margin, they are quite different. The search friction increases, while the bargaining friction decreases the market liquidity. This shows that greater transparency can hurt liquidity if it leads to lower search frictions in the market. we also show that because of the substitutability of asset classes, the OTC markets are prone to flights-to-liquidity and gradualism in the introduction of

[36]One also needs to put additional restrictions on payoff functions. In particular, Fuchs and Skrzypacz (2013) assume that $v(1) = c(1)$ (no-gap assumption).

transparency can have adverse effects for the market liquidity. Finally, we derive the asset price decomposition which holds for a variety of specifications of post-search delay.

A Appendix

Appendix $A.1$ contains the proofs for the micro-foundations of the SBS. Appendix $A.2$ contains the the analysis of the OTC model. The Online Appendix contains proofs of the auxiliary results.

A.1 Microfoundation for the Screening Bargaining Solution

Proof of Part 1 of Theorem 1

Denote $\overline{\alpha} = \frac{1-e^{-\rho\Delta_b}}{1-e^{-\rho\Delta}}$, $\underline{\alpha} = \frac{e^{-\rho\Delta_s}-e^{-\rho\Delta}}{1-e^{-\rho\Delta}}$, $\overline{\mathbf{p}}(\theta^s,\theta^b) = (1-\underline{\alpha})v(\theta^b)+\underline{\alpha}c(\theta^s)$, and $\underline{\mathbf{p}}(\theta^s,\theta^b) = (1-\overline{\alpha})v(\theta^b)+\overline{\alpha}c(\theta^s)$. Let $\underline{\ell}$ and $\overline{\ell}$, resp., be the minimum and maximum on [0,1] of derivatives of v. We first derive the following bounds on the price of trade.

Lemma 4. *Suppose after some history, the highest remaining types of the buyer and seller equal to $\hat{\theta}^b$ and $\hat{\theta}^s$, resp., and the lowest remaining types of the buyer and seller equal to $\check{\theta}^b$ and $\check{\theta}^s$, resp. Suppose $\hat{\theta}^b > \check{\theta}^b$ and $\hat{\theta}^s > \check{\theta}^s$. Then*

$$\underline{\mathbf{p}}(\check{\theta}^s,\check{\theta}^b) \le p_{F,\Delta}(\theta^s,\theta^b) \le \overline{\mathbf{p}}(\hat{\theta}^s,\hat{\theta}^b). \tag{A.1}$$

Moreover, any offer below $\underline{\mathbf{p}}(\check{\theta}^s,\check{\theta}^b)$ and any offer above $\overline{\mathbf{p}}(\hat{\theta}^s,\hat{\theta}^b)$ is accepted with probability 1 by the buyer and seller, resp.

Proof. Let P, resp. Q, be the supremum over all histories of price offers accepted by the buyer, resp. rejected by the seller, with positive probability in PBE. By the definition of Q, after any history the buyer's type θ^b can guarantee himself the utility arbitrarily close to $e^{-\rho\Delta_s}(v(\theta^b)-Q)$ by making an offer marginally above Q (that is guaranteed to be accepted by the seller). By the definition of P,

$$e^{-\rho\Delta_s}(v(\theta^b)-Q) \le v(\theta^b)-P. \tag{A.2}$$

Let $U(\theta^s)$ be the supremum over all histories of the continuation utilities of the seller's type θ^s after the rejection of the offer in the current round. If type θ^s rejects an offer, she cannot guarantee more than $\max\{e^{-\rho\Delta_b}(P-c(\theta^s)), e^{-\rho\Delta}U(\theta^s)\}$, which implies $U(\theta^s) \le e^{-\rho\Delta_b}(P-c(\theta^s))$. By the definition of Q,

$$Q-c(\theta^s) \le e^{-\rho\Delta_b}(P-c(\theta^s)). \tag{A.3}$$

By $(A.3)$, $Q \leq P$. Combining $(A.2)$ and $(A.3)$, we get

$$\begin{aligned} P &\leq (1-e^{-\rho\Delta_s})v(\overline{\theta}^b) + e^{-\rho\Delta_s}Q \\ &\leq (1-e^{-\rho\Delta_s})v(\overline{\theta}^b) + e^{-\rho\Delta_s}(1-e^{-\rho\Delta_b})c(\overline{\theta}^s) + e^{-\rho\Delta}P, \end{aligned} \tag{A.4}$$

where we used the fact that the support of beliefs does not expand to put an upper bound on $v(\theta^b)$ and $c(\theta^s)$ in $(A.2)$ and $(A.3)$. By iterating the inequality $(A.4)$, we obtain the upper bound in $(A.1)$. By the definition of Q, any offer above $\overline{\mathbf{p}}(\hat{\theta}^s, \hat{\theta}^b)$ is accepted with probability one by the seller. The argument for the lower bound is symmetric. □

Denote $D = \{(\theta^s, \theta^b) : |\theta^s - \theta^b| < \frac{1}{4}\varepsilon^2\}$, $\Omega = \{(\theta^s, \theta^b) : |p_{F,\Delta}(\theta^s, \theta^b) - \mathbf{p}(\theta^s, \theta^b)| < \frac{1}{4}\varepsilon^2\}$, and $\Theta = \Omega \cap D$. Fix $\varepsilon > 0$. For F sufficiently far in the sequence, $\mathbb{P}_F(D) > 1 - \frac{1}{4}\varepsilon^2$. Moreover, for any such F and Δ sufficiently small, there is a PBE in $\mathcal{G}(F, \Delta)$ such that $\mathbb{P}_F(\Omega) > 1 - \frac{1}{4}\varepsilon^2$. Define $B^{1-\varepsilon,s} = \{\theta^s : \mathbb{P}_F(\Theta|\theta^s) > 1 - \varepsilon\}$ and $B^{1-\varepsilon,b} = \{\theta^b : \mathbb{P}_F(\Theta|\theta^b) > 1 - \varepsilon\}$.

Lemma 5. *For any interval I such that $|I| < \varepsilon$, $I \cap B^{1-\varepsilon,s} \neq \emptyset$ and $I \cap B^{1-\varepsilon,b} \neq \emptyset$.*

Proof. We show that $I \cap B^{1-\varepsilon,s} \neq \emptyset$ ($I \cap B^{1-\varepsilon,b} \neq \emptyset$ is analogous). Let F^s be the marginal of F on θ^s. First, we show that $\mathbb{P}_{F^s}(B^{1-\varepsilon,s}) > 1 - \frac{\varepsilon}{2}$. Note that

$$\mathbb{P}_F(\Theta) = \mathbb{P}_F(\Omega) + \mathbb{P}_F(D) - \mathbb{P}_F(\Omega \cup D) \geq \mathbb{P}_F(\Omega) + \mathbb{P}_F(D) - 1 > 1 - \tfrac{\varepsilon^2}{2}.$$

Now,

$$\mathbb{P}_F(\Theta) = \int_0^1 \mathbb{P}_F(\Theta|\theta^s)dF^s(\theta^s) \leq (1-\varepsilon)(1 - \mathbb{P}_{F^s}(B^{1-\varepsilon,s})) + \mathbb{P}_{F^s}(B^{1-\varepsilon,s}) = 1 - \varepsilon + \varepsilon\mathbb{P}_{F^s}(B^{1-\varepsilon,s})$$

and so, $\mathbb{P}_{F^s}(B^{1-\varepsilon,s}) \geq 1 - \frac{1}{\varepsilon}(1 - \mathbb{P}_F(\Theta)) > 1 - \frac{\varepsilon}{2}$.

Note that $F \xrightarrow{p} F^* \implies F \xrightarrow{d} F^* \implies F^s \xrightarrow{d} F^{s*}$. Since F^s and F^{s*} are continuous, they converge uniformly as functions of θ^s. Thus, for F far enough in the sequence $|F^s(\theta^s) - F^{s*}(\theta^s)| < \frac{\varepsilon}{4}$ for all θ^s. Let $I = [\check{\theta}^s, \hat{\theta}^s]$. By the triangular inequality,

$$|\mathbb{P}_{F^s}(I) - \mathbb{P}_{F^{s*}}(I)| \leq |F^s(\hat{\theta}^s) - F^{s*}(\hat{\theta}^s)| + |F^s(\check{\theta}^s) - F^{s*}(\check{\theta}^s)| \leq \tfrac{\varepsilon}{2},$$

and so, $\mathbb{P}_{F^s}(I) \in [|I| - \frac{\varepsilon}{2}, |I| + \frac{\varepsilon}{2}]$. Therefore, $\mathbb{P}_{F^s}(I \cap B^{1-\varepsilon,s}) \geq \mathbb{P}_{F^s}(I) + \mathbb{P}_{F^s}(B^{1-\varepsilon,s}) - 1 \geq |I| - \varepsilon > 0$ which proves $I \cap B^{1-\varepsilon,s} \neq \emptyset$. □

Let $[\underline{s}_0^*, \overline{s}_0^*] = [\underline{b}_0^*, \overline{b}_0^*] = [0, 1]$. For $n = 1, \ldots, N$ (N to be specified below), define the nested intervals of seller's types $[\underline{s}_n^*, \overline{s}_n^*)$ as follows. For given $[\underline{s}_{n-1}^*, \overline{s}_{n-1}^*)$, let $\mathcal{S}_n^*$ be the collection of all intervals $[\underline{s}_{n-1}, \overline{s}_{n-1})$ such that

- before round n, seller's types in $[\underline{s}_n, \overline{s}_n)$ pool with types in $[\underline{s}^*_{n-1}, \overline{s}^*_{n-1})$, and in round n, they pool with each other and separate from types in $[\underline{s}^*_{n-1}, \overline{s}^*_{n-1})\backslash[\underline{s}_n, \overline{s}_n)$, and (since players use interval strategies, $[\underline{s}_n, \overline{s}_n)$ is well defined);

- $\underline{s}_n < 1 - 2\varepsilon$.

Let $[\underline{s}^*_n, \overline{s}^*_n)$ be the set in $\mathcal{S}^*_n$ such that $\underline{s}^*_n > \underline{s}_n$ for all intervals $[\underline{s}_n, \overline{s}_n) \in \mathcal{S}^*_n$. Analogously, for $n = 1, \ldots, N$, define the nested intervals of buyer's types $(\underline{b}^*_n, \overline{b}^*_n]$ as follows. For given $(\underline{b}^*_{n-1}, \overline{b}^*_{n-1}]$, let $\mathcal{B}^*_n$ be the collection of all $(\underline{b}_n, \overline{b}_n]$ of seller's types that satisfy

- buyer's types in $(\underline{b}_n, \overline{b}_n]$ pool with each other in round n and pool with types in $(\underline{b}^*_{n-1}, \overline{b}^*_{n-1}]$ before round n;

- $\overline{b}_n > 2\varepsilon$.

Let $(\underline{b}^*_n, \overline{b}^*_n]$ be the set in $\mathcal{B}^*_n$ such that $\overline{b}^*_n < \overline{b}_n$ for all $(\underline{b}_n, \overline{b}_n] \in \mathcal{B}^*_n$. Let round N be the first round n in which either $\frac{1}{3} \leq \underline{s}^*_n$ or $\overline{b}^*_n \leq \frac{2}{3}$.

Lemma 6. *For ε sufficiently small and any $n < N$,*

1. $\overline{s}^*_n \geq 1 - 2\varepsilon$ *and* $\underline{b}^*_n \leq 2\varepsilon$*;*

2. $\overline{s}^*_n - \underline{s}^*_n > \frac{1}{3}$ *and* $\overline{b}^*_n - \underline{b}^*_n > \frac{1}{3}$*;*

3. *types in* $(\underline{s}^*_n, \overline{s}^*_n]$ *and* $(\underline{b}^*_n, \overline{b}^*_n]$ *reject the opponent's offer and make some counter-offers* p^{s*}_n *and* p^{b*}_n*, resp.;*

4. *there is a positive constant C (independent of n) such that*

$$p^{b*}_n \leq \mathbf{p}(\tfrac{1}{3}) + C\varepsilon; \tag{A.5}$$

$$p^{s*}_n \geq \mathbf{p}(\tfrac{2}{3}) - C\varepsilon. \tag{A.6}$$

Proof. 1,2) Fix $n < N$. If $\overline{s}^*_n < 1 - 2\varepsilon$, then there is $[\underline{s}_n, \overline{s}_n) \in \mathcal{S}^*_n$ such that $\underline{s}_n = \overline{s}^*_n > \underline{s}^*_n$ which contradicts the definition of $[\underline{s}^*_n, \overline{s}^*_n)$. Thus, $\overline{s}^*_n \geq 1 - 2\varepsilon$, and analogously, $\underline{b}^*_n \leq 2\varepsilon$. Since $\underline{s}^*_n < \frac{1}{3}$ by the definition of N, $\overline{s}^*_n - \underline{s}^*_n \geq \frac{2}{3} - 2\varepsilon > \frac{1}{3}$ (when $\varepsilon < \frac{1}{6}$), and analogously, $\overline{b}^*_n - \underline{b}^*_n > \frac{1}{3}$.

3) Suppose before round n, both players pool with $[\underline{s}^*_n, \overline{s}^*_n)$ and $(\underline{b}^*_{n-1}, \overline{b}^*_{n-1}]$, resp., and suppose to contradiction that seller's types in $[\underline{s}^*_n, \overline{s}^*_n)$ accept p^{b*}_{n-1}. Let $\hat{\theta}^s = \sup\{(\frac{1}{3}, \frac{2}{3} - \frac{1}{4}\varepsilon^2) \cap B^{1-\varepsilon,s}\}$ and $\check{\theta}^s = \inf\{(\frac{1}{3}, \frac{2}{3} - \frac{1}{4}\varepsilon^2) \cap B^{1-\varepsilon,s}\}$ (these types also accept p^{b*}_{n-1} as $[\check{\theta}^s, \hat{\theta}^s] \subseteq [\underline{s}^*_n, \overline{s}^*_n)$). We first show that for some c_0, $|\mathbf{p}(\theta^s) - p^{b*}_{n-1}| \leq c_0\varepsilon$ for $\theta^s \in \{\check{\theta}^s, \hat{\theta}^s\}$. Consider $\theta^s \in \{\check{\theta}^s, \hat{\theta}^s\}$. Since $\theta^s \in B^{1-\varepsilon,s}$, type θ^s assigns probability at least $1-\varepsilon$ to Θ. Note that $[\theta^s - \frac{1}{4}\varepsilon^2, \theta^s + \frac{1}{4}\varepsilon^2] \subseteq$

$[2\varepsilon, \frac{2}{3}] \subseteq (\underline{b}^*_{n-1}, \overline{b}^*_{n-1}]$, and so after round $n-1$ the probability of Θ is at least $\frac{1-\varepsilon}{1-\beta}$ where $\beta = 1 - F(\overline{b}^*_{n-1}|\theta^s) + F(\underline{b}^*_{n-1}|\theta^s)$. Since type θ^s accepts p^{b*}_{n-1} and $\theta^s \in B^{1-\varepsilon,s}$, it is necessary

$$p^{b*}_{n-1} \in \left[\tfrac{1-\varepsilon}{1-\beta}(\mathbf{p}(\theta^s - \tfrac{1}{4}\varepsilon^2) - \tfrac{1}{4}\varepsilon^2) + \tfrac{\varepsilon-\beta}{1-\beta}\underline{\mathbf{p}}(0), \tfrac{1-\varepsilon}{1-\beta}(\mathbf{p}(\theta^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \tfrac{\varepsilon-\beta}{1-\beta}\overline{\mathbf{p}}(1)\right] \subseteq$$
$$\left[(1-\varepsilon)(\mathbf{p}(\theta^s - \tfrac{1}{4}\varepsilon^2) - \tfrac{1}{4}\varepsilon^2) + \varepsilon\underline{\mathbf{p}}(0), (1-\varepsilon)(\mathbf{p}(\theta^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \varepsilon\overline{\mathbf{p}}(1)\right] \quad (A.7)$$

and so, $|\mathbf{p}(\theta^s) - p^{b*}_{n-1}| \le c_0\varepsilon$ where c_0 is some positive constant. Thus,

$$\underline{\ell}(\hat{\theta}^s - \check{\theta}^s) \le \mathbf{p}(\hat{\theta}^s) - \mathbf{p}(\check{\theta}^s) \le |\mathbf{p}(\check{\theta}^s) - p^{b*}_{n-1}| + |\mathbf{p}(\hat{\theta}^s) - p^{b*}_{n-1}| \le 2c_0\varepsilon$$

and so, $\hat{\theta}^s - \check{\theta}^s \le \frac{2c_0}{\underline{\ell}}\varepsilon$. On the other hand, by Lemma 5, $\hat{\theta}^s - \check{\theta}^s \ge \frac{1}{3} - 3\varepsilon$ and so, for $\varepsilon < \frac{\underline{\ell}}{3(3\underline{\ell}+2c_0)}$ this leads to a contradiction. Therefore, types in $[\underline{s}^*_n, \overline{s}^*_n)$ reject p^{b*}_{n-1} and make a counter-offer p^{s*}_n. The argument is analogous for the buyer.

4) Consider type $\check{\theta}^s$ defined above. By Lemma 5, $\check{\theta}^s < \frac{1}{3} + \varepsilon$. By $(A.7)$,

$$p^{b*}_{n-1} \le (1-\varepsilon)(\mathbf{p}(\check{\theta}^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \varepsilon\overline{\mathbf{p}}(1) \le (1-\varepsilon)(\mathbf{p}(\tfrac{1}{3} + \varepsilon + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2) + \varepsilon\overline{\mathbf{p}}(1) \le \mathbf{p}(\tfrac{1}{3}) + C\varepsilon,$$

where C is some constant. □

Lemma 7. *For sufficiently small ε, one of the two obtains:*

- *there is the seller's type $\tilde{\theta}^s \in (0, 4\varepsilon) \cap B^{1-\varepsilon,s}$ and the seller's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(\tilde{\theta}^s))$ in the beginning of the game;*
- *there is the buyer's type $\tilde{\theta}^b \in (1-4\varepsilon, 1) \cap B^{1-\varepsilon,b}$ and the buyer's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \overline{\mathbf{p}}(1, \frac{2}{3}))$ in the beginning of the game.*

Proof. There are two cases depending on whether $\frac{1}{3} \le \underline{s}^*_N$ or $\overline{b}^*_N \le \frac{2}{3}$ and $\frac{1}{3} > \underline{s}^*_N$

Case 1: $\frac{1}{3} \le \underline{s}^*_N$. There are two possibilities:

1. First, suppose that there is $\eta \le N$ and an interval of seller's types $[\underline{s}_\eta, \overline{s}_\eta) \subset [\underline{s}^*_{\eta-1}, \overline{s}^*_{\eta-1})$ with $\underline{s}^*_N \le \underline{s}_\eta$ that reject $p^{b*}_{\eta-1}$ and make a counter-offer $\hat{p} > \underline{\mathbf{p}}(\underline{s}^*_N, 0)$ in round η. By Lemma 4, if such a counter-offer in round η is rejected, then the seller can guarantee to trade at price $\underline{\mathbf{p}}(\underline{s}_\eta, 0)$ in round $\eta + 1$. Consider the following strategy of the seller:
   - as long as the buyer pools with types in $[\underline{b}^*_{n-1}, \overline{b}^*_{n-1})$, the seller pools with types $[\underline{s}^*_n, \overline{s}^*_n)$ for $n \le \eta$, pools with $[\underline{s}_\eta, \overline{s}_\eta)$ in round η and, if in round n, offer $\hat{p}$ is rejected, offers $\underline{\mathbf{p}}(\underline{s}^*_\eta, 0)$ in round $\eta + 1$;
   - otherwise, the seller rejects all offers and makes unacceptable offers above $\overline{\mathbf{p}}(1)$.

Type $\tilde{\theta}^s \in (2\varepsilon+\frac{1}{4}\varepsilon^2, 4\varepsilon)\cap B^{1-\varepsilon,s}$ (this set is non-empty by Lemma 5) assigns probability at least $1-\varepsilon$ to the buyer's type belonging in $[\underline{b}^*_{\eta-1}, \overline{b}^*_{\eta-1})$, as $[\tilde{\theta}^s - \frac{1}{4}\varepsilon^2, \tilde{\theta}^s + \frac{1}{4}\varepsilon^2] \subseteq [2\varepsilon, \frac{2}{3}] \subseteq [\underline{b}^*_{\eta-1}, \overline{b}^*_{\eta-1})$. Therefore, by deviating to the described strategy, type $\tilde{\theta}^s$ can guarantee utility

$$\begin{aligned}(1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\underline{s}_\eta, 0) - c(\tilde{\theta}^s)) &\geq (1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\underline{s}^*_N, 0) - c(\tilde{\theta}^s)) \\ &\geq (1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\tfrac{1}{3}, 0) - c(\tilde{\theta}^s)).\end{aligned}$$

2. Now, suppose that for any $\eta \leq N$ and any interval of seller's types $[\underline{s}, \overline{s}) \subset [\underline{s}^*_{\eta-1}, \overline{s}^*_{\eta-1})$ with $\underline{s}^*_N \leq \underline{s}$ that pool with each other and separate from other types in $[\underline{s}^*_{\eta-1}, \overline{s}^*_{\eta-1})$, they either accept $p^{b*}_{\eta-1}$ or make a counter-offer below $\underline{\mathbf{p}}(\underline{s}^*_N, 0)$ in round η. Consider the following strategy of the buyer:

   - for $\eta \leq N$, as long as the seller pools with types in $[\underline{s}^*_\eta, \overline{s}^*_\eta)$, the buyer pools with types in $[\underline{b}^*_N, \overline{b}^*_N)$, unless the seller makes an offer below $\underline{\mathbf{p}}(\underline{s}^*_N, 0)$, in which case the buyer accepts it;
   - otherwise, the buyer rejects all offers and makes unacceptable offers below $\underline{\mathbf{p}}(0)$.

   By Lemma 5, there exists type $\tilde{\theta}^b \in (1-2\varepsilon+\frac{1}{4}\varepsilon^2, 1-\frac{1}{4}\varepsilon^2]\cap B^{1-\varepsilon,b}$ that assigns probability at least $1-\varepsilon$ to $\theta^s > 1-2\varepsilon$, as $[\tilde{\theta}^b - \frac{1}{4}\varepsilon^2, \tilde{\theta}^b + \frac{1}{4}\varepsilon^2] \subseteq [1-2\varepsilon, 1] \subseteq [\underline{s}^*_N, 1]$. Therefore, by deviating to the described strategy, type $\tilde{\theta}^b$ can guarantee utility

$$\begin{aligned}&(1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \max\{p^{b*}_1, \dots, p^{b*}_{N-1}, \underline{\mathbf{p}}(\underline{s}^*_N, 0)\}) \geq \\ &\qquad (1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \max\{\mathbf{p}(\tfrac{1}{3}) + C\varepsilon, \underline{\mathbf{p}}(\underline{s}^*_N, 0)\}),\end{aligned}$$

   where the inequality follows from Lemma 6.

Case 2: $\overline{b}^*_N \leq \frac{2}{3}$, but $\frac{1}{3} > \underline{s}^*_N$. By the symmetric argument as in case 1, one of the following holds:

- there is the seller's type $\tilde{\theta}^s \in (0, 4\varepsilon) \cap B^{1-\varepsilon,s}$ and the seller's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(\min\{\mathbf{p}(\frac{2}{3}) - C\varepsilon, \overline{\mathbf{p}}(1, \overline{b}^*_N)\} - c(\tilde{\theta}^s))$ in the beginning of the game;
- there is the buyer's type $\tilde{\theta}^b \in (1-4\varepsilon, 1) \cap B^{1-\varepsilon,b}$ and the buyer's strategy that guarantees the expected utility at least $(1-\varepsilon)e^{-\rho\Delta(N+1)}(v(\tilde{\theta}^b) - \overline{\mathbf{p}}(1, \frac{2}{3}))$ in the beginning of the game.

This completes the proof of the lemma. □

We can now prove part 1 of Theorem 1. Suppose in Lemma 7 the first case is realized (the argument is symmetric for the second case): there is a type $\tilde{\theta}^s \in (0, 4\varepsilon) \cap B^{1-\varepsilon,s}$ who is guaranteed utility $(1-\varepsilon)e^{-r\Delta(N+1)}(\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(\tilde{\theta}^s))$ in the beginning of the game. Since $\tilde{\theta}^s \in B^{1-\varepsilon,s}$, it is necessary that

$$(1-\varepsilon)e^{-\rho\Delta(N+1)}(\underline{\mathbf{p}}(\tfrac{1}{3}, 0) - c(\tilde{\theta}^s)) \leq (1-\varepsilon)(\mathbf{p}(\tilde{\theta}^s + \tfrac{1}{4}\varepsilon^2) - c(\tilde{\theta}^s)) + \varepsilon v(1)$$

and so,

$$e^{-\rho\Delta N} \leq e^{r\Delta}\frac{(1-\varepsilon)(\mathbf{p}(\tilde{\theta}^s + \frac{1}{4}\varepsilon^2) - c(\tilde{\theta}^s)) + \varepsilon v(1)}{(1-\varepsilon)(\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(\tilde{\theta}^s))} \underset{\varepsilon\to 0}{\to} \frac{\mathbf{p}(0) - c(0)}{\underline{\mathbf{p}}(\frac{1}{3}, 0) - c(0)} < 1.$$

Therefore, when ε is small, there is $\overline{T} > 0$ such that $\Delta N > \overline{T}$ and so, types in $(\frac{1}{3}, \frac{2}{3})^2$ trade with a delay at least $\overline{T}$. The probability of such types approaches $\frac{1}{3}$ as $F \overset{p}{\to} F^*$, and so there is $c_1 > 0$ such that when $\varepsilon < \frac{1}{6c_1}$,

$$\mathbb{E}_F[e^{-\rho t_{F,\Delta}}] \leq (\tfrac{1}{3} + c_1\varepsilon)e^{-\rho\overline{T}} \leq \tfrac{1}{2}e^{-\rho\overline{T}} \equiv x_l,$$

for all F sufficiently far in the sequence and all Δ sufficienly small. This proves (3.1).

To show (3.2), observe that every seller type can trade at price $\underline{\mathbf{p}}(0)$. Thus, for any $\theta^s \in B^{1-\varepsilon,s}$,

$$\begin{aligned} \underline{\mathbf{p}}(0) - c(\theta^s) &\leq (1-\varepsilon)\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta](\mathbf{p}(\theta^s + \tfrac{1}{4}\varepsilon^2) + \tfrac{1}{4}\varepsilon^2 - c(\theta^s)) + \varepsilon v(1) \\ &\leq \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta](1-\alpha)\xi + c_2\varepsilon, \end{aligned}$$

for some constant $c_2 > 0$. This implies that for any $\overline{\theta}^s$

$$\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta] \geq \frac{\underline{\mathbf{p}}(0) - c(\theta^s) - c_2\varepsilon}{(1-\alpha)\xi} > \frac{\underline{\mathbf{p}}(0) - c(\underline{\theta}^s) - c_2\varepsilon}{(1-\alpha)\xi}$$

for all $\theta^s \in B^{1-\varepsilon,s}$ below $\underline{\theta}^s$. By choosing $\underline{\theta}^s$ and ε sufficiently close to zero, we can guarantee that there is $\tilde{x} > x_l$ such that for all Δ sufficiently small, $\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta] \geq \tilde{x}$. Moreover,

$$\begin{aligned} \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s < \underline{\theta}^s] &= \int_0^{\underline{\theta}^s} \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s]dF^s(\theta^s) \\ &\geq \int_0^{\underline{\theta}^s} \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s, \Theta]\mathbb{P}_F(\Theta|\theta^s)dF^s(\theta^s) \\ &\geq \tilde{x}\int_0^{\underline{\theta}^s} \mathbb{P}_F(\Theta|\theta^s)dF^s(\theta^s) \\ &\geq (1-\varepsilon)\tilde{x}\mathbb{P}_{F^s}(B^{1-\varepsilon,s}) \geq (1-\varepsilon)(1-\tfrac{\varepsilon}{2})\tilde{x}. \end{aligned}$$

Thus, there is $x_h > x_l$ such that for F sufficiently close to F^* and all Δ sufficiently small, $\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s < \underline{\theta}^s] \geq x_h$.

The analogous argument applied to buyer's types close to 1 gives that there is $\overline{\theta}^s > \underline{\theta}^s$ such that $\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^b < \overline{\theta}^s] \geq x_h$. Observing that for fixed $\overline{\theta}^s$, $|\mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^b < \overline{\theta}^s] - \mathbb{E}_F[e^{-\rho t_{F,\Delta}}|\theta^s < \overline{\theta}^s]| \to 0$ as $F \xrightarrow{p} F^*$, we get the inequality (3.2).

Proof of Part 2 of Theorem 1

The proof proceeds as follows. We first introduce and analyze the continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$ which is a generalization of the game $\mathcal{G}(p^s_\cdot, p^b_\cdot)$ in subsection 2.1 to affiliated distributions of types F (thus, $\mathcal{G}(p^s_\cdot, p^b_\cdot) = \mathcal{G}(p^s_\cdot, p^b_\cdot|F^*)$). We then proceed with a series of approximations. First, we approximate the SBS outcome with the BNE outcome in $\mathcal{G}(\cdot|F^*)$. Second we approximate each BNE outcome of $\mathcal{G}(\cdot|F^*)$ with BNE outcomes of $\mathcal{G}(\cdot|F), F \xrightarrow{p} F^*$. Finally, we approximate each BNE outcome of $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$ with PBE outcome in the discrete-time bargaining game $\mathcal{G}(F, \Delta), \Delta \to 0$. Thus, we construct a sequence of PBE frequent-offer limits (indexed by $F \xrightarrow{p} F^*$) that approximates the SBS outcome, and hence prove the theorem.

Continuous-Time Bargaining Game $\mathcal{G}(\cdot|F)$ Consider a strictly decreasing function $p^s_\cdot$ and a strictly increasing function $p^b_\cdot$ and the following continuous-time bargaining game $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$. The buyer continuously makes offers p^b_t and the seller continuously makes offers p^s_t. Players can choose only the time when they accept the price offer of the opponent, and the trade happens once the first acceptance happens (at the accepted price). The difference from $\mathcal{G}(p^s_\cdot, p^b_\cdot)$ is that players' types are distributed according to a general affiliated distribution F, not F^*. We consider BNEs in threshold strategies. Let T be the first time when $p^s_t = p^b_t$. For any $t \in [0, T]$, let θ^s_t and θ^b_t be, resp., strictly increasing and strictly decreasing functions such that $\theta^s_0 = 0$ and $\theta^b_0 = 1$. At time t, all types of the seller below θ^s_t (resp.,, all types of the buyer above θ^b_t) accept the offer p^b_t (resp., p^s_t).

Lemma 8. *Suppose that a tuple* $(p^s_\cdot, p^b_\cdot, \theta^s_\cdot, \theta^b_\cdot)$ *satisfies the system of differential equations*

$$\begin{aligned} r(v(\theta^b_t) - p^s_t) + \dot{p}^s_t =& (p^s_t - p^b_t)\dot{\theta}^b_t \frac{f(\theta^s_t|\theta^b_t)}{1 - F(\theta^s_t|\theta^b_t)}, \\ -r(p^b_t - c(\theta^s_t)) + \dot{p}^b_t =& (p^s_t - p^b_t)\dot{\theta}^b_t \frac{f(\theta^b_t|\theta^s_t)}{F(\theta^b_t|\theta^s_t)}; \end{aligned} \tag{A.8}$$

with initial conditions $\theta^s_0 = 0$ *and* $\theta^b_0 = 1$*, and* $\theta^s_T < 1$ *and* $\theta^b_T > 0$*. Then threshold strategies* $(\theta^s_\cdot, \theta^b_\cdot)$ *constitute a BNE in* $\mathcal{G}(p^s_\cdot, p^b_\cdot|F)$*.*

Proof. We show that if $\theta^b_\cdot$ satisfies the first equation in the system $(A.8)$, then it is a best response to the threshold strategy $\theta^s_\cdot$. Buyer's type θ^b chooses the acceptance time t to maximize $u(\theta^b, t)$

given by

$$u(\theta^b, t) = \int_0^t e^{-ru}(v(\theta^b) - p_u^b)dF(\theta_u^s|\theta^b) + (1 - F(\theta_t^s|\theta^b))e^{-rt}(v(\theta^b) - p_t^s).$$

The first-order condition for this problem is

$$(p_t^s - p_t^b)f(\theta_t^s|\theta^b)\dot{\theta}_t^s = (1 - F(\theta_t^s|\theta^b))(r(v(\theta^b) - p_t^s) + \dot{p}_t^s).$$

From the first-order condition,

$$\begin{aligned} u(1, t(1)) - u(\tilde{\theta}^b, t(\tilde{\theta}^b)) &= \int_{\tilde{\theta}^b}^1 \left(\frac{\partial}{\partial \theta^b} u(\theta^b, t(\theta^b)) + \frac{\partial}{\partial t} u(\theta^b, t(\theta^b)) t'(\theta^b) \right) d\theta^b \\ &= \int_{\tilde{\theta}^b}^1 \frac{\partial}{\partial \theta^b} u(\theta^b, t(\theta^b)) d\theta^b, \end{aligned} \tag{A.9}$$

where $t(\theta^b)$ is the inverse of $\theta_\cdot^b$. In Claim 1 below, we show that $u(\theta^b, t)$ satisfies the smooth single crossing difference (SSCD) condition in $(\theta^b, -t)$. Together with the envelope formula $(A.9)$, this verifies the conditions of Theorem 4.2 in Milgrom (2004) and proves that $\theta_\cdot^b$ is a best response to $\theta_\cdot^s$. Therefore, $(\theta_\cdot^s, \theta_\cdot^b)$ constitute a BNE of $\mathcal{G}(F)$.

Claim 1. *$u(\theta^b, t)$ satisfies the SSCD condition in $(\theta^b, -t)$.*

Proof: We will show the following conditions are satisfied which imply the SSCD.

1. $u(\theta^b, t)$ satisfies the (strict) single crossing difference condition in $(\theta^b, -t)$, i.e. for all $\tilde{t} > t$ and $\tilde{\theta}^b > \theta^b$,

$$u(\theta^b, t) - u(\theta^b, \tilde{t}) \geq 0 \implies u(\tilde{\theta}^b, t) - u(\tilde{\theta}^b, \tilde{t}) > 0.$$

2. for all t, if $\frac{\partial}{\partial t} u(\theta^b, t) = 0$, then for all $\delta > 0$, $\frac{\partial}{\partial t} u(\theta^b, t - \delta) \geq 0$ and $\frac{\partial}{\partial t} u(\theta^b, t + \delta) \leq 0$.

Let us start with the single crossing difference condition. Consider $\theta^b < \tilde{\theta}^b$ and $t < \tilde{t} \leq T$ and suppose that

$$u(\theta^b, t) \geq u(\theta^b, \tilde{t}). \tag{A.10}$$

We will show that $u(\tilde{\theta}^b, t) > u(\tilde{\theta}^b, \tilde{t})$. Define function

$$g(u|\theta^b, t) = e^{-ru}(v(\theta^b) - p_u^b)1\{u < t)\} + e^{-rt}(v(\theta^b) - p_t^s)1\{u \geq t\}.$$

Then

$$\int_0^T g(u|\theta^b, t)dF(\theta_u^s|\theta^b) \geq \int_0^T g(u|\theta^b, \tilde{t})dF(\theta_u^s|\theta^b) \geq \int_0^T g(u|\theta^b, \tilde{t})dF(\theta_u^s|\tilde{\theta}^b),$$

where the first inequality follows from $(A.10)$, the second inequality follows from the fact that

$g(\cdot|\theta^b,\tilde{t})$ is decreasing and $F(\cdot|\tilde{\theta}^b)$ first-order stochastically dominates $F(\cdot|\theta^b)$ (as f is affiliated). This implies that

$$u(\theta^b,t)=\int_0^t e^{-ru}(v(\theta^b)-p_u^b)dF(\theta_u^s|\theta^b)+(1-F(\theta_t^s|\theta^b))e^{-rt}(v(\theta^b)-p_t^s)$$
$$\geq \int_0^{\tilde{t}} e^{-ru}(v(\theta^b)-p_u^b)dF(\theta_u^s|\tilde{\theta}^b)+(1-F(\theta_{\tilde{t}}^s|\tilde{\theta}^b))e^{-rt}(v(\theta^b)-p_t^s),$$

or equivalently,

$$v(\theta^b)\left(\int_0^t e^{-ru}dF(\theta_u^s|\theta^b)+(1-F(\theta_t^s|\theta^b))e^{-rt}-\int_0^{\tilde{t}} e^{-ru}dF(\theta_u^s|\tilde{\theta}^b)-(1-F(\theta_{\tilde{t}}^s|\tilde{\theta}^b))e^{-r\tilde{t}}\right)$$
$$\geq p_t^s-\int_0^{\tilde{t}} e^{-ru}p_u^B dF(\theta_u^s|\tilde{\theta}^b)-(1-F(\theta_{\tilde{t}}^s|\tilde{\theta}^b))e^{-r\tilde{t}}p_{\tilde{t}}^s. \quad \text{(A.11)}$$

We will show that the left-hand side of $(A.11)$ is positive and so, the left-hand side would increase if we substitute $v(\tilde{\theta}^b)$ instead of $v(\theta^b)$. This in turn implies that $u(\tilde{\theta}^b,t)>u(\tilde{\theta}^b,t')$ and completes the proof of the strict single crossing difference. Let $h(u|t)=e^{-ru}1\{u<t)\}+e^{-rt}1\{u\geq t\}$. Then the left-hand side of $(A.11)$ is equal to

$$\begin{aligned}&v(\theta^b)\left(\int_0^T h(u|t)dF(\theta_u^s|\theta^b)-\int_0^T h(u|\tilde{t})dF(\theta_u^s|\tilde{\theta}^b)\right)\\ \geq &v(\theta^b)\left(\int_0^T h(u|t)dF(\theta_u^s|\tilde{\theta}^b)-\int_0^T h(u|\tilde{t})dF(\theta_u^s|\tilde{\theta}^b)\right)\\ = &v(\theta^b)\int_0^T (h(u|t)-h(u|\tilde{t}))dF(\theta_u^s|\tilde{\theta}^b)>0,\end{aligned}$$

where the first inequality follows from $F(\cdot|\tilde{\theta}^b)$ first-order stochastically dominates $F(\cdot|\theta^b)$ and $h(\cdot|t)$ decreasing, and the last term is strictly positive by $t<\tilde{t}$.

Now, let us show the second requirement of the SSCD condition. Suppose $\frac{\partial}{\partial t}u(\theta^b,t)=0$. By taking the partial derivative

$$e^{rt}\frac{\partial}{\partial t}u(\theta^b,t)=(p_t^s-p_t^b)f(\theta_t^s|\theta^b)\dot{\theta}_t^s-(1-F(\theta_t^s|\theta^b))(r(v(\theta^b)-p_t^s)+\dot{p}_t^s),$$

we get that

$$\begin{aligned}&e^{rt}\frac{\partial}{\partial t}u(\theta^b-\delta,t)=\\ &\quad(p_t^s-p_t^b)f(\theta_t^s|\theta^b-\delta)\dot{\theta}_t^s-(1-F(\theta_t^s|\theta^b-\delta))(r(v(\theta^b-\delta)-p_t^s)+\dot{p}_t^s)=\\ &\quad(1-F(\theta_t^s|\theta^b-\delta))\left((p_t^s-p_t^b)\frac{f(\theta_t^s|\theta^b-\delta)}{1-F(\theta_t^s|\theta^b-\delta)}\dot{\theta}_t^s-(r(v(\theta^b-\delta)-p_t^s)+\dot{p}_t^s\right).\end{aligned}$$

Since $v(\theta^b - \delta) \leq v(\theta^b)$ and $\frac{f(\theta_t^s|\theta^b-\delta)}{1-F(\theta_t^s|\theta^b-\delta)} \geq \frac{f(\theta_t^s|\theta^b)}{1-F(\theta_t^s|\theta^b)}$ (by the affiliation of f), it follows that $\frac{\partial}{\partial t}u(\theta^b - \delta, t) \geq 0$. Showing that $\frac{\partial}{\partial t}u(\theta^b + \delta, t) \leq 0$ is analogous. *q.e.d.* □

Approximate the SBS with BNEs in $\mathcal{G}(\cdot|F^*)$ The next lemma constructs price-offer paths $p_\cdot^{s,\varepsilon}$ and $p_\cdot^{b,\varepsilon}$ and BNEs in $\mathcal{G}(p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon}|F^*)$ that approximate the SBS outcome.

Lemma 9. *For any $\varepsilon \geq 0$. Let*

$$\begin{aligned} p_t^{b,\varepsilon} &= \mathbf{p}(0) + (1-\alpha)\varepsilon + (1-\alpha)r(\xi+\varepsilon)t, \\ p_t^{s,\varepsilon} &= \mathbf{p}(1) - \alpha\varepsilon - \alpha r(\xi+\varepsilon)t, \\ \theta_t^{b,\varepsilon} &= v^{-1}\left(\mathbf{p}(1) + \alpha\xi - \alpha r(\xi+\varepsilon)t\right), \\ \theta_t^{s,\varepsilon} &= c^{-1}\left(\mathbf{p}(0) - (1-\alpha)\xi + (1-\alpha)r(\xi+\varepsilon)t\right); \end{aligned} \tag{A.12}$$

for $t \leq T^\varepsilon = \frac{\mathbf{p}(1)-\mathbf{p}(0)-\varepsilon}{r(\xi+\varepsilon)}$. Then

1. *$(\theta_\cdot^{s,\varepsilon}, \theta_\cdot^{b,\varepsilon})$ constitutes a BNE in $\mathcal{G}(p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon}|F^*)$;*

2. *the outcome of $(\theta_\cdot^{s,0}, \theta_\cdot^{b,0})$ in $\mathcal{G}(p_\cdot^{s,0}, p_\cdot^{b,0}|F^*)$ coincides with the SBS outcome;*

3. *$\theta_{T^\varepsilon}^b > \theta_{T^\varepsilon}^s + \varepsilon/\overline{\ell}$;*

4. *the outcome of $(\theta_\cdot^{s,\varepsilon}, \theta_\cdot^{b,\varepsilon}, p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon})$ converge uniformly to $(\theta_\cdot^{s,0}, \theta_\cdot^{b,0}, p_\cdot^{s,0}, p_\cdot^{b,0})$ as $\varepsilon \to 0$.*

Proof. 1) $(\theta_\cdot^{s,\varepsilon}, \theta_\cdot^{b,\varepsilon}, p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon})$ satisfy the premise of Lemma 8 and so, $(\theta_\cdot^{s,\varepsilon}, \theta_\cdot^{b,\varepsilon})$ constitutes a BNE in $\mathcal{G}(p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon}|F^*)$.

2) We can verify using $(A.12)$ that $\theta_{T^0}^{s,0} = \theta_{T^0}^{b,0}$ and

$$\begin{aligned} p_t^{s,0} &= v(\theta_t^{b,0}) - \alpha\xi = \mathbf{p}(\theta_t^{b,0}), \\ p_t^{b,0} &= c(\theta_t^{s,0}) + (1-\alpha)\xi = \mathbf{p}(\theta_t^{s,0}). \end{aligned}$$

Since types $\theta_t^{s,0}$ and $\theta_t^{b,0}$ accept offers $p_t^{b,0}$ and $p_t^{s,0}$, resp., (2.1) obtains. Since threshold types $\theta_t^{s,0}$ and $\theta_t^{b,0}$ prefer to accept at time t rather than any other type $t \leq T^0$, (2.2) and (2.3) obtain where $\theta^* = \theta_{T^0}^{s,0} = \theta_{T^0}^{b,0}$. Thus, the outcome of $(\theta_\cdot^{s,0}, \theta_\cdot^{b,0})$ in $\mathcal{G}(p_\cdot^{s,0}, p_\cdot^{b,0}|F^*)$ coincides with the SBS outcome.

3) $v(\theta_T^{b,\varepsilon}) - c(\theta_T^{s,\varepsilon}) = \xi + \varepsilon$ and so, $v(\theta_T^{b,\varepsilon}) = v(\theta_T^{s,\varepsilon}) + \varepsilon \geq v(\theta_T^{b,\varepsilon}) - \overline{\ell}(\theta_T^{b,\varepsilon} - \theta_T^{s,\varepsilon}) + \varepsilon$ which implies $\theta_T^{b,\varepsilon} > \theta_T^{s,\varepsilon} + \varepsilon/\overline{\ell}$.

4) From $(A.12)$, $(\theta_\cdot^{s,\varepsilon}, \theta_\cdot^{b,\varepsilon}, p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon})$ converge pointwise to $(\theta_\cdot^{s,0}, \theta_\cdot^{b,0}, p_\cdot^{s,0}, p_\cdot^{b,0})$ as $\varepsilon \to 0$ on a compact $[0, T^0]$, and by the continuity of $(\theta_\cdot^{s,\varepsilon}, \theta_\cdot^{b,\varepsilon}, p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon})$ and $(\theta_\cdot^{s,0}, \theta_\cdot^{b,0}, p_\cdot^{s,0}, p_\cdot^{b,0})$, the convergence is also uniform. □

Approximate the BNEs in $\mathcal{G}(\cdot|F^*)$ with BNEs in $\mathcal{G}(\cdot|F)$ For each $(\theta_\cdot^{s,\varepsilon},\theta_\cdot^{b,\varepsilon})$ in $\mathcal{G}(p_\cdot^{s,\varepsilon},p_\cdot^{b,\varepsilon}|F^*)$, we construct an approximating sequence of BNEs $(\theta_\cdot^s,\theta_\cdot^b)$ in $\mathcal{G}(p_\cdot^{s,\varepsilon},p_\cdot^{b,\varepsilon}|F)$,$F \xrightarrow{p} F^*$.[37]

Lemma 10. *Let $T=T^\varepsilon,\theta_\cdot^s=\theta_\cdot^{s,\varepsilon}$, $\theta_\cdot^b=\theta_\cdot^{b,\varepsilon}$ and p_t^s,p_t^b be given by the differential equations $(A.8)$ with the terminal condition $p_T^s=p_T^b$ and the initial condition $p_0^s=p_0^{s,\varepsilon}$. Then $(\theta_\cdot^s,\theta_\cdot^b)$ constitute BNE in $\mathcal{G}(p_\cdot^s,p_\cdot^b|F)$ and $(\theta_\cdot^s,\theta_\cdot^b,p_\cdot^s,p_\cdot^b)$ converge uniformly to $(\theta_\cdot^{s,\varepsilon},\theta_\cdot^{b,\varepsilon},p_\cdot^{s,\varepsilon},p_\cdot^{b,\varepsilon})$ as $F\xrightarrow{p}F^*$.*

Proof. To prove the convergence, we show that $p_\cdot^s,p_\cdot^b$ converge pointwise to $p_\cdot^{s,\varepsilon},p_\cdot^{b,\varepsilon}$ as $F\xrightarrow{p}F^*$. Denote

$$\begin{aligned}\psi_1(t) &= \frac{(1-\alpha)r(\xi+\varepsilon)}{c'(\theta_t^{s,\varepsilon})}\cdot\frac{f(\theta_t^{s,\varepsilon}|\theta_t^{b,\varepsilon})}{1-F(\theta_t^{s,\varepsilon}|\theta_t^{b,\varepsilon})},\\ \psi_2(t) &= \frac{\alpha r(\xi+\varepsilon)}{v'(\theta_t^{b,\varepsilon})}\cdot\frac{f(\theta_t^{b,\varepsilon}|\theta_t^{s,\varepsilon})}{F(\theta_t^{b,\varepsilon}|\theta_t^{s,\varepsilon})}.\end{aligned}$$

Using $(A.12)$ to compute $\dot\theta_t^{s,\varepsilon}$ and $\dot\theta_t^{b,\varepsilon}$, we rewrite the system $(A.8)$ as

$$\begin{aligned}\dot p_t^s &= p_t^s(\psi_1(t)+r)-p_t^b\psi_1(t)-rv(\theta_t^{b,\varepsilon}),\\ \dot p_t^b &= p_t^b(\psi_2(t)+r)-p_t^s\psi_2(t)-rc(\theta_t^{s,\varepsilon}),\end{aligned}$$

By subtracting the second equation from the first and denoting $\Delta p_t=p_t^s-p_t^b$, we get

$$\Delta\dot p_t=\Delta p_t(\psi_1(t)+\psi_2(t)+r)-r(v(\theta_t^{b,\varepsilon})-c(\theta_t^{s,\varepsilon}))$$

with the terminal condition $\Delta p_T=0$, which we can solve to get

$$\Delta p_t=r\int_t^{T_\varepsilon}(v(\theta_u^{b,\varepsilon})-c(\theta_u^{s,\varepsilon}))e^{-\int_t^u(\psi_1(s)+\psi_2(s)+r)ds}du.$$

We can now solve for individual price-offer paths. We have

$$\dot p_t^s=rp_t^s+\Delta p_t\psi_1(t)-rv(\theta_t^{b,\varepsilon}),$$

from which we get

$$\begin{aligned}p_t^s &= e^{rt}(\mathbf{p}(1)-\alpha\varepsilon)+\int_0^t(\Delta p_u\psi_1(u)-rv(\theta_u^{b,\varepsilon}))e^{r(t-u)}du,\\ p_t^b &= p_t^s-\Delta p_t.\end{aligned}$$

By Lemma 9, $\theta_t^b=\theta_t^{b,\varepsilon}>\theta_t^{s,\varepsilon}+\varepsilon/\bar\ell=\theta_t^s+\varepsilon/\bar\ell$ for all $t\le T$, and so, for F sufficiently far in

[37] We do not explicitly index the sequence by corresponding F and ε and simply write $(\theta_\cdot^s,\theta_\cdot^b)$.

the sequence, for all $t \leq T$, $\frac{f(\theta_t^{s,\varepsilon}|\theta_t^{b,\varepsilon})}{1-F(\theta_t^{s,\varepsilon}|\theta_t^{b,\varepsilon})}$ and $\frac{f(\theta_t^{b,\varepsilon}|\theta_t^{s,\varepsilon})}{F(\theta_t^{b,\varepsilon}|\theta_t^{s,\varepsilon})}$ are bounded from above by $c_0\varepsilon$ for some constant c_0. This together with the fact that v' and c' are bounded from below by $\underline{\ell}$ implies that $|\psi_1(t)|$ and $|\psi_2(t)|$ converge to zero pointwise on $[0,T]$ as $F \xrightarrow{p} F^*$. Therefore, price-offer paths and their derivatives converge pointwise on $[0,T]$ as $F \xrightarrow{p} F^*$ and so, by the continuity of $p_\cdot^s, p_\cdot^b, p_\cdot^{s,\varepsilon}, p_\cdot^{b,\varepsilon}$ and their derivatives on the compact $[0,T]$, the convergence is also uniform.

The derivatives of $p_\cdot^{s,\varepsilon}$ and $p_\cdot^{b,\varepsilon}$ are bounded away from zero (from above and below, resp.), and so for F sufficiently far in the sequence, $p_\cdot^s$ and $p_\cdot^b$ are strictly decreasing and increasing, resp. This together with the construction of $(\theta_\cdot^s, \theta_\cdot^b, p_\cdot^s, p_\cdot^b)$, implies that $(\theta_\cdot^s, \theta_\cdot^b, p_\cdot^s, p_\cdot^b)$ satisfies the conditions of Lemma 8 and so, $(\theta_\cdot^s, \theta_\cdot^b)$ constitutes a BNE in $\mathcal{G}(p_\cdot^s, p_\cdot^b|F)$. □

Approximate BNEs in $\mathcal{G}(\cdot|F)$ with PBEs in $\mathcal{G}(F,\Delta)$ We use grim trigger strategies to construct PBEs in $\mathcal{G}(F,\Delta)$ that approximate each BNE $(\theta_\cdot^s, \theta_\cdot^b)$ in $\mathcal{G}(p_\cdot^s, p_\cdot^b|F)$. On the equilibrium path, the seller makes decreasing offers q_n^s and the buyer makes increasing offers q_n^b. Offers do not depend on the type, but the acceptance of the opponent's offer does. Specifically, players follow threshold strategies on-path: in round n, all types of the seller below s_n accept q_{n-1}^b and types above s_n reject it and counter-offer q_n^s; all types of the buyer above b_n accept q_n^s and types below b_n reject it and counter-offer q_n^b. If there is a deviation from the equilibrium path, the play switches to the punishing path (described below) that punishes the deviator.

Construction of the Equilibrium Path: We first construct on-path strategies and show that no type wants to mimic another type in the acceptance decision. We construct the discrete-time approximation of $\theta_\cdot^s$ and $\theta_\cdot^b$ (defined in the previous step) using the Euler method: $s_{N+1} = 1$,$s_N = \theta_T^s$, $b_N = \theta_T^b$, and for $n < N = \lfloor \frac{T}{\Delta} \rfloor$, $s_n = s_{n+1} - \dot{\theta}_{(n+1)\Delta}^s \Delta$ and $b_n = b_{n+1} - \dot{\theta}_{(n+1)\Delta}^b \Delta$. We construct price-offer paths $q_\cdot^s$ and $q_\cdot^b$ backwards in time starting from N and $q_{N+1}^s = q_N^b = q_T^s$ as follows: for $n \leq N$,

$$v(b_n) - q_n^s = e^{-r\Delta_s} \frac{F(s_{n+1}|b_n) - F(s_n|b_n)}{1 - F(s_n|b_n)} (v(b_n) - q_n^b) + e^{-r\Delta} \frac{1 - F(s_{n+1}|b_n)}{1 - F(s_n|b_n)} (v(b_n) - q_{n+1}^s), \tag{A.13}$$

$$q_{n-1}^b - c(s_n) = e^{-r\Delta_b} \frac{F(b_n|s_n) - F(b_{n-1}|s_n)}{F(b_{n-1}|s_n)} (q_n^s - c(s_n)) + e^{-r\Delta} \frac{F(b_n|s_n)}{F(b_{n-1}|s_n)} (q_n^b - c(s_n)). \tag{A.14}$$

Denote by $(\overline{s}_\cdot, \overline{b}_\cdot, \overline{q}_\cdot^s, \overline{q}_\cdot^b)$ the linear extrapolation of $(s_\cdot, b_\cdot, q_\cdot^s, q_\cdot^b)$ to the continuos time on $[0,T]$.

Lemma 11. *$(\overline{s}_\cdot, \overline{b}_\cdot, \overline{q}_\cdot^s, \overline{q}_\cdot^b)$ converges uniformly to $(\theta_\cdot^s, \theta_\cdot^b, p_\cdot^s, p_\cdot^b)$ as $F \xrightarrow{p} F^*$. When both players use threshold strategies, no player wants to deviate to a different acceptance strategy.*

Proof. To prove the first part, the convergence of $\overline{s}_\cdot$ and $\overline{b}_\cdot$ is by construction. Next, rewrite equation $(A.13)$ as follows

$$\frac{1-e^{-r\Delta}}{\Delta}(v(b_n)-q^s_n)-e^{-r\Delta}\frac{q^s_n-q^s_{n+1}}{\Delta}=\frac{F(s_{n+1}|b_n)-F(s_n|b_n)}{\Delta(1-F(s_n|b_n))}(e^{-r\Delta_s}(v(b_n)-q^b_n)-e^{-r\Delta}(v(b_n)-q^s_{n+1})).$$

Since $\dot{\theta}^s_\cdot$ is positive and bounded uniformly on $[0,T)$, by the construction of s_n, there is an upper bound c_0 on $\frac{1}{\Delta}|s_{n+1}-s_n|$ for $n<N$. Choose F sufficiently close to F^* so that $\sup_{(\theta^s,\theta^b):|\theta_b-\theta_s|>\varepsilon/\bar{\ell}} f(\theta^s|\theta^b)<\varepsilon/\bar{\ell}$. Then since $s_n\le s_N=\theta^s_T<\theta^b_T-\varepsilon/\bar{\ell}=b_N-\varepsilon/\bar{\ell}\le b_n-\varepsilon/\bar{\ell}$, we have

$$0\le\frac{F(s_{n+1}|b_n)-F(s_n|b_n)}{\Delta(1-F(s_n|b_n))}\le\frac{|s_{n+1}-s_n|\varepsilon}{\Delta\bar{\ell}(1-\varepsilon/\bar{\ell})}\le\frac{c_0\varepsilon}{\bar{\ell}(1-\varepsilon/\bar{\ell})}\equiv C\varepsilon,$$

for $n<N$. Thus,

$$\frac{1-e^{-r\Delta}}{\Delta}(v(b_n)-q^s_n)-e^{-r\Delta}\frac{q^s_n-q^s_{n+1}}{\Delta}=C(\varepsilon) \tag{A.15}$$

where $C(\varepsilon)$ is some function that is bounded in absolute value by $C\varepsilon$. This implies that $\overline{q}^s_\cdot$ converges pointwise to $p^s_\cdot$ as $F\xrightarrow{p}F^*$, and also uniformly by the continuity of $\overline{q}^s_\cdot$ and $p^s_\cdot$ on a compact $[0,T]$. Moreover, the left-derivative of $\overline{q}^s_\cdot$ equals to $\frac{q^s_n-q^s_{n+1}}{\Delta}$ for some n, and from equation $(A.15)$, it converges uniformly to $\dot{p}^s_\cdot$ as $F\xrightarrow{p}F^*$. This implies that $\overline{q}^s_\cdot$ is strictly decreasing for F sufficiently far in the sequence. The uniform convergence of $\overline{q}^b_\cdot$ and its left-derivative is proven analogously.

To prove the second part, first observe that we can follows the same line of argument as the proof of the single crossing difference condition in Lemma 8 (Claim 1) to prove the following claim.

Claim 2. *Buyer's types $\theta^b>b_n$ and seller's types $\theta^s<s_n$, strictly prefer to accept in round $n-1$ to accepting in round n. Buyer's types $\theta^b<b_n$ and seller's types $\theta^s>s_n$, strictly prefer to accept in round $n+1$ to accepting in round n.*

Claim 2 implies that no player wants to deviate from the threshold strategy in the acceptance decision. □

Construction of the Punishing Path: We now construct punishing path for deviations to offers that are off the equilibrium path and show that such are not profitable. Suppose that the seller deviates from the price-offer path $q^s_\cdot$ in round n (the construction of the punishing path for the buyer is symmetric). Then specify that the buyer assigns probability 1 to the lowest remaining seller's type, s_{n-1}. The next lemma states that there is a continuation equilibrium that is efficient in deterring deviations to off-path offers.

Lemma 12 (Coasian Property). *Let $\underline{s}<\theta^s_T$ and $\overline{b}>\theta^b_T$. Suppose after some history, the buyer assigns probability 1 to type $\underline{s}$ and the seller's beliefs are $F(\theta^b|\theta^s,\theta^b<\overline{b})$. Then for any*

$\varepsilon_0 > 0$ *there is* $\overline{\Delta}$ *(that does not depend on* $\underline{s}$ *and* $\overline{b}$*) such that for all* $\Delta < \overline{\Delta}$ *, there is a continuation PBE strategies in which the seller's initial offer is below* $\mathbf{p}(\underline{s}, 0) + \varepsilon_0$*.*

Proof. We construct the continuation PBE in which the buyer's types pool on the unacceptable offer above $\overline{\mathbf{p}}(1)$ and the seller (with commonly known type $\underline{s}$) makes offers to screen the buyer's type. We can follow the step in the proof of Proposition 1 in Fudenberg et al. (1985) to construct the screening path of the seller in which the last seller's offer equals $\overline{\mathbf{p}}(\underline{s}, 0)$. By the Uniform Coase Conjecture in Ausubel and Deneckere (1989), there is $\overline{\Delta}$ (that does not depend on $\underline{s}$ and $\overline{b}$) such that for all $\Delta < \overline{\Delta}$, there is a PBE in which the seller's initial offer is below $\mathbf{p}(\underline{s}, 0) + \varepsilon$. To guarantee that the buyer does not have incentives to make acceptable offers, we specify that if the buyer deviates from making the unacceptable offer, the seller assigns probability 1 to type $\overline{b}$. Specify that after histories in which the seller assigns probability 1 to a certain type of the buyer (there are two possibilities: either 0 or $\overline{b}$), the continuation play is as in the complete information game.[38] Thus, if the buyer deviates, he trades at price at best $\underline{\mathbf{p}}(\underline{s}, \overline{b})$ which is strictly higher than $\mathbf{p}(\underline{s}, 0) + \varepsilon_0$ when $\overline{b} > \theta_T^b > \underline{s}$ and ε is small. Therefore, such a deviation is not profitable for sufficiently small Δ. □

We now show that players do not have incentives to deviate to off-path price offers. By Lemma 12, if such a type deviates in round n, she trades at a price at best $\mathbf{p}(s_n, 0) + \varepsilon_0$. Thus, the price of the punishing path converges uniformly to $\mathbf{p}(\theta_t^s)$ as $\Delta \to 0$ by Lemma 11. On the other hand, on the equilibrium path the buyer offers p_n^b converge uniformly to $p_t^s = \mathbf{p}(\theta_t^s) + (1-\alpha)\varepsilon > \mathbf{p}(\theta_t^s)$ by Lemmas 9 and 10. Therefore, any deviation from the on-path offers is not profitable when Δ is sufficiently small. This completes the construction of the PBEs in $\mathcal{G}(F, \Delta)$ and completes the proof of part 2 of Theorem 1.

A.2 Proofs for the OTC Model

Steady-State Distribution

Proof of Lemma 1. We first derive the steady-state distribution of times spent in the match. For $\theta \in \Theta_L \cup \Theta_M$ and $u \in [0, t(\theta)]$, let $G(\theta, u)$ be the mass of sellers that have spent time u negotiating the price of an asset with quality θ. During the time interval du, a fraction $(y_u + y_d)du$ of matches is destroyed due to the switching of intrinsic types, and for an asset with quality θ, a mass $\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)du$ of agents enters the bargaining stage. Hence, the change in the mass of sellers that have spent in the match less than u is $(1 - y_u du - y_d du)G(\theta, u - du) + \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)du - G(\theta, u)$, which equals 0 in the steady-state. Thus,

$$\frac{\partial}{\partial u} G(\theta, u) = -(y_u + y_d)G(\theta, u) + \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta), \tag{A.16}$$

[38] This is the only place where we use the weakening of the support restriction on beliefs.

which together with $G(\theta,0)=0$ gives:

$$G(\theta,u)=\frac{1-e^{-(y_u+y_d)u}}{y_u+y_d}\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta).$$

The total mass of sellers in the bargaining stage for asset θ is equal to $\mu_{sm}(\theta)$ which translates into $G(\theta,t(\theta))=\mu_{sm}(\theta)$ or equivalently

$$\mu_{sm}(\theta)=\frac{1-e^{-(y_u+y_d)t(\theta)}}{y_u+y_d}\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta). \tag{A.17}$$

Let $\gamma(\theta)$ be the intensity with which agents leave the match. During the time interval du, sellers that have already spent time $[t(\theta)-du,t(\theta)]$ in the bargaining stage complete their trades. Thus, $\gamma(\theta)=\frac{\partial}{\partial u}G(\theta,t(\theta))$ or

$$\gamma(\theta)=\lambda M_{bu}(\phi)\mu_{su}(\theta)e^{-(y_u+y_d)t(\theta)}\sigma(\theta). \tag{A.18}$$

Now, we derive the distribution M. For $\theta\in\Theta_I$, $\mu_{su}(\theta)=\frac{y_d}{y_u+y_d}$, $\mu_{bu}(\theta)=\frac{y_u}{y_u+y_d}$, $\mu_{sm}(\theta)=\mu_{bm}(\theta)=0$ and we only consider $\theta\in\Theta_L\cup\Theta_M$. In the steady state, $\mu_{su}(\theta),\mu_{bu}(\theta),M_{bu}(\phi),M_{su}(\phi)$ stay constant over time:

$$\begin{cases} y_d\mu_{sm}(\theta)+y_d\mu_{bu}(\theta)=y_u\mu_{su}(\theta)+\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta),\\ y_u\mu_{sm}(\theta)+y_u\mu_{su}(\theta)+\gamma(\theta)=y_d\mu_{bu}(\theta),\\ y_uM_{bm}(\Theta_L\cup\Theta_M)+y_uM_{su}(\phi)=y_dM_{bu}(\phi)+\lambda M_{bu}(\phi)\int_0^1\mu_{su}(\theta)\sigma(\theta)d\theta,\\ y_dM_{bm}(\Theta_L\cup\Theta_M)+y_dM_{bu}(\phi)+\int_0^1\gamma(\theta)d\theta=y_uM_{su}(\phi), \end{cases} \tag{A.19}$$

where the left-hand sides are the inflows into and the right-hand sides are the outflows from $\mu_{su}(\theta),\mu_{bu}(\theta),M_{bu}(\phi),M_{su}(\phi)$, respectively. Combining the system $(A.19)$ with the balance conditions $(2.5)-(2.7)$ and $(A.17)-(A.18)$, we get:

$$\begin{cases} y_d\mu_{sm}(\theta)+y_d\mu_{bu}(\theta)-y_u\mu_{su}(\theta)-\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)=0,\\ y_u\mu_{sm}(\theta)+y_u\mu_{su}(\theta)-y_d\mu_{bu}(\theta)+\lambda M_{bu}(\phi)\mu_{su}(\theta)e^{-(y_u+y_d)t(\theta)}\sigma(\theta)=0,\\ \mu_{su}(\theta)+\mu_{bu}(\theta)+\mu_{sm}(\theta)=1,\\ y_uM_{sm}(\Theta_L\cup\Theta_M)+y_uM_{su}(\phi)-y_dM_{bu}(\phi)-\lambda M_{bu}(\phi)\left(\int_0^1\mu_{su}(\theta)\sigma(\theta)d\theta\right)=0,\\ M_{su}(\phi)+M_{bu}(\phi)+M_{sm}(\Theta_L\cup\Theta_M)=a-1,\\ (y_u+y_d)\mu_{sm}(\theta)-(1-e^{-(y_u+y_d)t(\theta)})\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)=0,\\ y_dM_{sm}(\Theta_L\cup\Theta_M)+y_dM_{bu}(\phi)-y_uM_{su}(\phi)+\lambda M_{bu}(\phi)\int_0^1\mu_{su}(\theta)e^{-(y_u+y_d)t(\theta)}\sigma(\theta)d\theta=0. \end{cases} \tag{A.20}$$

The rank of the system is five and we eliminate the last two equations to make the system have a full rank. From the first three equations in $(A.20)$,

$$\begin{cases} \mu_{su}(\theta) &= \dfrac{y_d}{y_u + y_d + \lambda M_{bu}(\phi)\sigma(\theta)}, \\ \mu_{bm}(\theta) &= \dfrac{\lambda M_{bu}(\phi)\sigma_\theta(1 - e^{-(y_u+y_d)t(\theta)})y_d}{(y_u + y_d)(y_u + y_d + \lambda M_{bu}(\phi)\sigma(\theta))}, \\ \mu_{bu}(\theta) &= \dfrac{y_u(y_u + y_d) + \lambda M_{bu}(\phi)\sigma(\theta)(y_u + y_d e^{-(y_u+y_d)t(\theta)})}{(y_u + y_d)(y_u + y_d + \lambda M_{bu}(\phi)\sigma(\theta))}. \end{cases} \tag{A.21}$$

From the forth and fifth equations in $(A.20)$,

$$\begin{cases} y_u M_{sm}(\Theta_L \cup \Theta_M) + y_u M_{su}(\phi) - y_d M_{bu}(\phi) - \lambda M_{bu}(\phi) \int_{\Theta_L \cup \Theta_M} \mu_{su}(\theta)\sigma(\theta)d\theta = 0, \\ M_{su}(\phi) + M_{bu}(\phi) + M_{sm}(\Theta_L \cup \Theta_M) = a - 1. \end{cases} \tag{A.22}$$

In $(A.22)$, subtracting the first equation from the second equation multiplied by y_u, plugging in $\mu_{su}(\theta)$ from the first line of $(A.21)$, and making the change of variables $M_{bu}(\phi) = \frac{\Lambda_s}{\lambda}$, we get the equation (4.1). The left-hand side of (4.1) is strictly increasing in Λ_s and the right-hand side is strictly decreasing in Λ_s unless $\sigma(\theta) = 0$ for all θ which does not hold in equilibrium. At $\Lambda_s = 0$, the left-hand side is zero and the right-hand side equals $\frac{y_u}{y_u+y_d}(a - 1) > 0$. Thus, equation (4.1) has a unique positive solution. This completes the characterization of M. Quantities $\mu_{su}(\theta), \mu_{bu}(\theta), \mu_{bm}(\theta), \mu_{sm}(\theta)$ are given by $(A.21)$, $M_{bu}(\phi) = \frac{\Lambda_s}{\lambda}$ and $M_{su}(\phi)$ is found from $(A.22)$. Finally, using the expression for $\mu_{su}(\theta)$ in the first line of $(A.21)$, we find that Λ_b is given by (4.2) and that $F_L(\theta) = \frac{\int_{[0,\theta]\cap\Theta_L} \mu_{su}(\theta)d\theta}{M_{su}(\Theta_L)}$ is uniform conditional on $\theta \in \Theta_L$. □

By definition, $\gamma(\theta)$ gives the trade volume and since each asset is in the unit supply, it is also the asset turnover. Using $(A.18)$ and the characterization of M above, we get that $\gamma(\theta)$ is given by (2.8).

Analysis of Value Functions

I first express value functions through $\Lambda_s, \Lambda_b, \sigma$ and x.

Denote by U_s the utility of the seller who owns an asset and does not participate in the search market. U_s can be found from equation (4.6) by setting $\sigma(\theta) = 0$:

$$U_s(\theta) = \frac{1}{r}\left(k\theta - \frac{r + y_d}{r + y_u + y_d}\ell\right). \tag{A.23}$$

The utility of sellers of illiquid assets is given by $V_{su}(\theta) = U_s(\theta)$ for $\theta \in \Theta_I$. The next lemma simplifies equations $(4.3), (4.4), (4.5), (4.6)$ and shows that V_{bu} and $V_{su}(\phi)$ can be expressed through $V_{bu}(\phi)$ and V_{su}.

Lemma 13. *For all* $\theta \in [0,1]$,

$$V_{bu}(\theta) = \frac{k\theta + y_d V_{su}(\theta)}{r + y_d}, \tag{A.24}$$

$$V_{su}(\phi) = \frac{y_u V_{bu}(\phi)}{r + y_u}, \tag{A.25}$$

$$V_{bu}(\phi) = \Lambda_b \frac{r + y_u}{r\rho} \left(\mathbb{E}\left[V_{bm}(\theta) | \theta \in \Theta_L \right] - V_{bu}(\phi) \right), \tag{A.26}$$

$$V_{su}(\theta) = U_s(\theta) + \sigma(\theta)\Lambda_s \frac{r + y_d}{r\rho} \left(V_{sm}(\theta) - V_{su}(\theta) \right). \tag{A.27}$$

I next turn to the outcome of the bargaining stage and express value functions V_{bm} and V_{sm} of matched agents through $V_{bu}(\phi)$ and V_{su}. In Subsection 4.2, we introduced functions $\hat{v}$ and $\hat{c}(\theta)$ as the value functions of the buyer and the seller who remain in the match and never trade with the current partner. By the definition, $\hat{c}(\theta)$ is given by the Bellman equation

$$r\hat{c}(\theta) = k\theta - \ell + y_u(V_{bu}(\theta) - \hat{c}(\theta)) + y_d(V_{su}(\theta) - \hat{c}(\theta)),$$

and so, using $(A.23)$ and $(A.24)$,

$$\hat{c}(\theta) = \frac{1}{\rho}(\underline{v}(\theta) + y_u V_{bu}(\theta) + y_d V_{su}(\theta)) = \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta).$$

Analogously, $\hat{v}$ is given by the Bellman equation

$$r\hat{v} = y_u(V_{bu}(\phi) - \hat{v}) + y_d(V_{su}(\phi) - \hat{v}),$$

and so, using $(A.25)$,

$$\hat{v} = \frac{1}{\rho}(y_u V_{bu}(\phi) + y_d V_{su}(\phi)) = \frac{y_u}{r + y_u} V_{bu}(\phi).$$

Therefore, functions v and c introduced in (4.8) and (4.7) are given by

$$c(\theta) = \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi), \tag{A.28}$$

$$v(\theta) = \frac{k\theta}{r + y_d} + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi). \tag{A.29}$$

Observe that $v(\theta) - c(\theta) = \frac{\ell}{\rho}$. The next lemma expresses value functions of matched agents through x, V_{su} and $V_{bu}(\phi)$.

Lemma 14. *For any $\theta \in [0,1]$,*

$$V_{bm}(\theta) = \alpha\xi x(\theta) + \frac{y_u}{r+y_u} V_{bu}(\phi), \tag{A.30}$$

$$V_{sm}(\theta) = (1-\alpha)\xi x(\theta) + \frac{r}{r+y_d} U_s(\theta) + \frac{y_d}{r+y_d} V_{su}(\theta). \tag{A.31}$$

Proof. Given that the trade at the bargaining stage is not immediate, the utility of matched agents depends on time and for $\tau \in \{bm, sm\}$ we index $V_\tau(t,\theta)$ by time t. Observe that $V_{bm}(t(\theta),\theta) = V_{bu}(\theta) - p(\theta)$ and $V_{sm}(t(\theta),\theta) = p(\theta) + V_{su}(\phi)$. Moreover, the following Bellman equation holds for $V_{bm}(t,\theta)$:

$$rV_{bm}(t,\theta) = y_u(V_{bu}(\phi) - V_{bm}(t,\theta)) + y_d(V_{su}(\phi) - V_{bm}(t,\theta)) + \tfrac{\partial}{\partial t} V_{bm}(t,\theta).$$

I solve this differential equation to get

$$V_{bm}(t,\theta) = (V_{bu}(\theta) - p(\theta))\, e^{-\rho(t(\theta)-t)} + \frac{y_u V_{bu}(\phi)}{r+y_u}\left(1 - e^{-\rho(t(\theta)-t)}\right).$$

Using (2.1), $(A.24)$, and $V_{bm}(0,\theta) = V_{bm}(\theta)$, we get $(A.30)$. Symmetrically, the Bellman equation for $V_{sm}(t,\theta)$ is

$$rV_{sm}(t,\theta) = k\theta - \ell + y_u(V_{bu}(\theta) - V_{sm}(t,\theta)) + y_d(V_{su}(\theta) - V_{sm}(t,\theta)) + \tfrac{\partial}{\partial t} V_{sm}(t,\theta),$$

which has solution

$$V_{sm}(\theta) = (p(\theta) + V_{su}(\phi))\, e^{-\rho(t(\theta)-t)} + \frac{1}{\rho}\left(k\theta - \ell + y_u V_{bu}(\theta) + y_d V_{su}(\theta)\right)\left(1 - e^{-\rho(t(\theta)-t)}\right).$$

Using (2.1), $(A.24)$, $(A.25)$, and $V_{sm}(0,\theta) = V_{sm}(\theta)$, we get $(A.31)$. □

Proof of Lemma 2. Combining $(A.24)$ and $(A.30)$, we get (4.10). The buyer prefers to trade with the seller of asset θ if and only if $V_{bm}(\theta) \geq V_{bu}(\phi)$, or combining $(A.30)$ and (4.10), we get the condition (4.9). □

It follows from $(A.27)$ and $(A.31)$ that for $\theta \in \Theta_L \cup \Theta_M$ function V_{su} is given by

$$V_{su}(\theta) = U_s(\theta) + (1-\alpha)\frac{r+y_d}{r}\frac{\sigma(\theta)\Lambda}{\rho + \sigma(\theta)\Lambda}\xi x(\theta) \tag{A.32}$$

Equation $(A.32)$ implies that $V_{su}(\theta) > U_s(\theta)$ whenever $x(\theta) > 0$ and so, sellers always prefer to trade.

Solution of the Model

I first derive equilibrium strategy σ and liquidity profile x.

Lemma 15. *x is given by* (4.11)*.*

Proof of Lemma 15. we first show that (2.2) and (2.3) implies (4.11). Consider $\theta > \theta^*$ and rewrite the maximization problem in (2.2) as follows

$$\theta \in \arg\max_{\theta' \in [\theta^*, 1]} x(\theta')(v(\theta) - p(\theta')). \tag{A.33}$$

By the envelope theorem (Milgrom and Segal (2002)), function $x(\theta)(v(\theta) - p(\theta))$ is absolutely continuous and at differentiability points satisfies

$$x'(\theta)(v(\theta) - p(\theta)) - x(\theta)p'(\theta) = 0, \tag{A.34}$$

or using (2.1),

$$\frac{x'(\theta)}{x(\theta)} = \frac{v'(\theta)}{\alpha\xi}. \tag{A.35}$$

Together with the $x(1) = 1$, $(A.35)$ implies (4.11). The argument for $\theta < \theta^*$ is analogous. In this case, x is given by

$$\frac{x'(\theta)}{x(\theta)} = -\frac{c'(\theta)}{(1-\alpha)\xi}, \tag{A.36}$$

which together with $x(0) = 1$ implies (4.11). □

Corollary 1. *For differentiability points $\theta > \theta^*$ of v, $x'(\theta) = 0$ if and only if $v'(\theta) = 0$, and for differentiability points $\theta < \theta^*$ of c, $x'(\theta) = 0$ if and only if $c'(\theta) = 0$.*

Proof. Follows immediately from $(A.35)$ and $(A.36)$. □

Proof of Lemma 3. The analysis proceeds in a series of claims.

Claim 3. *If $x(\theta) = \underline{x}$ for some set (θ', θ''), then $\sigma(\theta) \in (0, 1)$ for almost every $\theta \in (\theta', \theta'')$.*

Proof. Suppose that $x(\theta) = \underline{x}$, but $\sigma(\theta) = 0$ for $\theta \in (\theta', \theta'')$ (the argument is identical for $\sigma(\theta) = 1$). By $(A.32)$, V_{su} is strictly increasing on (θ', θ'') and so, by $(A.28)$ and $(A.29)$, v and c are strictly increasing, which contradicts Corollary 1. *q.e.d.*

Claim 4. *There exist $\check{\theta} \leq \underline{\theta} \leq \theta^* \leq \overline{\theta} \leq \check{\theta}$ such that $\Theta_L = [0, \check{\theta}] \cup [\hat{\theta}, 1]$, $\Theta_M = (\check{\theta}, \underline{\theta}] \cup [\overline{\theta}, \hat{\theta})$,* and *$\Theta_I = (\underline{\theta}, \overline{\theta})$.*

Proof. By Lemma 2, buyers accept only asset qualities with $x(\theta) \geq \underline{x}$. By Lemma 15, x has a U-shape and so, there exist $\check{\theta} \leq \underline{\theta} \leq \theta^* \leq \overline{\theta} \leq \check{\theta}$ such that $x(\theta) \geq \underline{x}$ on $[0, \underline{\theta}] \cup [\overline{\theta}, 1]$ and $x(\theta) > \underline{x}$ on $[0, \check{\theta}] \cup [\hat{\theta}, 1]$, which combined with Claim 3 gives the result. *q.e.d.*

Claim 5. $\check{\theta} < \hat{\theta}$ *implies* $\check{\theta} < \underline{\theta}$.

Proof. Suppose to contradiction there exists $\check{\theta} = \underline{\theta} < \hat{\theta}$. Then there is an increasing sequence of $\{\theta_i'\} \subset [0, \check{\theta}]$ and a decreasing sequence $\{\theta_i''\} \subset \Theta_I$ both converging to θ'. From $(A.28)$ and $(A.32)$, this implies that $c(\theta_i') > c(\theta_i'')$ while $\theta_i' < \theta_i''$, which contradicts the monotonicity of c. *q.e.d.*

Claim 6. $\overline{\theta} = \hat{\theta}$.

Proof. Suppose to contradiction there exists $\overline{\theta} = \hat{\theta}$. Then there is a decreasing sequence of $\{\theta_i'\} \subset [\hat{\theta}, 1]$ and an increasing sequence $\{\theta_i''\} \subset \Theta_M$ both converging to θ'. Corollary 1 implies that $\{v(\theta_i'')\}$ is constant, and so, from $(A.29)$ and $(A.32)$, $\{\sigma(\theta_i'')\}$ is decreasing. On the other hand, $\sigma(\theta_i') = 1$ which contradicts the continuity of v at θ'. *q.e.d.*

It follows from Claims 1-4 that the only possibilities are: a) $\check{\theta} = \underline{\theta} = \overline{\theta} = \check{\theta}$, b) $\check{\theta} < \underline{\theta} = \overline{\theta} = \check{\theta}$, c) $\check{\theta} < \underline{\theta} < \overline{\theta} = \check{\theta}$. □

Lemma 16.

$$\theta = 1 + \frac{r}{k}\alpha\xi \ln x(\theta) + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho+\Lambda_s}(1-x(\theta)), \ \textit{for } \theta \geq \hat{\theta}, \tag{A.37}$$

$$\theta = -\frac{r}{k}(1-\alpha)\xi \ln x(\theta) + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho+\Lambda_s}(1-x(\theta)), \ \textit{for } \theta \leq \check{\theta}. \tag{A.38}$$

Moreover, v and c are strictly increasing on Θ_L.

Proof. For almost every $\theta > \hat{\theta}$, plugging $v'(\theta)$ from $(A.29)$ into $(A.35)$, we get

$$\frac{x'(\theta)}{x(\theta)} = \frac{\overline{v}'(\theta) + y_d V_{su}'(\theta)}{\alpha\xi(r+y_d)}.$$

By $(A.32)$,

$$V_{su}'(\theta) = \frac{k}{r} + (1-\alpha)\frac{r+y_d}{r}\frac{\Lambda_s}{\rho+\Lambda_s}\xi x'(\theta) \tag{A.39}$$

and so,

$$x'(\theta) = \frac{k}{\xi r\left(\frac{\alpha}{x(\theta)} - \frac{y_d}{r}\frac{\Lambda_s}{\rho+\Lambda_s}(1-\alpha)\right)}, \tag{A.40}$$

which together with $x(1) = 1$ gives $(A.37)$. From (2.4), the denominator of $(A.40)$ is positive.[39] Therefore, $x'(\theta) > 0$ and so $(A.35)$ implies $v'(\theta) > 0$.

Analogously, plugging in $c'(\theta)$ from $(A.28)$ into $(A.36)$,

[39] Indeed,

$$\frac{\alpha}{x(\theta)} \geq \alpha \geq \frac{y_d}{r}(1-\alpha) > \frac{y_d}{r}\frac{\Lambda_s}{\rho+\Lambda_s}(1-\alpha).$$

$$\frac{x'(\theta)}{x(\theta)} = -\frac{rU_s'(\theta) + y_d V_{su}'(\theta)}{(1-\alpha)\xi(r+y_d)},$$

and using $(A.39)$ for $V_{su}'(\theta)$, we get

$$x'(\theta) = -\frac{k}{\xi r(1-\alpha)\left(\frac{1}{x(\theta)} + \frac{y_d}{r}\frac{\Lambda_s}{\rho+\Lambda_s}\right)}, \tag{A.41}$$

which together with $x(0) = 1$ gives $(A.38)$. From $(A.41)$, $x'(\theta) < 0$ and so $(A.36)$ implies $c'(\theta) < 0$. □

It next to find conditions to determine thresholds $\hat{\theta}, \check{\theta}, \underline{\theta}$. By Lemma 16 and the fact that $\underline{x} = x(\check{\theta}) = x(\hat{\theta})$:

$$\hat{\theta} = 1 + \frac{r}{k}\alpha\xi \ln \underline{x} + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho+\Lambda_s}(1-\underline{x}), \tag{A.42}$$

$$\check{\theta} = -\frac{r}{k}(1-\alpha)\xi \ln \underline{x} + \frac{y_d}{k}(1-\alpha)\xi\frac{\Lambda_s}{\rho+\Lambda_s}(1-\underline{x}). \tag{A.43}$$

For each $\theta \in \Theta_M$, $x(\theta) = \underline{x}$ and so, $c(\theta) = c(\check{\theta})$ by Corollary 1. Therefore, by $(A.28)$,

$$V_{su}(\theta) = V_{su}(\check{\theta}) - \frac{r}{y_d}(U_s(\theta) - U_s(\check{\theta})). \tag{A.44}$$

Using $(A.32)$ and $x(\check{\theta}) = \underline{x}$,

$$V_{su}(\theta) - U_s(\theta) = \frac{r+y_d}{r}\left(\frac{k}{y_d}(\check{\theta} - \theta) + (1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}\right). \tag{A.45}$$

Threshold $\underline{\theta}$ is determined as the minimum of $\hat{\theta}$ and the solution to the equation $U_s(\underline{\theta}) = V_{su}(\underline{\theta})$ and so, from $(A.45)$,

$$\underline{\theta} = \min\left\{\hat{\theta}, \check{\theta} + (1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}\right\}. \tag{A.46}$$

This completes the description of x for given Λ_s and $\underline{x}$. The next lemma determines σ.

Lemma 17. *For given Λ_s and $\underline{x}$,*

$$\sigma(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, \check{\theta}] \cup [\hat{\theta}, 1], \\ 0, & \text{if } \theta \in [\underline{\theta}, \hat{\theta}), \\ \frac{\rho}{\Lambda_s}\frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x} - \frac{k}{y_d}(\theta - \check{\theta})}{(1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi\underline{x} + \frac{k}{y_d}(\theta-\check{\theta})}, & \text{if } \theta \in (\check{\theta}, \underline{\theta}). \end{cases} \tag{A.47}$$

Proof. I only need to determine $\sigma(\theta)$ for $\theta \in \Theta_M$. It follows from $(A.32), (A.45)$ and $x(\theta) = \underline{x}$:

$$\sigma(\theta) = \frac{\rho}{\Lambda_s} \frac{\frac{r}{r+y}(V_{su}(\theta) - U_s(\theta))}{(1-\alpha)\xi\underline{x} - \frac{r}{r+y}(V_{su}(\theta) - U_s(\theta))} = \frac{\rho}{\Lambda_s} \left(\frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x} - \frac{k}{y_d}(\theta - \check{\theta})}{(1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi\underline{x} + \frac{k}{y_d}(\theta - \check{\theta})} \right).$$

□

Proof of Theorem 2. For given Λ_s and $\underline{x}$, we can determine equilibrium strategy σ from Lemma 17 and x from Lemma 16 where $\check{\theta}, \underline{\theta}$, and $\hat{\theta}$ are expressed from $(A.42), (A.43)$, and $(A.46)$. Lemma 1 describes the steady-state distribution for given σ and x. Hence, we need to show that there is unique pair of Λ_s and $\underline{x}$ that satisfies the equilibrium.

Lemma 19 in the Online Appendix shows that equation (4.1) implies equation (4.13). Next, we derive (4.12). Combining (4.2) and (4.9), we get

$$\rho \quad \geq \quad \frac{\lambda y_d}{y_u + y_d + \Lambda_s} \int_{x(\theta) > \underline{x}} \left(\frac{x(\theta)}{\underline{x}} - 1 \right) d\theta, \tag{A.48}$$

which holds as equality whenever $L < 1$. It follows from $(A.42)$ and $(A.43)$ that

$$L \quad = \quad 1 - \hat{\theta} + \check{\theta} = -\frac{r\xi}{k} \ln \underline{x} \tag{A.49}$$

From $(A.49)$ it follows that alternatively the equilibrium is pinned down by the pair Λ_s and L. Given $(A.37)$ and $(A.38)$, we can explicitly calculate

$$X = \int_{x(\theta) > \underline{x}} x(\theta) d\theta = \int_{\hat{\theta}}^{1} x(\theta) d\theta + \int_{0}^{\check{\theta}} x(\theta) d\theta = \int_{\underline{x}}^{1} x \frac{d\theta(x)}{dx} dx - \int_{\underline{x}}^{1} x \frac{d\theta(x)}{dx} dx = \frac{r\xi}{k}(1 - \underline{x}) \tag{A.50}$$

and combined with $(A.49)$ and $(A.48)$ this gives (4.12). Therefore, equilibrium Λ_s and L are pinned down by (4.12) and (4.13). Denote by Λ_s^1, Λ_s as a function of L expressed from equation (4.12), and by Λ_s^2, Λ_s as a function of L expressed from equation (4.13). Lemma 21 in the Online Appendix shows that there is a unique solution to this system. □

Proof of Proposition 2. To get (5.1), plug functions c and v from $(A.28)$ and $(A.29)$ into (2.1) and then substitute $V_{bu}(\phi)$ and $V_{su}(\theta)$ from (4.10) and $(A.32)$. □

Comparative Statics

Proof of Proposition 3. Suppose k increases. The equilibrium is characterized by (4.12) and (4.13). Consider functions Λ_s^1 and Λ_s^2 introduced in the proof of Theorem 2. Since $\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L} - 1\right)\right)'_k = \frac{\xi r}{k^2}\left(1 + e^{\frac{k}{\xi r}L}\left(\frac{k}{\xi r}L - 1\right)\right) > 0$, Λ_s^1 is increasing in k and so, an increase in k leads to the upward shift of Λ_s^1 and as a result, to an increase in Λ_s and a decrease in L. To show that $\underline{x}$ decreases, we use $(A.49)$ to reformulate equilibrium conditions in terms of $(\Lambda_s, \underline{x})$:

$$\begin{cases} \Lambda_s = \frac{\xi r \lambda y_d}{k\rho}\left(\frac{1}{\underline{x}} - 1 + \ln \underline{x}\right) - (y_u + y_d), \\ -\ln \underline{x} = \frac{k(y_u+y_d+\Lambda_s)}{r\xi y_d \Lambda_s}\left(y_u(a-1) - (y_u+y_d)\frac{\Lambda_s}{\lambda} - h(\Lambda_s)\right). \end{cases} \tag{A.51}$$

Let functions $\tilde{\Lambda}_s^1$ and $\tilde{\Lambda}_s^2$ by such that $\Lambda_s = \tilde{\Lambda}_s^1(\underline{x})$ solves the first equation in the system and $\Lambda_s = \tilde{\Lambda}_s^2(\underline{x})$ solves the second equation. It follows from the monotonicity of Λ_s^1 and Λ_s^2 that $\tilde{\Lambda}_s^1$ is decreasing and $\tilde{\Lambda}_s^2$ is increasing. Since the right-hand side of the first equation is decreasing in k and the right-hand side of the second equation is is increasing in k, an increase in k leads to a downward shift of $\tilde{\Lambda}_s^1$ and an upward shift of $\tilde{\Lambda}_s^2$ and so, to a decrease in $\underline{x}$. From $(A.49)$ and $(A.50)$, $\overline{x} = \frac{1-\underline{x}}{-\ln \underline{x}}$ is increasing in $\underline{x}$, and so, also decreases with an increase in k.

To derive the comparative statics in λ, we express equilibrium conditions (4.12) and (4.13) in terms of variables L and $M_{bu}(\phi)$ as follows[40]

$$\begin{cases} M_{bu}(\phi) = \frac{y_d}{\rho}\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L} - 1\right) - L\right) - \frac{y_u+y_d}{\lambda}, \\ L = \frac{(y_u+y_d)/\lambda + M_{bu}(\phi)}{y_d M_{bu}(\phi)}\left(y_u(a-1) - (y_u+y_d)M_{bu}(\phi) - h(\lambda M_{bu}(\phi))\right). \end{cases} \tag{A.52}$$

The right-hand side of the first equation in $(A.52)$ is increasing in L and increasing in λ, while the right-hand side of the second equation in $(A.52)$ is decreasing in $M_{bu}(\phi)$ and decreasing in λ. Therefore, an increase in λ leads to a decrease in L and so, an increase in $\underline{x}$ by $(A.49)$. □

Proof of Proposition 5. Consider functions Λ_s^1 and Λ_s^2 introduced in the proof of Theorem 2. Consider an increase in y_d and a decrease in y_u so that $y_d + y_u$ does not change. This leads to an upward shift of Λ_s^1 and a downward shift of Λ_s^2, and so, a decrease in L. □

Proof of Proposition 9. Since (4.12) and (4.13) do not depend on α, L and Λ_s are independent of α. From $(A.42)$ and $(A.43)$, $\hat{\theta}$ and $\check{\theta}$ are decreasing in α. □

Proof of Proposition 6. An increase in a leads to an upward shift of Λ_s^2 and so, an increase in Λ_s and L, and by $(A.49)$, decrease in $\underline{x}$. □

Two-Class Extension

Proof of Theorem 3. Under the assumption of the theorem, equilibrium quantities $(\Lambda_{s,1}, \underline{x}_1)$ and $(\Lambda_{s,2}, \underline{x}_2)$ are determined by the unique solution to the system $(A.51)$ with $a = a_1$ and $a = a_2$, respectively. Denote by $\underline{x}(a)$ the equilibrium threshold of the buyer's strategy given that the

[40]Again we consider only cases where before and after an increase in λ, $L < 1$, as other cases are straightforward to show from (4.1).

mass of agents is a. Equations in the system $(A.51)$ are continuous in parameters and so, the solution $(\Lambda_s, \underline{x})$ varies continuously with a and an increase in a leads to a decrease in $\underline{x}$ by Proposition 6. Thus, $\underline{x}(\cdot)$ is continuous and decreasing in a. By (6.1), a_1 is determined by $\underline{x}(a_1) = \underline{x}(a - a_1)$ which has a unique solution. □

Proof of Proposition 7. Suppose k_1 increases and/or k_2 decreases. We show that as a result a_1 decreases and a_2 increases. Suppose to contradiction that a_1 increases and a_2 decreases. By Propositions 3 and 6, $\underline{x}_1$ decreases and $\underline{x}_2$ increases which contradicts the indifference of buyers (6.1). □

References

Amihud, Y. and Mendelson, H.: 1986, Asset pricing and the bid-ask spread, *Journal of Financial Economics, 17, 223-249* .

Ashcraft, A., Goldsmith-Pinkham, P. and Vickery, J.: 2010, MBS ratings and the mortgage credit boom, *Working Paper* .

Asquith, P., Covert, T. and Pathak, P.: 2013, The effects of mandatory transparency in financial market design: Evidence from the corporate bond market, *Working Paper, MIT* .

Asriyan, V., Fuchs, W. and Green, B.: 2015, Information spillovers in asset markets with correlated values, *Working Paper, Berkeley* .

Atakan, A. and Ekmekci, M.: 2014, Bargaining and reputation in search markets, *Review of Economic Studies, 81, 1-29* .

Aumann, R.: 1999, Interactive epistemology I: Knowledge, *International Journal of Game Theory, 28, 263-300* .

Ausubel, L. M. and Deneckere, R. J.: 1989, Reputation in bargaining and durable goods monopoly, *Econometrica* pp. 511–531.

Bao, J., Pan, J. and Wang, J.: 2011, The illiquidity of corporate bonds, *Journal of Finance, 66, 911-946* .

Beber, A., Brandt, M. and Kavajecz, K.: 2009, Flight-to-quality or flight-to-liquidity? Evidence from the euro-area bond market, *Review of Financial Studies, 22, 925-957* .

Benmelech, E. and Dlugosz, J.: 2010, The credit rating crisis, *NBER Macroeconomics Annual 2009, 24, 161-207* .

Bessembinder, H., Maxwell, W. and Venkataraman, K.: 2006, Market transparency, liquidity externalities, and institutional trading costs in corporate bonds, *Journal of Financial Economics, 82, 251-288* .

Bikhchandani, S.: 1992, A bargaining model with incomplete information, *Review of Economic Studies, 59, 187-203* .

Binmore, K.: 1987, Nash bargaining theory I, II, *in Binmore, K. and P. Dasgupta (eds.): The Economics of Bargaining Cambridge: Basic Blackwell.* .

Binmore, K., Rubinstein, A. and Wolinsky, A.: 1986, The Nash bargaining solution in economic modeling, *RAND Journal of Economics, 176-188* .

Brunnermeier, M. K.: 2008, Deciphering the liquidity and credit crunch 2007-08, *Technical report*, National Bureau of Economic Research.

Chang, B.: 2014, Adverse selection and liquidity distortions, *Working Paper, Wisconsin School of Business* .

Chen, H., Cui, R., He, Z. and Milbradt, K.: 2014, Quantifying liquidity and default risks of corporate bonds over the business cycle, *Working Paper* .

Constantinides, G.: 1986, Capital market equilibrium with transactions costs, *Journal of Political Economy, 94, 842-862* .

Deneckere, R. and Liang, M.: 2006, Bargaining with interdependent values, *Econometrica, 74, 1309-1364* .

Dick-Nielsen, J., Feldhutter, P. and Lando, D.: 2012, Corporate bond liquidity before and after the onset of the subprime crisis, *Journal of Financial Economics, 103, 471-492* .

Duffie, D., Dworczak, P. and Zhu, H.: 2015, Benchmarks in search markets, *Working Paper, Stanford* .

Duffie, D., Gârleanu, N. and Pedersen, L.: 2005, Over-the-counter markets, *Econometrica, 73, 1815-1847* .

Duffie, D., Gârleanu, N. and Pedersen, L. H.: 2007, Valuation in over-the-counter markets, *Review of Financial Studies, 20, 1865-1900* .

Duffie, D. and Sun, Y.: 2007, Existence of independent random matching, *Annals of Applied Probability, 17, 386-419* .

Edwards, A., Harris, L. and Piwowar, M.: 2007, Corporate bond market transaction costs and transparency, *Journal of Finance, 62, 1421-1451* .

Ericsson, J. and Renault, O.: 2006, Liquidity and credit risk, *Journal of Finance, 61, 2219-2249* .

Friewald, N., Jankowitsch, R. and Subrahmanyam, M.: 2012, Illiquidity or credit deterioration: A study of liquidity in the us corporate bond market during financial crises, *Journal of Financial Economics, 105, 18-36* .

Fuchs, W. and Skrzypacz, A.: 2013, Bridging the gap: Bargaining with interdependent values, *Journal of Economic Theory, 148, 1226-1236* .

Fudenberg, D., Tirole, J. and Levine, D.: 1985, Infinite horizon models of bargaining with one sided incomplete information, *Game Theoretic Models of Bargaining, ed. by A. Roth. Cambridge University Press, Cambridge, UK* .

Gale, D.: 2000, *Strategic Foundations of General Equilibrium: Dynamic Matching and Bargaining Games*, Cambridge University Press.

Goldstein, M. A., Hotchkiss, E. S. and Sirri, E. R.: 2007, Transparency and liquidity: A controlled experiment on corporate bonds, *Review of Financial Studies* **20**(2), 235–273.

Grossman, S. J. and Perry, M.: 1986, Sequential bargaining under asymmetric information, *Journal of Economic Theory* **39**(1), 120–154.

Guerrieri, V. and Shimer, R.: 2014, Dynamic adverse selection: A theory of illiquidity, fire sales, and flight to quality, *American Economic Review, 1875-1908* .

Guerrieri, V., Shimer, R. and Wright, R.: 2010, Adverse selection in competitive search equilibrium, *Econometrica, 78, 1823-1862* .

Gul, F. and Sonnenschein, H.: 1988, On delay in bargaining with one-sided uncertainty, *Econometrica* pp. 601–611.

He, Z. and Milbradt, K.: 2014, Endogenous liquidity and defaultable bonds, *Econometrica, 82, 1443-1508* .

Heaton, J. and Lucas, D.: 1996, Evaluating the effect of incomplete markets on risk sharing and asset pricing, *Journal of Political Economy, 104, 443-487* .

Huang, M.: 2003, Liquidity shocks and equilibrium liquidity premia, *Journal of Economic Theory, 109, 104-129* .

Hugonnier, J., Lester, B. and Weill, P.: 2014, Heterogeneity in decentralized asset markets, *Working Paper, National Bureau of Economic Research* .

Kurlat, P.: 2013, Lemons markets and the transmission of aggregate shocks, *American Economic Review, 103, 1463-1489* .

Kurlat, P.: n.d., Asset markets with heterogeneous information, *Econometrica, Forthcoming* .

Lagos, R. and Rocheteau, G.: 2007, Search in asset markets: Market structure, liquidity, and welfare, *American Economic Review, 198-202* .

Lagos, R. and Rocheteau, G.: 2009, Liquidity in asset markets with search frictions, *Econometrica, 77, 403-426* .

Lauermann, S. and Wolinsky, A.: 2014, Search with adverse selection, *Working Paper* .

Lester, B., Shourideh, A., Venkateswaran, V. and Zetlin-Jones, A.: 2015, Screening and adverse selection in frictional markets, *Working Paper* .

Lewis, M.: 2011, *The Big Short: Inside the Doomsday Machine*, WW Norton & Company, New York.

Lo, A., Mamaysky, H. and Wang, J.: 2004, Asset prices and trading volume under fixed transactions costs, *Journal of Political Economy, 112, 1054-1090* .

Longstaff, S., Mithal, S. and Neis, E.: 2005, Corporate yield spreads: Default risk or liquidity? new evidence from the credit default swap market, *Journal of Finance, 50, 2213-2253* .

Morris, S. and Shin, H. S.: 1998, Unique equilibrium in a model of self-fulfilling currency attacks, *American Economic Review* pp. 587–597.

Nash, J.: 1950, The bargaining problem, *Econometrica, 155-162* .

Osborne, M. and Rubinstein, A.: 1990, *Bargaining and Markets*, San Diego: Academic Press.

Roth, A.: 1979, *Axiomatic Models of Bargaining*, Berlin: Springer-Verlag.

Rubinstein, A.: 1982, Perfect equilibrium in a bargaining model, *Econometrica, 97-109* .

Rubinstein, A. and Wolinsky, A.: 1985, Equilibrium in a market with sequential bargaining, *Econometrica, 53, 1133-1150* .

Satterthwaite, W. and Shneyerov, A.: 2007, Dynamic matching, two-sided incomplete information, and participation costs: Existence and convergence to perfect competition, *Econometrica, 75, 155-200* .

Shen, J., Wei, B. and Yan, H.: 2015, Financial intermediation chains in a search market, *Working Paper, Yale School of Management* .

Simon, L. K. and Stinchcombe, M. B.: 1989, Extensive form games in continuous time: Pure strategies, *Econometrica* pp. 1171–1214.

Tsoy, A.: 2014, Alternating-offer bargaining with a global games information structure, *Working Paper, MIT* .

Vayanos, D. and Wang, T.: 2007, Search and endogenous concentration of liquidity in asset markets, *Journal of Economic Theory, 136, 66-104* .

Vayanos, D. and Weill, P.: 2008, A search-based theory of the on-the-run phenomenon, *Journal of Finance, 63, 1361-1398* .

Weill, P.: 2008, Liquidity premia in dynamic bargaining markets, *Journal of Economic Theory, 140, 66-96* .

B Online Appendix (Not for Publication)

Example of F

Here, we provide an example of a sequence of F approximating F^* that satisfies our assumptions. Fix $\gamma > 0$. Suppose θ normally distributed with zero mean and variance $\gamma^2 - \frac{1}{\gamma}$, and ε_b and ε_s are independent normals with zero mean and variance $\frac{1}{\gamma}$. Let F be the distirbution of $(\theta+\varepsilon^s, \theta+\varepsilon^b)$ conditional on $(\theta+\varepsilon^s, \theta+\varepsilon^b) \in [0,1]$.

Proposition 8. *1. F is affiliated;*

2. $F \xrightarrow{p} F^*$ as $\gamma \to \infty$;

3. for any $\varepsilon > 0$, there is $\overline{\gamma}$ so that for all $\gamma > \overline{\gamma}$,

$$\sup_{(\theta^s,\theta^b):|\theta^s-\theta^b|>\varepsilon} \max\{f(\theta^b|\theta^s), f(\theta^s|\theta^b)\} < \varepsilon. \tag{B.1}$$

Proof. 1) By definition, F is a bivariate normal distribution with zero mean and covariance matrix $\Sigma = \begin{pmatrix} \gamma^2 & \gamma^2 - \frac{1}{\gamma} \\ \gamma^2 - \frac{1}{\gamma} & \gamma^2 \end{pmatrix}$ conditional on $(\theta^s, \theta^b) \in [0,1]^2$. Since the density of the positively correlated bivariate normal distribution is log-supermodular so is f. Thus, F is affiliated.

2) The densify f is given by

$$f(\theta^s,\theta^b) = \frac{\exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)}{\int_0^1\int_0^1 \exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right) d\theta^s d\theta^b}$$
$$= \frac{\exp\left(-\frac{\gamma(\theta^s-\theta^b)^2+2\gamma^{-2}\theta^s\theta^b}{2(2-\gamma^{-3})}\right)}{\int_0^1\int_0^1 \exp\left(-\frac{\gamma(\theta^s-\theta^b)^2+2\gamma^{-2}\theta^s\theta^b}{2(2-\gamma^{-3})}\right) d\theta^s d\theta^b}.$$

After the change of variables $x=\theta^s-\theta^b, y=\theta^s\theta^b$, we have

$$f(x,y) = \frac{\exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)}{2\int_0^1\int_0^{1-x} \exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right) \frac{dydx}{\sqrt{x^2+4y}}}.$$

We next construct upper and lower bounds on the nominator and the denominator of f. For the nominator, the bounds are

$$\exp\left(-\tfrac{1}{3}\gamma x^2 - \tfrac{2}{3}\gamma^{-2}\right) \le \exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right) \le \exp\left(-\tfrac{1}{5}\gamma x^2\right).$$

For the denominator, for any $\varepsilon_0\in(0,\frac{1}{2})$ the upper bound is

$$\begin{aligned}\int_0^1\int_0^{1-x} \exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right) \frac{dydx}{\sqrt{x^2+4y}} \le& \int_0^1\int_0^{1-x} \exp\left(-\frac{\gamma x^2}{5}\right) \frac{dydx}{\sqrt{x^2+4y}}\\ =&\int_0^1 \exp\left(-\tfrac{1}{5}\gamma x^2\right)(1-x)dx\\ \le&\int_0^1 \exp\left(-\tfrac{1}{5}\gamma x^2\right)dx\\ \le&\exp\left(-\tfrac{1}{5}\gamma^{2\varepsilon_0}\right)(1-\gamma^{-1/2+\varepsilon_0})+\gamma^{-1/2+\varepsilon_0}\\ \le& c_1\gamma^{-1/2+\varepsilon_0},\end{aligned}$$

and the lower bound is

$$\begin{aligned}\int_0^1\int_0^{1-x} \exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right) \frac{dydx}{\sqrt{x^2+4y}} \ge& \int_0^1\int_0^{1-x} \exp\left(-\tfrac{1}{3}\gamma x^2-\tfrac{2}{3}\gamma^{-2}\right) \frac{dydx}{\sqrt{x^2+4y}}\\ =&\int_0^1 \exp\left(-\tfrac{1}{3}\gamma x^2-\tfrac{2}{3}\gamma^{-2}\right)(1-x)dx\\ \ge&\gamma^{-1/2}\exp\left(-\tfrac{2}{3}\right)(1-\gamma^{-1/2})\\ \ge& c_2\gamma^{-1/2}.\end{aligned}$$

Thus,

$$\frac{1}{c_1}\gamma^{1/2-\varepsilon}\exp\left(-\tfrac{1}{3}\gamma x^2-\tfrac{2}{3}\gamma^{-2}\right)\le \quad f(x,y)\quad \le \frac{1}{c_2}\gamma^{1/2}\exp\left(-\tfrac{1}{5}\gamma x^2\right),$$

and so,

- for all $|x|>\gamma^{-1/4}$, $f(x,y)<\frac{1}{c_2}\gamma^{1/2}\exp\left(-\frac{1}{5}\gamma^{1/2}\right)\underset{\gamma\to\infty}{\to}0$;
- for all $|x|<\gamma^{-1}$, $f(x,y)>\frac{1}{c_1}\gamma^{1/2-\varepsilon_0}\exp\left(-\frac{1}{3}\gamma^{-1}-\frac{2}{3}\gamma^{-2}\right)\underset{\gamma\to\infty}{\to}\infty$;
- for any x,

$$1\le\frac{\max_{y\in[0,|x|]}f(x,y)}{\min_{y\in[0,|x|]}f(x,y)}=\frac{\max_{y\in[0,|x|]}\exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)}{\min_{y\in[0,|x|]}\exp\left(-\frac{\gamma x^2+2\gamma^{-2}y}{2(2-\gamma^{-3})}\right)}\le\exp\left(\frac{\gamma^{-2}}{2-\gamma^{-3}}\right)\underset{\gamma\to\infty}{\to}1.$$

This implies that $F\overset{p}{\to}F^*$ as $\gamma\to\infty$.

3) For $|\theta^s-\theta^b|>\gamma^{-1/4}$,

$$\begin{aligned}
f(\theta^s|\theta^b)=\frac{f(\theta^s,\theta^b)}{\int_0^1 f(\theta^s,\theta^b)d\theta^s}&=\frac{\exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)}{\int_0^1\exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)d\theta^s}\\
&\le\frac{\exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)}{\int_{\max\{0,\theta^b-\gamma^{-1/4}\}}^{\min\{1,\theta^b+\gamma^{-1/4}\}}\exp\left(-\frac{(\theta^s)^2+(\theta^b)^2-2(1-\gamma^{-3})\theta^s\theta^b}{2\gamma^2(1-(1-\gamma^{-3})^2)}\right)d\theta^s}\\
&\le\frac{\frac{1}{c_2}\gamma^{1/2}\exp\left(-\frac{1}{5}\gamma(\theta^s-\theta^b)^2\right)}{\frac{2}{c_1}\gamma^{1/2-\varepsilon_0}\exp\left(-\frac{1}{3}\gamma^{-1/2}-\frac{2}{3}\gamma^{-2}\right)\gamma^{-1/4}}\\
&=\frac{c_1}{2c_2}\gamma^{1/4+\varepsilon_0}\exp\left(-\tfrac{1}{5}\gamma x^2+\tfrac{1}{3}\gamma^{-1/2}+\tfrac{2}{3}\gamma^{-2}\right)\\
&\le\frac{c_1}{2c_2}\gamma^{1/4+\varepsilon_0}\exp\left(-\tfrac{1}{5}\gamma^{1/2}+\tfrac{1}{3}\gamma^{-1/2}+\tfrac{2}{3}\gamma^{-2}\right)\underset{\gamma\to\infty}{\to}0,
\end{aligned}$$

and the symmertic argument holds for $f(\theta^b|\theta^s)$. Thus, $(B.1)$ obtains. □

B.1 Proofs for OTC Model

Lemma 18. $\int_{\breve{\theta}}^{\overline{\theta}}\frac{\Lambda_s\sigma(\theta)}{y_u+y_d+\Lambda_s\sigma(\theta)}d\theta=h(L,\Lambda_s)$.

Proof. Expressing $\sigma(\theta)$ from $(A.47)$, $\check{\theta}$ from $(A.43)$, and $\overline{\theta}$ from $(A.46)$, we get

$$
\begin{aligned}
& \int_{\check{\theta}}^{\overline{\theta}} \frac{\Lambda_s \sigma(\theta)}{y_u + y_d + \Lambda_s \sigma(\theta)} d\theta \\
= \; & \int_{\check{\theta}}^{\check{\theta} + \min\{1 + \frac{r\xi}{k} \ln \underline{x}, (1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\xi \underline{x}\}} \frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x} - \frac{k}{y_d}(\theta - \check{\theta})}{\frac{y_u+y_d}{\rho}((1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi\underline{x} + \frac{k}{y_d}(\theta-\check{\theta})) + (1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x} - \frac{k}{y_d}(\theta-\check{\theta})} d\theta \\
= \; & \int_0^{\min\left\{\frac{1+\frac{r\xi}{k}\ln \underline{x}}{(1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}}, 1\right\}} \frac{1-s}{1 + \frac{y_u+y_d}{\Lambda_s}s - (1 - \frac{y_u+y_d}{\rho})s} ds,
\end{aligned}
$$

where in the second line we make a change of variables $s = \frac{\frac{k}{y_d}(\theta-\check{\theta})}{(1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\xi\underline{x}}$. After expressing $\underline{x}$ from $(A.49)$ we get the desired conclusion. □

Lemma 19. *Equations* $(4.1), (A.47), (A.42), (A.43), (A.46)$ *imply equation* (4.13)

Proof. From (4.1),

$$
\frac{\Lambda_s}{\lambda} \quad = \quad \frac{y_u}{y_u + y_d}(a-1) - \frac{y_d}{y_u + y_d}\int_0^1 \frac{\Lambda_s \sigma(\theta)}{y_u + y_d + \Lambda_s \sigma(\theta)} d\theta. \tag{B.2}
$$

Expressing σ from $(A.47)$,

$$
\frac{\Lambda_s}{\lambda} = \frac{y_u}{y_u + y_d}(a-1) - \frac{y_d}{y_u + y_d}\left(\frac{\Lambda_s L}{y_u + y_d + \Lambda_s} + \int_{\check{\theta}}^{\underline{\theta}} \frac{\Lambda_s \sigma(\theta)}{y_u + y_d + \Lambda_s \sigma(\theta)} d\theta\right),
$$

which together with Lemma 18 gives equation (4.13). □

Denote by Λ_s^1, Λ_s as a function of L expressed from equation (4.12), and by Λ_s^2, Λ_s as a function of L expressed from equation (4.13).

Lemma 20. Λ_s^2 *is strictly decreasing.*

Proof. Consider an increase in L to L' so that before and after the increase $\underline{\theta} < \hat{\theta}$. It is easy to see that in this case the right-hand side of (4.13) is strictly decreasing in Λ_s and so $\Lambda_s^2(L) < \Lambda_s^2(L')$.

Now, suppose that before and after an increase in L to L', $\hat{\theta} = \underline{\theta} > \check{\theta}$. We use equation $(A.49)$ to rewrite equations $(A.42), (A.43), (A.46), (A.47)$ in terms of L as follows:

$$
\begin{aligned}
\check{\theta}(L) \quad &= \quad (1-\alpha)\left(L + \frac{y_d}{k}\xi\frac{\Lambda_s}{\rho + \Lambda_s}(1 - e^{-\frac{k}{r\xi}L})\right), && \text{(B.3)} \\
\hat{\theta}(L) \quad &= \quad \check{\theta} + 1 - L, && \text{(B.4)} \\
\underline{\theta}(L) \quad &= \quad \check{\theta} + \min\left\{1 - L, (1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho+\Lambda_s}\xi e^{-\frac{k}{r\xi}L}\right\}, && \text{(B.5)}
\end{aligned}
$$

$$\Lambda_s\sigma(\theta, L) \quad = \quad \begin{cases} \Lambda_s, & \text{if } \theta \in [0, \check{\theta}] \cup [\hat{\theta}, 1], \\ 0, & \text{if } \theta \in [\underline{\theta}, \hat{\theta}), \\ \rho \frac{(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi e^{-\frac{k}{r\xi}L} - \frac{k}{y_d}(\theta - \check{\theta})}{(1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi e^{-\frac{k}{r\xi}L} + \frac{k}{y_d}(\theta - \check{\theta})}, & \text{if } \theta \in (\check{\theta}, \underline{\theta}). \end{cases} \tag{B.6}$$

Let $\check{\theta} \equiv \check{\theta}(L), \hat{\theta} \equiv \hat{\theta}(L), \underline{\theta} \equiv \underline{\theta}(L)$ and $\check{\theta}' \equiv \check{\theta}(L'), \hat{\theta}' \equiv \hat{\theta}(L'), \underline{\theta}' \equiv \underline{\theta}(L')$. The fact that $\hat{\theta} = \underline{\theta} > \check{\theta}$ implies that

$$1 - L < (1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho + \Lambda_s}\xi e^{-\frac{k}{r\xi}L} \tag{B.7}$$

and the same inequality holds for L', as $\hat{\theta}' = \underline{\theta}' > \check{\theta}'$.

Recall that $\Lambda_s^2(L)$ solves equation $(B.2)$. We next show that after the increase in L, the integral in $(B.2)$ increases. We first show the following claim

Claim 7. Then $\sigma(\hat{\theta} - y, L) \leq \sigma(\hat{\theta}' - y, L')$ for $y \in (0, 1 - L)$ with a strict inequality for $y \in (0, 1 - L')$.

Proof: Observe that for $y \in [1 - L', 1 - L)$, $\sigma(\hat{\theta}' - y, L') = 1$. Now, using $(B.4)$ and $(B.6)$, for $y \in (0, 1 - L')$

$$\sigma(\hat{\theta} - y, L) = \frac{\rho}{\Lambda_s}\frac{\frac{y_d}{k}(1-\alpha)\frac{\Lambda_s}{\rho+\Lambda_s}\xi e^{-\frac{k}{r\xi}L} - 1 + L + y}{\frac{y_d}{k}(1-\alpha)\frac{\rho}{\rho+\Lambda_s}\xi e^{-\frac{k}{r\xi}L} + 1 - L - y}.$$

Differentiating $\sigma(\hat{\theta} - y, L)$ with respect to L, we get

$$\text{sgn}\left(\frac{\partial\sigma(\hat{\theta} - y, L)}{\partial L}\right) = \text{sgn}\left(\frac{r\xi}{k} - 1 + L + y\right).$$

It follows from $(B.7)$ and (2.4) that

$$\begin{aligned} 1 - L &< (1-\alpha)\frac{y_d}{k}\frac{\Lambda_s}{\rho + \Lambda_s}\xi e^{-\frac{k}{r\xi}L} \\ &< (1-\alpha)\frac{y_d}{k}\xi \\ &\leq \alpha\frac{r\xi}{k}, \end{aligned}$$

and so, $\frac{\partial\sigma(\hat{\theta}-y,L)}{\partial L} > 0$. Thus, $\sigma(\hat{\theta} - y, L) < \sigma(\hat{\theta}' - y, L')$ for $y \in (0, 1 - L')$.

q.e.d.

Outside the interval $(\check{\theta}, \hat{\theta})$, $\sigma(\theta, L) = 1$, and outside $(\hat{\theta}' - 1 + L, \hat{\theta}')$, $\sigma(\theta, L') = 1$. Together with Claim 7 this implies that the integral in $(B.2)$ is larger for L' and so, the right-hand side of $(B.2)$ strictly decreases when L increases to L'.

It follows from $(B.6)$ that $\Lambda_s\sigma(\theta)$ is strictly increasing in Λ_s and so, the right-hand side of $(B.2)$ is strictly decreasing in Λ_s, while the left-hand side of $(B.2)$ is strictly increasing in Λ_s. Combined with the fact that the right-hand side of $(B.2)$ is strictly decreasing with L, we get that $\Lambda_s^2(L) < \Lambda_s^2(L')$. □

Lemma 21. *There is a unique solution* (Λ_s, L) *to* (4.12) *and* (4.13).

Proof. I show that Λ_s^1 is strictly increasing and Λ_s^2 is strictly decreasing, which implies that the solution is unique. Lemma 20 implies that Λ_s^2 is strictly decreasing. The right-hand of equation (4.12) is increasing in L, as $\left(\frac{\xi r}{k}\left(e^{\frac{k}{\xi r}L}-1\right)-L\right)'_L = e^{\frac{k}{\xi r}L}-1>0$. Therefore, Λ_s^1 is strictly increasing. □

Surplus Share Here, we explore the effect of the split of the surplus.

Proposition 9. *An increase in* α *does not change* $L, \underline{x}$ *and* Λ_s*, but leads to an increase in* $\hat{\theta}$ *and* $\check{\theta}$.

Notice that even though the market liquidity does not depend on α, the composition of liquid assets depends on the split of surplus. The greater the share of the buyer, the higher the fraction of high-quality assets (above θ^*) in the set of liquid assets. The higher share of the surplus makes an agent more impatient and increases for him the costs of delay. As a result, higher α gives the buyer additional incentives to accept faster in the SBS which in turn increases the liquidity of high-quality assets. For low-quality assets (below θ^*), the logic is the opposite. The seller bears a smaller fraction of the delay costs, which increases his incentives to wait longer, and hence, decreases the liquidity of such asset qualities.