

# Dynamic Prudential Regulation\*

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## Abstract

We investigate the design of prudential bank regulation and its effects on the real and financial decisions of banks in a continuous-time structural framework. In our model, the regulator controls the dynamic risk-shifting incentives of a representative bank through the threat of intervention in the bank's operations. The optimal regulatory policy, which we characterize analytically, entails an optimal combination of a capital requirement, intervention to control the risk of the bank's portfolio, capital injection, and liquidation of the bank. Under the optimal risk intervention policy, the regulator intervenes when the bank's capital ratio lies inside a "band" consisting of two triggers. We calibrate the model and show that the optimal capital requirement is 23%, which supports the substantially higher capital requirements being proposed in the Basel III accords. Relative to a benchmark unregulated bank, regulation increases bank leverage by 12%, while lowering its credit spread. Capital injection or risk intervention alone has modest impact on the bank's social value. The optimal combination of regulatory policies, however, significantly improves social value by 4% and bank value by 11%. Optimal capital requirements should be counter-cyclical and should be stricter for large banks. Optimal regulation is also significantly affected by monetary and fiscal policies. Overall, our analysis highlights the importance of considering the interactions among different individual regulatory policies as well as the economic environment in designing optimal regulation.

*Key Words:* capital regulation, asset substitution, bank bailout, prudential regulation

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## I. Introduction

The 2007–2009 financial crisis has engendered a vigorous debate on the prudential regulation of financial institutions. Despite substantial progress in regulation such as the 2010 Dodd-Frank Act and the Basel III agreement, a broad consensus has yet to emerge on a number of issues. What forms should regulatory intervention take? At what points should regulators intervene in banks’ operations to control their risk, force liquidation or recapitalization of distressed banks, or bail them out with public funds? What are the relative impacts of different components of regulation—capital regulation, intervention and control of banks’ risk-taking, and capital injection—on banks’ private and social values? How sensitive is optimal prudential regulation to the economic environment? Given the inter-temporal nature of banks’ operations and regulatory intervention policies, a dynamic model that is both analytically tractable and calibratable to data can potentially provide qualitative and quantitative guidance on the design and impact of prudential regulation.

We take initial steps towards addressing the above issues by developing a structural model to analyze the design of prudential bank regulation and its effects on the real and financial decisions of banks. We seek to obtain insights into how existing regulatory policies affect the investment and financing decisions of banks as well as the optimal design of prudential regulation. The optimal regulatory policy, which we characterize analytically, is multi-pronged in nature in that it entails imposition of a capital requirement, intervention to control the risk of a bank’s portfolio, injection of capital after the bank becomes insolvent, and liquidation of the bank when it is no longer optimal to keep it active. Our analysis of the calibrated model shows that, relative to a benchmark unregulated bank, regulation increases bank leverage substantially, while lowering its credit spread. The optimal capital requirement is consistent with the substantially higher capital requirements proposed in the Basel III accords and advocated by recent studies such as Admati, DeMarzo, Hellwig and Pfleiderer (2011). While capital injection and risk intervention *independently* have modest effects on social value, their optimal combination significantly improves social and bank values suggesting that regulation has substantial economic benefits provided that the entire gamut of intervention tools are appropriately employed. Regulation should not be independent of monetary and fiscal policies, optimal capital requirements should be counter-cyclical, and larger banks should be subject to stricter capital requirements.

In our continuous-time model, a representative bank, whose capital structure consists of equity and debt, operates in a regulated environment. As in prior literature, there are additional dead-weight costs of equity issuance relative to debt issuance (e.g. Giammarino, Lewis and Sappington (1993)). The bank can dynamically alter the distribution of its earnings by changing its investment portfolio. For simplicity, the bank can choose one of two “projects” at any date, where the term “project” is a metaphor for the bank’s portfolio of investments. The first project has a higher NPV and lower volatility than the second one, thereby creating incentives for asset substitution. We allow for a broad gamut of regulatory intervention tools that include the imposition of a capital

requirement, intervention in the bank’s project choices, capital injection when the bank is insolvent, and liquidation of the bank. Regulation has potential benefits because, while the bank’s management cares about shareholder value, the regulator cares about the bank’s “social value” that internalizes the claims of shareholders, debtholders and tax-payers. Further, the regulator internalizes the externality that a bank imposes on the economy through its financial distress.

We first analyze the benchmark scenario in which the bank is unregulated. The bank chooses its capital structure and dynamically chooses its projects to maximize shareholder value. We solve the bank’s stochastic control problem and obtain a closed-form analytical characterization of the unregulated bank’s optimal behavior. The optimal strategy of the unregulated bank is characterized by three earnings thresholds: the *switching threshold* at which the bank switches from the low-risk to the high-risk project, the *liquidity threshold* at which debt payments consume all of the bank’s earnings, and the *insolvency threshold* at which the bank’s equity value endogenously falls to zero. Because shareholders effectively hold a convex claim, it is optimal for the bank to increase risk when its earnings are sufficiently low to lower the probability of insolvency. We rigorously establish the existence of a *single* threshold at which the bank optimally switches projects that is not a priori obvious given the dynamic nature of the bank’s problem, and the fact that the insolvency threshold is also endogenous.<sup>1</sup>

Next, we analyze the *fully regulated* environment in which the bank is completely controlled by the regulator whose objective is to maximize the bank’s “social” value that internalizes the claims of shareholders, debt holders, and tax payers. To incorporate the systemic effects of the bank on the economy, we also allow the equity issuance, financial distress and liquidation costs in the regulator’s objective function to differ from those of the unregulated bank. We analytically show that the optimal policy of the regulator is described by three thresholds as in the unregulated bank’s problem (their values, however, differ from those for the unregulated bank) and an additional *liquidation threshold*. The regulator takes over the bank and injects capital in the region between the insolvency and liquidation thresholds. In particular, our results demonstrate that, in a dynamic environment, it could also be optimal for the regulator to permit asset substitution by increasing risk when the bank’s earnings are sufficiently low. The intuition is that, because the regulator optimally injects capital when the bank’s earnings fall below the insolvency threshold, the regulator effectively becomes the residual claimant and, therefore, has a “locally” convex objective function akin to that of shareholders of an unregulated bank. Consequently, it could be optimal for the regulator to increase risk to reduce the probability of liquidation and the associated liquidation costs. Nevertheless, because the regulator cares about the bank’s social value, it has lower incentives to engage in asset substitution than the bank’s shareholders so that the regulator’s optimal switching threshold is always lower than that of the unregulated bank.

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<sup>1</sup>Subramanian and Yang (2012) analyze a more general model of dynamic risk-taking by an agent with multiple projects and a payoff function that belongs to a broader class. They provide counter-examples to show that, if the agent’s payoff does not satisfy additional conditions (that are satisfied in our setting), there could be multiple switching triggers even with two available projects.

By comparing the unregulated bank’s problem with that of the regulator, we derive the optimal *regulatory policy* that comprises of three components: *capital injection, capital regulation, and risk intervention*. Capital injection by the regulator prevents costly premature liquidation of banks and the associated social costs. In the presence of a capital injection policy, the regulator imposes an initial capital requirement by setting a cap on the initial leverage ratio of the bank to prevent ex ante risk-taking of the bank in the form of excessive borrowing. The risk intervention policy reduces costs of ex post risk-shifting of the bank and has a “band” structure. The regulator intervenes and forces the bank to choose low risk project when the bank’s earnings lie in between the unregulated bank’s optimal switching threshold and the regulator’s optimal switching threshold because it is in this interval where their respective optimal project choices differ.<sup>2</sup>

To quantitatively analyze the impact of regulation, we calibrate the model to data on U.S. banks over the period 1991 – 2008. In the aftermath of the financial crisis, it has become clear that banks were poorly regulated during this period, but many banks were subsequently “bailed out” through injection of public funds. Consequently, the model that we calibrate to data is an “intermediate” one where the bank is free to choose its capital structure and projects, but nevertheless implicitly enjoys the protection provided by public capital injection after the bank is no longer able to raise capital to meet debt requirements. In our calibration, we match banks’ average earnings to asset ratio, risk, leverage, and yield spread. In addition, we use the ratio of the economic costs to the fiscal costs of banking crises reported in Hoggarth, Reis, and Saporta (2002) to calibrate the parameters in the model that reflect the systemic effects of the bank on the economy. Our calibration approach, therefore, allows us to differentiate the private and systemic financial distress costs.

Regulation has a substantial impact on bank leverage. Under the calibrated baseline parameters, the fully regulated bank has an optimal leverage of 77%, while the unregulated bank’s leverage is 65%. The credit spread (39 bp) of the regulated bank is, however, lower than that of the unregulated bank (67 bp). The reduction in the bank’s asset risk due to the regulator’s intervention together with the regulator’s commitment of capital injection enable the regulated bank to issue much more debt with a lower spread than the unregulated bank. The optimal capital requirement, which is equal to the initial equity ratio in the fully regulated environment, is 23%, substantially higher than that suggested by the Basel II accords, and consistent with the significantly higher capital requirements proposed by the Basel III committee. This finding also supports the proposal of Admati et al (2011) that the capital requirement be set much higher than current regulatory levels (they recommend capital ratios between 20% and 30% in their conclusions). The optimal intervention ratio, which is the level below which the regulator intervenes in the bank’s management, is 6.9%. The predicted intervention ratio is broadly consistent with the FDIC definitions of “undercapitalization” and “significant undercapitalization” ratios (8% and 6%) that trigger monitoring

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<sup>2</sup>In our model, the regulator/social planner provides implicit deposit insurance by injecting capital into an insolvent bank and thus embodies the role of the deposit insurer. We also show that incorporating explicit deposit insurance does not change the optimal regulatory policies.

and restrictions on banks' activities.<sup>3</sup>

From an *ex ante* perspective, optimal regulation increases a bank's value by 11% relative to the unregulated bank, while it increases the bank's "social" value (that also incorporates the costs arising from systemic risk) by 4%. Regulation has an even bigger quantitative impact *ex post*. We also analyze the relative quantitative impacts of different regulatory tools. Risk intervention on its own improves a bank's social value by only 0.4%, while capital injection on its own reduces it by 1.9%. Together, the evidence suggests that there are significant interactive effects of different regulatory tools.

The analytical characterizations of the bank's and regulator's policies greatly facilitate the exploration of various "comparative static" relationships that generate implications for the sensitivity of optimal regulation to changes in the economic environment. Our results suggest that optimal capital requirements should be counter-cyclical (e.g. Dewatripont, Rochet, and Tirole (2010)), and should decrease with the tax rate and with the risk-free rate. Bank regulation, therefore, should not be independent of monetary and fiscal policies as suggested by Admati et al. (2011). Larger banks should be subject to stricter capital requirements. We also show that the capital requirement increases with the social cost of financial distress, but is virtually independent of the bank's private distress cost, suggesting that regulatory policy should be mainly concerned with the systemic risk of the banking sector (e.g. Acharya, Cooley, Richardson, and Walter (2010)).

Finally, we examine the robustness of our qualitative and quantitative results to extensions of our basic model. First, we extend the model to incorporate the possibility that the bank's debt can be dynamically restructured either upward or downward when its earnings either increase or decrease sufficiently, respectively. Our main implications are robust to the extended model. In fact, the possibility of future upward leverage adjustments causes the bank's optimal capital requirement to be even higher than in the base model without dynamic debt restructuring. The basic model incorporates *implicit* deposit insurance because of the possibility of bailouts by the regulator/social planner. We also extend the model to allow for *explicit* deposit insurance, where the bank insures a portion of its debt by paying a premium. Our results are robust to this extension too.

## II. Related Literature

We contribute to the literature by developing a dynamic model to qualitatively and quantitatively investigate the design and impact of prudential regulation. We build on insights provided by a number of prior studies. One stream of the literature stresses the role of bank capital in inducing banks' monitoring efforts (Diamond (1984), Giammarino, Lewis, and Sappington (1993), Holmström and Tirole (1997), Allen, Carletti, and Marquez (2011), Mehran and Thakor (2011), Acharya, Mehran, and Thakor (2011)). Another strand of the literature considers the banks' liquid-

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<sup>3</sup>Details of the relevant rules and regulation can be found on the FDIC website at <http://www.fdic.gov/regulations/laws/rules/2000-1300.html>

ity provision role in accepting demand deposits (Diamond and Dybvig (1983), Gorton and Winton (1995), Diamond and Rajan (2000)). In our model, the bank’s special roles in the economy such as liquidity provision are embodied in the excess costs of equity relative to debt and the systemic costs of the bank’s financial distress.

A substantial literature studies the risk-shifting incentives of banks stemming from the early studies of Merton (1977), Kareken and Wallace (1978), and Sharpe (1978).<sup>4</sup> While Koehn and Santomero (1980) and Kim and Santomero (1988) show that a risk-averse banker may actually select a riskier portfolio when facing capital adequacy requirements, Rochet (1992) points out that the limited liability of the bank implies that capital regulation is still helpful in reducing risk-taking behavior. Dewatripont and Tirole (1994) propose the “representation hypothesis” that regulators monitor and supervise banks on behalf of small depositors who lack the incentives or resources to do so. Marshall and Prescott (2001, 2006) consider state-contingent regulation when the bank has moral hazard in project selection. Also related are studies of shareholder-debtholder conflicts and asset substitution for non-financial firms (Jensen and Meckling (1976), Leland (1998)). In our model, we consider the risk-shifting incentives of banks both in terms of increasing leverage ex ante and increasing portfolio risk ex post.

We contribute to the above streams of the literature in a number of respects. First, we develop a continuous-time model in which the regulator has a wide gamut of intervention tools consistent with those observed in reality. Second, the bank can dynamically engage in asset substitution, and its capital structure reflects the tradeoff between the excess costs of issuing equity against financial distress costs that include the agency costs arising from asset substitution and liquidation costs. Third, we analytically characterize the optimal regulatory policy, and calibrate our model to data to obtain quantitative insights into the design and value of prudential regulation as well as the relative effects of different components of regulation on the bank’s value and its social value.

A few recent studies quantitatively analyze the impact of bank capital regulation, using dynamic models with calibrated parameters. Zhu (2008) and Van den Heuvel (2009) develop dynamic models to evaluate the macroeconomic impact of capital regulation. De Nicolò, Gamba, and Lucchetta (2011) considers the impacts of capital regulation, liquidity requirements, and taxation on banks’ behavior and social welfare. They abstract away from the effects of asset substitution. Analogous to the above studies, we too develop a dynamic “calibratable” model that can be used to carry out a quantitative analysis of the design and impact of prudential regulation. Because our model allows analytical solutions, we are able to obtain additional insights on the optimal prudential regulation policies and the behavior of banks in response to these policies.

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<sup>4</sup>See Santos (2001, 2006) for reviews of this literature.

### III. The Model

We develop a continuous-time model of a financial institution that we refer to as a “bank.” The time horizon is  $[0, \infty)$ . Our focus is on shareholder-debtholder agency conflicts so we assume that the bank’s management (hereafter, the “bank”) behaves in the interests of its shareholders.

#### A. The Bank’s Earnings and Project Choices

The bank can dynamically alter the distribution of its earnings by choosing its portfolio of projects (hereafter its “project”). To keep the model parsimonious and computationally tractable, we assume that the bank can choose one of two possible projects, 1 or 2, at any date  $t$ .<sup>5</sup> The bank’s *total earnings* (before debt payments)  $C$  evolve as follows under the two projects:

$$dC_t = \bar{\mu}_1 C_t dt + \sigma_1 C_t d\bar{B}_t, \text{ if project 1 is chosen,} \quad (1)$$

$$dC_t = \bar{\mu}_2 C_t dt + \sigma_2 C_t d\bar{B}_t, \text{ if project 2 is chosen.} \quad (2)$$

In the above,  $\bar{\mu}_1$  and  $\bar{\mu}_2$  are the drifts or expected growth rates of earnings under projects 1 and 2, respectively;  $\sigma_1$  and  $\sigma_2$  are the volatilities, and  $\bar{B}_t$  is a standard Brownian motion that we assume to be the same for the two projects (without loss of generality) to simplify the notation.

The market prices of risk,  $\varsigma_1, \varsigma_2$ , of the two projects and the risk-free rate,  $r$ , are constants. By the principle of *risk-neutral valuation* (see Duffie (2001)), the market prices of claims to the bank’s earnings are given by their expected discount payoffs where the discounting is at the risk-free rate and the expectation is under the risk-neutral measure. Moreover, by Girsanov’s theorem, the bank’s earnings evolve as follows under the two projects in the risk-neutral measure:

$$dC_t = (\bar{\mu}_1 - \varsigma_1 \sigma_1) C_t dt + \sigma_1 C_t dB_t, \text{ if project 1 is chosen,} \quad (3)$$

$$dC_t = (\bar{\mu}_2 - \varsigma_2 \sigma_2) C_t dt + \sigma_2 C_t dB_t, \text{ if project 2 is chosen.} \quad (4)$$

In the above,  $B$  is a Brownian motion under the risk-neutral measure and

$$\begin{aligned} \mu_1 &= \bar{\mu}_1 - \varsigma_1 \sigma_1 \\ \mu_2 &= \bar{\mu}_2 - \varsigma_2 \sigma_2 \end{aligned} \quad (5)$$

are the projects’ *risk-neutral* or *risk-adjusted* drifts. To ensure that asset prices are well-defined, we assume that

$$r > \mu_1, \mu_2 \quad (6)$$

Note from (5) that the risk-neutral drifts could well be close to zero or even negative. As our

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<sup>5</sup>Subramanian and Yang (2012) analyze a more general model with an arbitrary, but finite number of projects and show that the analytical results developed in this study have natural extensions to the general setting.

focus is on shareholder-debtholder agency conflicts arising from *asset substitution* or risk-shifting, we assume that project 1 has a higher risk-neutral drift and lower volatility than project 2, that is,

$$\mu_1 > \mu_2 \text{ and } \sigma_1 < \sigma_2. \quad (7)$$

It is easy to see from (3), (4) and (7) that the NPV of the total future earnings from project 1 at any date is greater than that of project 2. Project 2, therefore, has higher risk and lower NPV than project 1. Our analysis can be easily generalized to accommodate the scenario where  $\mu_1 > \mu_2$  and  $\sigma_1 > \sigma_2$ , but the tradeoffs are less interesting and relevant to the real-world regulation of institutions where the mitigation of value-destroying asset substitution plays an important role. Note from (1), (2) and (5) that we could have the *actual drift*,  $\bar{\mu}_1$ , of project 1 being less than the *actual drift*,  $\bar{\mu}_2$ , of project 2, consistent with the usual risk-return tradeoff (“high risk- high return”) in the physical probability measure, while the *risk-neutral drift* of project 1 is greater. Note that, in contrast with Leland (1998), the projects’ risk-neutral drifts differ from each other.

Henceforth, without loss of generality, we work under the risk-neutral measure. The bank’s earnings as well as the project parameters  $\mu_1, \mu_2, \sigma_1, \sigma_2$  are observable by all market participants. The bank’s actual project choices are, however, only observable by the regulator if it intervenes in the bank’s operations. We show later that the regulatory intervention policy can be designed so that the bank’s dynamic project choices are optimal from the standpoint of the regulator. Consequently, there is no deadweight loss due to the fact that the regulator can only observe the bank’s project choices when it intervenes in the bank’s management. The information filtration generated by the earnings process  $C$  is  $\mathcal{F}_t$ .

## B. Capital Structure

The bank’s capital structure comprises of equity and debt. For simplicity, we assume that the bank’s debt structure requires it to make a constant payment of  $\theta$  per unit time to debtholders.<sup>6</sup> The bank’s capital structure is determined by the parameter  $\theta$ . Note that we make no assumptions about the maturity structure of the bank’s debt; we simply assume that the bank’s total debt structure is such that it makes a payment  $\theta$  per unit time to debtholders.

As in several prior studies (e.g. Giammarino et al. (1993)), equity issuance is costly relative to debt issuance. These costs include direct costs such as the tax disadvantage of equity relative to debt as well as *shadow* costs arising from various sources such as the comparative advantage that banks possess in raising capital through deposits rather than equity, investors’ demand for liquid demand deposits, etc. Some of these costs are more naturally interpreted as “flow” costs, while others are easier to interpret as “lump sum” costs that are incurred whenever equity is issued. It would greatly complicate the exposition to separately model “lump sum” and “flow” costs, and it

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<sup>6</sup>More generally, a bank’s debt structure includes demand deposits as well as longer-term debt obligations. The explicit incorporation of the different components of a bank’s debt structure is well beyond the scope of this study.



is also hard to independently calibrate the corresponding parameters in our quantitative analysis. Consequently, we choose to model the excess costs of equity as “flow” costs throughout. It should be borne in mind, however, that we can alternately model these as “lump sum” costs or a combination of “flow” and “lump sum” costs. The alternate modeling choices do not affect the main insights of the study.

In any period  $[t, t + dt]$ , as long as the bank is solvent, the cash flow to debt holders is  $\theta dt$  and the cash flow to equity holders is

$$\text{Net cash flow to equity holders} = (C_t - \theta)dt - \lambda_1(C_t - \theta)dt \text{ if } C_t \geq \theta \quad (8)$$

$$= (C_t - \theta)dt - \lambda_2(\theta - C_t)dt \text{ if } C_t < \theta, \quad (9)$$

where

$$\lambda_1, \lambda_2 > 0 \quad (10)$$

In (8),  $\lambda_1$  is the excess cost of equity modeled as a flow cost. In (9),  $\lambda_2$  is the proportional cost associated with the bank’s financial distress when its earnings are insufficient to meet debt payments. In the spirit of studies such as Leland (1994), we model the excess costs of equity issuance directly as “calibratable” parameters. A major extension of our framework, which we leave for future research, would be to endogenize the excess costs in a general equilibrium framework.

We allow for the bank to raise funds in financial distress by issuing equity and/or borrowing from the central bank. To keep the model parsimonious, we assume that either channel is equally costly for the bank so that we can assume henceforth that the bank issues additional equity in financial distress. We can also allow for the bank to liquidate assets in financial distress, but this is usually costlier than issuing equity because it lowers the bank’s cash flow generating capacity (see Leland (1994)). The bank services debt payments entirely as long as it is solvent. If the bank’s earnings are insufficient to meet debt payments, it can issue additional equity. After the equity value falls to zero, the regulator services debt payments by injecting capital until it is no longer optimal to do so.

In the main model, we assume that the bank cannot restructure its debt for simplicity. It could be optimal for a bank to increase its debt when its earnings improve sufficiently. In contrast, it is very costly for a bank to lower its debt burden in reality. A substantial portion of bank debt comprises of demand deposits that are difficult to close down without the bank incurring substantial costs such as those arising from a loss of reputation because of its important role as a provider of liquidity. Further, if the bank retires debt (either demand deposits or other debt), it usually must do so at par value that is significantly higher than the market value of its debt when it is in financial distress, that is, the value of continuing to service its existing debt as long as it is able to do so. As argued by Leland (1994) and Admati et al. (2012), the banks’ shareholders will be worse off if they retire debt at par value or at the market value after debt retirement (Leland (1994), Admati

et al (2012)). Nevertheless, it could still be optimal for a regulator to force a recapitalization of the bank by lowering its debt burden and, thereby, reduce the social costs of financial distress. In Section E and the online Appendix, we extend the model to allow for costly dynamic upward and downward debt restructuring. We show that the qualitative and quantitative implications of the main model remain robust to the extended model.

The bank services debt by issuing equity until an  $\mathcal{F}_t$ -stopping time  $\tau_B$  when it becomes insolvent (recall that  $\mathcal{F}_t$  is the information filtration generated by the bank's earnings process.) For  $t > \tau_B$ , the regulator takes control of the bank (in effect, it becomes the owner of the bank), injects capital, and continues to do so until a stopping time  $\tau_L \geq \tau_B$ . Hence, the bank services debt for  $t < \tau_B$ , and the regulator services debt by injecting capital for  $t \in (\tau_B, \tau_L)$ . Note that, in the region  $(\tau_B, \tau_L)$ , the bank's equity value is negative, that is, the regulator effectively holds a negative equity stake in the firm. The bank's assets are liquidated at  $t = \tau_L$  when the regulator decides to stop injecting capital, and debtholders receive the resulting liquidation payoff.

There are deadweight liquidation costs that we model by assuming that debtholders receive a proportion  $1 - \alpha$ ;  $\alpha \in (0, 1)$  of the *unlevered bank value* at the liquidation threshold. More precisely, it follows immediately from (3) that the value of the (hypothetically) unlevered bank at the liquidation threshold is

$$V_{\tau_L}^{\text{unlevered}} = \frac{(1 - \lambda_1)C_{\tau_L}}{r - \mu_1}. \quad (11)$$

The payoff to debtholders is

$$D_{\tau_L} = (1 - \alpha) \frac{(1 - \lambda_1)C_{\tau_L}}{r - \mu_1} \quad (12)$$

In reality, bank debt comprises of demand deposits that are protected by deposit insurance (up to a limit) as well as longer-term debt obligations that are not insured. Our framework actually incorporates the possibility of (implicit) deposit insurance because we allow the regulator to inject capital after the bank goes bankrupt, and this is also the version of the model that we calibrate. In our setting, the regulator is essentially the “social planner” who also embodies the role of the deposit insurer. In the online Appendix, we show that the optimal regulatory policies and thus the normative implications of our study are not affected by the *explicit* incorporation of deposit insurance.

### C. Regulation

The regulator observes the bank's earnings and can choose to intervene in the bank's management. The regulator's intervention policy could involve controlling the bank's risk as well as injecting capital to keep the bank solvent. As discussed in the previous section, the regulator injects capital in the region  $(\tau_B, \tau_L)$ . We denote the bank's strategy by an  $\mathcal{F}_t$ -adapted process

$$\Pi = (P, \tau_B). \quad (13)$$

In the above,  $P_t \in \{1, 2\}$  is the bank's project choice at date  $t$ , and  $\tau_B$  is the  $\mathcal{F}_t$ -stopping time at which the bank stops issuing equity to service debt and declares insolvency .

We denote the regulator's intervention policy by an  $\mathcal{F}_t$ -adapted process

$$\Pi^{reg} \equiv (P^{reg}, \tau_L). \quad (14)$$

In the above,  $P_t^{reg} \in \{1, 2\}$  is the regulator's project choice at date  $t$  and  $\tau_L$  is the  $\mathcal{F}_t$ -stopping time at which the regulator stops injecting capital and liquidates the bank. If the regulator forces the bank to choose a particular project at date  $t$ , then  $P_t = P_t^{reg}$ . Note that the above formulation is quite general and encompasses the possibilities that the regulator chooses not to intervene in the bank's operations at a particular date  $t$  and control its project/risk choice.

The proportional deadweight *social cost* of capital injection by the regulator is  $\lambda_2^{social}$ , which could differ from the cost of capital injection by the bank in financial distress. In other words, if the regulator injects capital  $k_t$ , the social cost is  $\lambda_2^{social} k_t$ . The wedge between  $\lambda_2^{social}$  and  $\lambda_2$  reflects the effects of the externality that the bank imposes on the economy because of its financial distress that the regulator internalizes.

#### D. Equity, Debt, Bank and Social Values

Let  $\Pi$  and  $\Pi^{reg}$  denote strategies for the bank and the regulator, respectively, as defined in the previous section. If the bank's debt level is  $\theta$ , the bank's debt value at any date  $t$  is

$$D_t(\theta, \Pi, \Pi^{reg}) = E_t \int_t^{\tau_L} e^{-r(u-t)} \theta du + E_t [e^{-r(\tau_L-t)} D_{\tau_L}], \quad (15)$$

where  $E_t$  denotes the conditional expectation at date  $t$ . The debt payoff  $D_{\tau_L}$  upon liquidation is given by (12).

By (9), the bank's equity value at date  $t$  is

$$S_t(\theta, \Pi, \Pi^{reg}) = E_t \int_t^{\tau_B} e^{-r(u-t)} \left[ \begin{array}{l} 1_{C_u(\Pi; \Pi^{reg}) \geq \theta} (1 - \lambda_1) (C_u(\Pi, \Pi^{reg}) - \theta) + \\ 1_{C_u(\Pi; \Pi^{reg}) < \theta} (1 + \lambda_2) (C_u(\Pi, \Pi^{reg}) - \theta) \end{array} \right] du. \quad (16)$$

The upper limit of the integral above is the stopping time  $\tau_B \leq \tau_L$  at which the bank becomes insolvent. In the integrand, we explicitly indicate the fact that the earnings process  $C_u$  depends on the bank's and the regulator's strategies.

The bank's total value at date  $t$  is the sum of its equity and debt values, and is given by

$$\begin{aligned}
F_t(\theta, \Pi, \Pi^{reg}) &= D_t(\theta, \Pi, \Pi^{reg}) + S_t(\theta, \Pi, \Pi^{reg}) \\
&= E_t \int_t^{\tau_B} e^{-r(u-t)} \left\{ \left[ \begin{array}{l} 1_{C_u(\Pi, \Pi^{reg}) \geq \theta} ((1 - \lambda_1)C_u(\Pi, \Pi^{reg}) + \lambda_1\theta) + \\ 1_{C_u(\Pi, \Pi^{reg}) < \theta} ((1 + \lambda_2)C_u(\Pi, \Pi^{reg}) - \lambda_2\theta) \end{array} \right] \right\} ds \\
&\quad E_t \int_{\tau_B}^{\tau_L} \theta ds + E_t[e^{-r(\tau_L-t)} D_{\tau_L}].
\end{aligned} \tag{17}$$

Finally, we define the bank's *social* value at date  $t$ , which also includes the net cash flows to the regulator when it takes control of the bank during the interval  $[\tau_B, \tau_L]$  and injects capital.

$$\begin{aligned}
F_t^{social}(\theta, \Pi, \Pi^{reg}) &= E_t \int_t^{\tau_L} e^{-r(u-t)} \left\{ \left[ \begin{array}{l} 1_{C_u(\Pi, \Pi^{reg}) \geq \theta} \left( (1 - \lambda_1^{social})C_u(\Pi, \Pi^{reg}) + \lambda_1^{social}\theta \right) + \\ 1_{C_u(\Pi, \Pi^{reg}) < \theta} \left( (1 + \lambda_2^{social})C_u(\Pi, \Pi^{reg}) - \lambda_2^{social}\theta \right) \end{array} \right] \right\} ds \\
&\quad + E_t[e^{-r(\tau_L-t)} D_{\tau_L}^{social}],
\end{aligned} \tag{18}$$

where

$$D_{\tau_L}^{social} = (1 - \alpha^{social}) \frac{(1 - \lambda_1^{social})C_{\tau_L}}{r - \mu_1}. \tag{19}$$

The bank's social value (18) differs from its total value (17) in a number of important dimensions. First, the excess cost of equity in “good” states from the regulator's standpoint,  $\lambda_1^{social}$ , could differ from  $\lambda_1$ , reflecting the fact that a portion of the excess cost of equity, such as the tax disadvantage of equity, are simply wealth transfers that do not affect the bank's social value (see Duffie (2001, Chapter 8) and Admati et al. (2011)). However, other costs of equity issuance such as those affecting a bank's role as an intermediary in issuing deposits are deadweight costs that are not simply transfers. The regulator, therefore, acknowledges the important role that banks play as financial intermediaries and, in particular, providers of liquidity. Second, the financial distress cost  $\lambda_2^{social}$  from the regulator's standpoint differs, in general, from  $\lambda_2$  because the regulator incorporates the effects of the externality that the bank imposes on the economy through its financial distress. For similar reasons, the proportional liquidation cost  $\alpha^{social}$  as seen by the regulator also differs from the proportional liquidation cost,  $\alpha$ , incurred by the bank. As with the bank's “private” cost parameters,  $\lambda_1, \lambda_2, \alpha$ , we model the “social” cost parameters,  $\lambda_1^{social}, \lambda_2^{social}, \alpha^{social}$ , directly to create an analytically and computationally tractable model. In our numerical analysis, we examine the implications of varying our parameters, including the social cost parameters, about their baseline values.

We first analyze the benchmark scenario where the bank is completely unregulated, that is, the regulator's intervention policy is the “null” policy of no intervention at all. Hence, the bank's equity, debt and total values are only functions of the debt level  $\theta$  and the bank's project choices

II. Further, because the regulator injects no capital at all, the liquidation time  $\tau_L$  coincides with the insolvency time  $\tau_B$ . Given a debt level  $\theta$ , the bank chooses its optimal strategy—its dynamic project choices and the insolvency time—to maximize its equity value, that is, the bank’s optimal strategy (which depends on its debt level) solves

$$\Pi^*(\theta) = \arg \max_{\Pi} S_0(\theta, \Pi). \quad (20)$$

Note that the regulator’s intervention strategy does not appear at all as an argument because it is the “null” policy of no intervention. The bank’s capital structure, which is determined by the debt level  $\theta$  maximizes its total value at date zero, that is,

$$\theta^* = \arg \max_{\theta} F_0(\theta, \Pi^*(\theta)) \quad (21)$$

Next, we consider the “full regulation” scenario in which the regulator completely controls the bank and chooses its capital structure as well as its projects to maximize the bank’s social value. Given a debt level  $\theta$ , the regulator’s optimal strategy—its project choices and liquidation time—solves

$$\Pi^{*reg}(\theta) = \arg \max_{\Pi} F_0^{social}(\theta, \Pi^{reg}). \quad (22)$$

The regulator chooses the bank’s capital structure to maximize the bank’s social value, that is,

$$\theta^{*reg} = \arg \max_{\theta} F_0^{social}(\theta, \Pi^{*reg}(\theta)) \quad (23)$$

To analyze the effects of different regulatory intervention tools—capital regulation, intervention in the bank’s project choices, and capital injection—we also explore a number of “intermediate” scenarios in which one or more of these tools are “turned off.” To avoid complicating the exposition here, we discuss these scenarios when we analyze them later.

#### IV. The Unregulated Bank

In the hypothetical scenario where the bank is completely unregulated, the bank’s dynamic project choices, the insolvency time, and capital structure solve (20) and (21). We first derive the bank’s optimal dynamic project and insolvency time choices for a given capital structure described by the debt level  $\theta$ . We then derive the bank’s optimal capital structure choice. The following proposition analytically characterizes the bank’s equity and debt values as well as the insolvency threshold for a *given* switching policy described by a trigger  $C_S$ .

**Proposition 1** [*Equity and Debt Values for Given Switching Policy*] *Suppose the debt level is  $\theta$  and the bank adopts a policy where it chooses project 1 if  $C_t \geq C_S$  and project 2 for  $C_t < C_S$ . Suppose that  $C_S \geq \theta$ . (For brevity, we show here the representation for the case  $C_S \geq \theta$ . In the*

case where  $C_S < \theta$ , equity values have similar representations that we present in the appendix.)

(i) The bank's equity value at any date  $t$  is a continuously differentiable function of the current earnings level  $C_t$  and is given by

$$S_t = S(C_t) = \begin{cases} \frac{(1-\lambda_1)C_t}{r-\mu_1} - \frac{(1-\lambda_1)\theta}{r} + A_1(C_S, \theta)C_t^{\gamma_1^-}, & \text{if } C_t \geq C_S, \\ \frac{(1-\lambda_1)C_t}{r-\mu_2} - \frac{(1-\lambda_1)\theta}{r} + A_2(C_S, \theta)C_t^{\gamma_2^-} + A_3(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } C_S > C_t \geq \theta \\ \frac{(1+\lambda_2)C_t}{r-\mu_2} - \frac{(1+\lambda_2)\theta}{r} + A_4(C_S, \theta)C_t^{\gamma_2^-} + A_5(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } \theta > C_t \geq C_B(C_S, \theta), \end{cases} \quad (24)$$

Furthermore, the equity value satisfies the following value matching and smooth pasting conditions at the insolvency threshold  $C_B(C_S, \theta)$

$$S(C_B(C_S, \theta)) = S'(C_B(C_S, \theta)) = 0. \quad (25)$$

(ii) The bank's debt value at any date  $t$  is continuously differentiable and given by

$$D_t = D(C_t) = \begin{cases} \frac{\theta}{r} + B_1(C_S, \theta)C_t^{\gamma_1^-}, & \text{if } C_t \geq C_S, \\ \frac{\theta}{r} + B_2(C_S, \theta)C_t^{\gamma_2^-} + B_3(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } C_S > C_t \geq C_B(C_S, \theta). \\ D(C_B(C_S, \theta)) = \frac{(1-\alpha)(1-\lambda_1)C_B(C_S, \theta)}{r-\mu_1} \end{cases} \quad (26)$$

The coefficients  $A_i(C_S, \theta); i = 1, \dots, 5$ ;  $B_j(C_S, \theta); j = 1, \dots, 3$ , and the insolvency level  $C_B(C_S, \theta)$  are determined by the conditions that the equity and debt values are continuously differentiable for all  $C_t > C_B(C_S, \theta)$  and the value-matching and smooth-pasting conditions (25). In the above,  $\gamma_i^+, \gamma_i^-; i = 1, 2$  are the positive and negative roots, respectively, of the equation

$$\frac{\sigma_i^2}{2}\gamma^2 + (\mu_i - \frac{\sigma_i^2}{2})\gamma - r = 0; i = 1, 2. \quad (27)$$

Note that the coefficients  $A_i(\cdot), B_i(\cdot)$ , and the insolvency level  $C_B(\cdot)$  depend on the switching trigger  $C_S$  and the debt level  $\theta$ . The asset values and the insolvency level can be analytically characterized in closed form. The following result shows that the optimal dynamic project choice policy of the bank has the structure assumed in Proposition 1, that is, there is a single time-independent trigger at which the bank switches between the two projects.

**Theorem 2** [Optimal Switching Policy] *Given an initial choice of debt level  $\theta$ , there exists a unique cash flow threshold  $C_S^*(\theta)$  such that the bank optimally chooses project 1 when  $C_t \geq C_S^*(\theta)$  and project 2 when  $C_S^*(\theta) > C_t > C_B^*(\theta)$ .  $C_B^*(\theta)$  is the insolvency trigger that is determined in Proposition 1 when the switching trigger is  $C_S^*(\theta)$ . The unique switching threshold  $C_S^*(\theta)$  is characterized by the second-order ‘‘super contact’’ condition,*

$$\frac{\partial^2 S}{\partial C^2} \Big|_{C=C_S^*(\theta)+} = \frac{\partial^2 S}{\partial C^2} \Big|_{C=C_S^*(\theta)-} . \quad (28)$$

By the discussion in Section A, in the absence of any frictions, the bank would always choose project 1 because it has a higher NPV than project 2. In the presence of shareholder-debtholder agency conflicts, however, the bank's shareholders have the incentive to engage in asset substitution, thereby diverting cash flows from debtholders. Asset substitution potentially occurs when the bank's earnings are below a threshold so that the bank is sufficiently leveraged. Note that, depending on the drifts and volatilities of the two projects, it may be optimal for the bank to always choose project 1, that is, not engage in risk-shifting at all. This scenario corresponds to the case where the switching threshold  $C_S^*(\theta)$  coincides with the insolvency threshold  $C_B^*(\theta)$ . Condition (28) is equivalent to requiring that the equity value is twice differentiable throughout. As we show in the proof of Theorem 2, the optimality of a "single switching threshold" policy is far from obvious and depends on the fact that the excess cost of equity is a general concave function in the level of cash flows. Subramanian and Yang (2012) analyze a more general model of dynamic risk-taking with an arbitrary number  $N$  of projects and a broader class of payoff functions. Under additional conditions on the payoff function (that are satisfied in the context of this study), they show that the optimal policy is characterized by at most  $N - 1$  switching triggers. Moreover, they provide counterexamples to show that, if the additional conditions are not satisfied, the optimal policy in the case of two projects could be characterized by multiple switching triggers.

Given the debt structure  $\theta$ , the initial leverage ratio of the bank is  $Lev_0 = D(C_0; \theta)/F(C_0; \theta)$ , where we also indicate the dependence of the debt value and bank value on the debt level  $\theta$ . The debt value and bank value are determined by Proposition 1 with the switching trigger set to the optimal switching trigger  $C_S^*(\theta)$ . We define the bank's leverage ratio at the switching point  $C_S^*(\theta)$  as the *switching leverage*. The switching leverage is given by

$$\text{Switching Leverage} = D(C_S^*(\theta); \theta)/F(C_S^*(\theta); \theta). \quad (29)$$

As described by Theorem 2, the bank has an incentive to engage in asset substitution by selecting the riskier, lower NPV project 2 once its leverage reaches the switching leverage. Therefore, the switching leverage is of importance to regulators who would like to limit the agency costs of ex post asset substitution. The following proposition provides a simple and useful characterization of the switching point and leverage for a given debt level  $\theta$ .

**Proposition 3** 1) The default boundary  $C_B^*(\theta)$  and switching trigger  $C_S^*(\theta)$  are linear functions of  $\theta$ , that is,

$$C_S^*(\theta) = c_S\theta; \quad C_B^*(\theta) = c_B\theta, \quad . \quad (30)$$

2) The switching leverage is a constant that does not depend on the initial choice of  $\theta$ .

Intuitively, the banker chooses the switching point in proportion to the initial debt level. When the debt level is high, agency costs are larger, and the banker would choose to switch to the risky project at a higher threshold. The switching leverage, however, is not sensitive to the initial choice of debt level. Rather, it is sensitive to the relative characteristics (drifts and volatilities) of the two projects. The result of Proposition 3 is useful because it provides a justification for some aspects of the FDIC rules and Basel accords where intervention by regulators can be triggered when a bank’s leverage reaches a pre-specified threshold *regardless of its initial leverage level*.

The unregulated bank’s optimal capital structure choice is determined by solving (21). The switching and insolvency triggers at the optimal capital structure,  $\theta^{unreg}$  (the superscript indicates that this is the scenario with no regulation) are, therefore, given by

$$C_S^{unreg} = C_S^*(\theta^{unreg}) = c_S \theta^{unreg}; \quad C_B^{unreg} = C_B^*(\theta^{unreg}) = c_B \theta^{unreg}, \quad . \quad (31)$$

where the constants  $c_S$  and  $c_B$  are defined in Proposition 3. Since the optimal capital structure cannot be analytically characterized in closed form, we compute it numerically in our subsequent analysis of the calibrated model.

## V. The Fully Regulated Bank

In the scenario in which the regulator completely controls the bank’s operations, the regulatory intervention policy is the policy of “total intervention” that is the polar opposite of the “null” intervention policy considered in the previous section. Such a policy achieves the first-best outcome for social welfare. From the regulator’s standpoint, the bank’s capital structure, project choices, the insolvency time, and the liquidation time should maximize the bank’s social value, that is, they should solve

$$(\theta^{reg}, \Pi^{reg}) = \arg \max_{(\theta, \Pi)} F_0^{social}(\theta, \Pi). \quad (32)$$

In the above, the superscript “reg” indicates the fact that this is the scenario where the regulator completely controls the bank’s operations.

As in the previous section, we first derive the regulator’s dynamic project choices for a given capital structure, and then derive the optimal capital structure. It turns out that the regulator’s optimal project choice policy has a similar structure to that of the bank in the previous section. There exists a threshold level of the bank’s earnings such that it is optimal for the regulator to choose project 1 above the threshold and project 2 below the threshold. The regulator’s optimal “switching threshold” differs, in general, from the bank’s switching threshold derived in the previous section. The following proposition analytically characterizes the bank’s social value for a given switching policy described by a trigger  $C_S$ . We omit its proof because it follows using exactly the same arguments used to prove Proposition 1.



**Proposition 4** [*Equity, Debt and social Values for Given Switching Policy*] Suppose the debt level is  $\theta$  and the regulator adopts a policy where it chooses project 1 if  $C_t \geq C_S$  and project 2 for  $C_t < C_S$ . Suppose that  $C_S \geq \theta$ . (For simplicity, we show here the representation for the case  $C_S \geq \theta$ . In the case where  $C_S < \theta$ , social values have similar representations that we present in the appendix.)

(i) The bank's social value at any date  $t$  is a continuously differentiable function of the current earnings level  $C_t$  and is given by

$$F_t^{\text{social}} = F^{\text{social}}(C_t) = \begin{cases} \frac{(1-\lambda_1^{\text{social}})C_t}{r-\mu_1} + \frac{\lambda_1^{\text{social}}\theta}{r} + X_1(C_S, \theta)C_t^{\gamma_1^-}, & \text{if } C_t \geq C_S, \\ \frac{(1-\lambda_1^{\text{social}})C_t}{r-\mu_2} + \frac{\lambda_1^{\text{social}}\theta}{r} + X_2(C_S, \theta)C_t^{\gamma_2^-} + X_3(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } C_S > C_t \geq \theta, \\ \frac{(1+\lambda_2^{\text{social}})C_t}{r-\mu_2} - \frac{\lambda_2^{\text{social}}\theta}{r} + X_4(C_S, \theta)C_t^{\gamma_2^-} + X_5(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } \theta > C_t \geq C_L(C_S, \theta). \end{cases} \quad (33)$$

Furthermore, the social value satisfies the following value matching and smooth pasting boundary conditions at the liquidation threshold  $C_L(C_S, \theta)$ ,

$$F^{\text{social}}(C_L(C_S, \theta)) = \frac{(1 - \alpha^{\text{social}})(1 - \lambda_1^{\text{social}})C_L(C_S, \theta)}{r - \mu_1}, \quad (34)$$

$$F^{\text{social}'}(C_L(C_S, \theta)) = \frac{(1 - \alpha^{\text{social}})(1 - \lambda_1^{\text{social}})}{r - \mu_1}. \quad (35)$$

The coefficients  $X_i(C_S, \theta); i = 1, \dots, 5$  in the above are determined by the conditions that the social value is continuously differentiable throughout and the value matching boundary condition (34). The liquidation level  $C_L(C_S, \theta)$  is determined by the smooth pasting condition (35) for the social value at the liquidation level.

(ii) The bank's equity value at any date  $t$  is given by (24) where the switching trigger is  $C_S$ . The endogenous insolvency threshold at which the equity value falls to zero is  $C_B(C_S, \theta)$ , which is determined by the smooth pasting condition in (25).

(iii) The bank's debt value at any date  $t$  is

$$D_t = D(C_t) = \begin{cases} \frac{\theta}{r} + B_1(C_S, \theta)C_t^{\gamma_1^-}, & \text{if } C_t \geq C_S, \\ \frac{\theta}{r} + B_2(C_S, \theta)C_t^{\gamma_2^-} + B_3(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } C_S > C_t \geq C_L(C_S, \theta). \\ D(C_L(C_S, \theta)) = \frac{(1-\alpha^{\text{social}})(1-\lambda_1^{\text{social}})C_L(C_S, \theta)}{r-\mu_1} \end{cases} \quad (36)$$

Comparing Proposition 1 and 4, we note that the insolvency trigger is now replaced by the liquidation trigger  $C_L(C_S, \theta)$ .

The following result shows that the optimal dynamic project choice policy of the regulator has a structure similar to that of the bank, that is, there is a single time-independent trigger at which the regulator switches between the two projects.

**Theorem 5** [*Regulator's Optimal Switching Policy*] Given an initial choice of debt level  $\theta$ , there

exists a unique cash flow threshold  $C_S^{*reg}(\theta) \leq \theta$  such that the regulator optimally chooses project 1 when  $C_t \geq C_S^{*reg}(\theta)$  and project 2 when  $C_S^{*reg}(\theta) > C_t > C_L^{*reg}(\theta)$ .  $C_L^{*reg}(\theta)$  is the liquidation trigger that is determined in Proposition 1 when the switching trigger is  $C_S^{*reg}(\theta)$ . The unique switching threshold  $C_S^{*reg}(\theta)$  is characterized by the second-order “super contact” condition,

$$\left. \frac{\partial^2 F^{social}}{\partial C^2} \right|_{C=C_S^{*reg}(\theta)+} = \left. \frac{\partial^2 F^{social}}{\partial C^2} \right|_{C=C_S^{*reg}(\theta)-}. \quad (37)$$

The above theorem shows that, contrary to received wisdom in anecdotal discussions of prudential regulation, it could actually be optimal for the regulator too to engage in asset substitution if the bank’s earnings are below a threshold. The theorem, however, shows that the regulator’s optimal switching trigger is never greater than the debt level,  $\theta$ . If the regulator finds it optimal to engage in asset substitution, therefore, it always does so when the bank is in financial distress, which is not the case for the unregulated bank as shown by Theorem 2 and its proof. It is worth mentioning that the regulator’s optimal switching trigger could actually lie below the insolvency threshold at which the bank’s equity value falls to zero, that is, it could lie in the “capital injection” region.

The intuition for why the regulator could also find it optimal to engage in asset substitution is as follows. First, when it injects capital, the regulator effectively holds a (negative) *equity stake* in the bank. The regulator’s objective function in this region is, therefore, akin to that of shareholders of the unregulated bank in the previous section. Consequently, when the bank’s earnings fall sufficiently far, it could be optimal for the regulator to increase risk to lower the expected costs of capital injection after insolvency. Second, by increasing risk, the regulator also delays liquidation and, thereby, lowers the probability of liquidation of the bank and the associated liquidation costs.

Because the regulator cares about the bank’s social value rather than just its equity value, one might expect that it has lower incentives to engage in asset substitution than the bank’s shareholders. As the following proposition shows, however, this is only ensured under additional conditions on the characteristics of the bank’s projects as well as its private and social cost parameters.

**Proposition 6** [*Comparison of Bank’s and Regulator’s Optimal Switching Policies*]

Suppose  $\frac{1-\lambda_1^{social}}{1-\lambda_1} \geq \frac{1+\lambda_2^{social}}{1+\lambda_2}$  and  $\gamma_2^+ > \gamma_1^+$ , then the optimal switching trigger for the regulator is less than or equal to the optimal switching trigger for the unregulated bank, that is,

$$C_S^{*reg}(\theta) \leq C_S^*(\theta). \quad (38)$$

If the conditions of the proposition are violated, it could actually be optimal for the regulator to switch to the high-risk project earlier than the bank’s shareholders. In particular, if  $\frac{1-\lambda_1^{social}}{1-\lambda_1} < \frac{1+\lambda_2^{social}}{1+\lambda_2}$ , which is true, for example, when the social distress cost  $\lambda_2^{social}$  is large, the incidence of asset substitution could actually be *greater* in the regulator’s problem than the bank’s problem.

Consequently, rather surprisingly, asset substitution might have significantly positive *social value*. Interestingly, the prediction that it could be optimal for the regulator too to engage in asset substitution is consistent with anecdotal evidence of increased risk-taking by Irish banks even after being bailed out.

Analogous to Proposition 3, the following proposition shows that the switching, insolvency, and liquidation thresholds are linear in the debt level  $\theta$ , and the switching leverage is constant. We omit the proof because it follows along the lines of the proof of Proposition 3.

**Proposition 7** 1) *The insolvency threshold  $C_B^{*reg}(\theta)$ , the switching threshold  $C_S^{*reg}(\theta)$ , and the liquidation threshold  $C_L^{*reg}(\theta)$  are all proportional to the debt level  $\theta$ ,*

$$C_B^{*reg}(\theta) = c_B^{reg}\theta, \quad C_S^{*reg}(\theta) = c_S^{reg}\theta, \quad C_L^{*reg}(\theta) = c_L^{reg}\theta. \quad (39)$$

where  $c_B^{reg}, c_S^{reg}, c_L^{reg}$  are constants independent of  $\theta$ .

2) *The switching leverage is a constant that does not depend on the initial choice of  $\theta$ .*

The regulator chooses the bank's capital structure to maximize the social value, that is, the optimal debt level  $\theta^{reg}$  solves

$$\theta^{reg} = \arg \max_{\theta} F_0^{social}(\theta), \quad (40)$$

where the bank's social value,  $F_0^{social}(\theta)$ , is given by Proposition 4 with the switching trigger set to  $C_S^{*reg}(\theta)$ . The switching, insolvency and liquidation triggers at the optimal capital structure are, therefore, given by

$$C_S^{reg} = C_S^{*reg}(\theta^{reg}) = c_S^{reg}\theta^{reg}, \quad C_B^{reg} = C_B^{*reg}(\theta^{reg}) = c_B^{reg}\theta^{reg}, \quad C_L^{reg} = C_L^{*reg}(\theta^{reg}) = c_L^{reg}\theta^{reg}. \quad (41)$$

## VI. Optimal Regulatory Policy

We now use the results of the previous two sections to describe the optimal intervention policy of the regulator.

First, by (40), it is optimal for the regulator to force the bank to choose a debt level  $\theta^{reg}$ . For future reference, we define the *initial capital requirement* to be  $1 - D(\theta^{reg})/F(\theta^{reg})$ , i.e., one minus the initial leverage ratio or the equity capital ratio.

Second, by Theorem 2, given the debt level  $\theta^{reg}$ , the bank chooses a project choice policy where the switching trigger is  $C_S^{*bank} = C_S^*(\theta^{reg})$ . However, by Proposition 5, from the standpoint of the regulator, it is optimal to switch projects at  $C_S^{reg}$ . By Proposition 6,  $C_S^{reg} < C_S^{*bank}$ . In other words, the regulator has lower incentives to shift risk than the bank's shareholders. To align the bank's project choices with that of the regulator, the regulator intervenes when the bank's earnings cross the threshold  $C_S^{*bank}$  and forces the bank to choose the lower risk project 1. We define the *intervention ratio* to be  $1 - Lev_{C_S^*(\theta^{reg})}$ , or the capital ratio of the bank at the switching leverage.

Third, by Proposition 5, it is optimal for the regulator to choose the higher risk project 2 when earnings fall below the threshold  $C_S^{reg}$ . Consequently, it is optimal for the regulator to stop intervening in the bank’s operations and allow it to shift risk in this region. Hence, the regulator follows a “band policy,” that is, it intervenes in the bank’s project choice in the region  $(C_S^{reg}, C_S^{*bank})$ .

Fourth, by (41), the insolvency level at which the equity value falls to zero is  $C_B^{reg}$  and the liquidation level is  $C_L^{reg}$ . The regulator, therefore, allows the bank to issue equity to service debt payments as long as the earnings exceed  $C_B^{reg}$ . When earnings fall below  $C_B^{reg}$ , but exceed  $C_L^{reg}$ , it is optimal for the regulator to inject capital to keep the bank afloat.

Fifth, if earnings hit the level  $C_L^{reg}$ , it is optimal for the regulator to liquidate the bank’s assets and pay off debtholders with the liquidation proceeds.

Figure 1 plots sample paths for a fully regulated bank to illustrate the above optimal regulatory policy. The bank begins with an earnings level above the debt level  $\theta^{reg}$  and chooses the low-risk project 1. When earnings hit the *liquidity threshold*  $\theta^{reg}$ , the bank begins to issue external equity in order keep servicing debt. When earnings further declines to *the bank’s switching threshold*  $C_S^{*bank}$ , despite the bank’s incentive to switch to the high-risk project 2, the regulator steps in and forces the bank to continue with the low-risk project 1. For illustrative purposes, the figure plots a particular case where asset substitution occurs when the bank is in financial distress, i.e.,  $\theta^{reg} > C_S^{*bank}$ . In general, however, it is possible for the bank to engage in asset substitution even when it is not in financial distress, i.e.,  $\theta^{reg} \leq C_S^{*bank}$ . When the bank’s earnings reaches the *insolvency threshold*  $C_B^{reg}$ , the bank’s equity value falls to zero and the regulator bails out the bank by starting to inject capital into the bank. The regulator switches the bank’s project to the high-risk project 2 when earnings reaches *the regulator’s switching threshold*  $C_S^{reg}$ , and liquidates the bank if earnings further deteriorates to the *liquidation threshold*  $C_L^{reg}$ . The figure shows that the regulator intervenes in the bank’s project choices when its earnings lie in the “risk intervention” region  $[C_S^{reg}, C_S^{*bank}]$ . The regulator injects capital into the bank when its earnings lie in the “capital injection” region  $[C_L^{reg}, C_B^{reg}]$ . In another sample path plotted in the figure, the bank’s earnings improve and the bank stops issuing equity when earnings rise to the liquidity threshold  $\theta^{reg}$ .

## VII. Quantitative Analysis

To obtain a reasonable set of baseline parameter values for our quantitative analysis of the effects of regulation, we calibrate the model to data on U.S. banks over the period 1991–2008. It has become evident that, over most of this time period, U.S. banks were only weakly regulated, if at all. Yet, many distressed banks were bailed out with public funds after the crisis. Moreover, a substantial portion of bank liabilities comprises of demand deposits that are insured (up to a limit). Consequently, the model that we calibrate to the data is an “intermediate” one where the bank chooses its capital structure and projects, but enjoys the “safety net” provided by capital injection after the bank’s equity value falls to zero.

Without loss of generality, we normalize the bank’s initial earnings,  $C_0$ —the state variable in the model—to 1. The set of model parameters that need to be calibrated is

$$\Gamma \equiv \left\{ r, \mu_1, \mu_2, \sigma_1, \sigma_2, \lambda_1, \lambda_1^{social}, \lambda_2, \lambda_2^{social}, \alpha, \alpha^{social} \right\}$$

We set the risk-free rate  $r$  to 0.035 to match the average short-term U.S. treasury rate over the period 1991 – 2008.<sup>7</sup> Tax shields on debt interest payments represent one of the sources of equity issuance costs faced by individual banks. However, from a social standpoint, tax shields are merely wealth transfers, which would suggest that  $\lambda_1^{social}$  should be less than  $\lambda_1$ . However, the bank’s personal costs of equity issuance may not completely reflect the “general equilibrium” effects of the redistribution of wealth, and the relative composition of securities issued by the bank among heterogeneous investors some of whom prefer information-insensitive debt-like securities, while others prefer risky equity-like securities. These effects could cause  $\lambda_1^{social}$  to even exceed  $\lambda_1$ . Since it is difficult (if not impossible) to calibrate  $\lambda_1^{social}$ , we set it equal to  $\lambda_1$  in our baseline analysis, and subsequently explore the effects of varying its value. The liquidation cost  $\alpha = 0.20$  is in the ranges estimated by Andrade and Kaplan (1998) and Davydenko, Strebulaev, and Zhao (2011).

We calibrate the social distress and liquidation cost parameters  $\lambda_2^{social}$  and  $\alpha^{social}$  by trying to match the ratio of economic costs to fiscal costs of bailouts incurred by a banking crisis. By examining a sample of banking crises around the world, Hoggarth, Reis, and Saporta (2002) estimate the average annual fiscal costs incurred in a banking crisis is 3.8% of the country’s GDP and the average annual economic costs (GDP losses) of a banking crisis is 4.6% of GDP. Hoggarth et al (2002) also estimate that about 85% of the economic costs are directly caused by the banking crisis. Therefore, the ratio of economic costs to fiscal costs of a banking crisis is on average  $4.6\% \times 0.85 / 3.8\% = 1.02$ . In our model, we proxy for the fiscal and economic costs by using the ex ante expected costs of capital injection and the expected social distress costs, respectively, that are given by

$$InjCost_0 = E_0 \left[ \int_{\tau_B}^{\tau_L} e^{-rt} (C_t - \theta) dt \right], \quad (42)$$

$$DistressCost_0^{social} = E_0 \left[ \int_0^{\tau_L} e^{-rt} \lambda_2^{social} (C_t - \theta) 1_{C_t \leq \theta} dt \right]. \quad (43)$$

There are closed-form representations of the above two quantities in our model (the formulae are provided in the online appendix). To reduce the number of free parameters, we also assume that

$$\frac{\alpha^{social}}{\alpha} = \frac{\lambda_2^{social}}{\lambda_2}, \quad (44)$$

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<sup>7</sup>We consider this time period because the Basel I Accord was published in 1988 and enforced by G-10 countries in 1992. The risk-free rates are downloaded from Kenneth French’s website.

so that the ratio of the social to the private liquidation costs are equal to the ratio of the social to the private financial distress costs.

To calibrate the project return and risk characteristics, we use Compustat data for U.S. banks over the sample time period . In the model, the bank chooses the low risk project except possibly in financial distress. Accordingly, we set the risk,  $\sigma_1$ , of the low-risk project to 0.126 to match the asset risk of the average bank in the sample.<sup>8</sup> We set the risk of the high-risk project  $\sigma_2$  to 0.148, which corresponds to the 75th percentile of the asset risk of banks in the sample. In the absence of default risk, the bank's value is

$$V_0 = \frac{(1 - \lambda_1)C_0}{r - \mu_1} + \frac{\lambda_1\theta}{r}, \quad (45)$$

where  $\theta$  is the interest payment. The ratio of the average bank's earnings to assets and interest payments to assets are 3.8% and 2.8%, respectively. Setting  $C_0/V_0$  to 3.8% and  $\theta/V_0$  to 2.8% in (45), respectively, we obtain

$$\mu_1 = r \left( 1 - \frac{(1 - \lambda_1)0.038}{r - 0.028\lambda_1} \right) \quad (46)$$

To avoid introducing additional arbitrary parameters in our analysis, we assume that there is a single parameter  $\pi$  such that the projects' risk-neutral drifts are given by

$$\mu_1 = r - \pi\sigma_1 \quad (47)$$

$$\mu_2 = r - \pi\sigma_2, \quad (48)$$

From (48), we have

$$\frac{r - \mu_1}{\sigma_1} = \frac{r - \mu_2}{\sigma_2} \quad (49)$$

We then calibrate the projects' risk-neutral drifts  $\mu_1, \mu_2$ , and the cost parameters  $\lambda_1, \lambda_2, \lambda_2^{social}$ , and  $\alpha^{social}$  by matching the average leverage and credit spread of banks and the economic-fiscal cost ratio. The average leverage ratio of banks in the sample is 87%. Krishnan, Ritchken, and Thomson (2006) find that the the credit spread of 10-year bonds issued by average highly rated (A- or better) banks is 106 bp. We choose  $\mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_2^{social}$ , and  $\alpha^{social}$  so that the equations (44), (46) and (49) are satisfied, and the optimal initial leverage ratio, the credit spread, and the ratio of expected social distress costs to expected capital infusion costs match the empirical values (87%, 106 bp, and 1.02) as closely as possible. We thereby obtain  $\mu_1 = -0.02\%$ ,  $\mu_2 = -0.62\%$ ,  $\lambda_1 = 0.288$ ,  $\lambda_2 = 0.442$ ,  $\lambda_2^{social} = 0.6$ , and  $\alpha^{social} = 0.272$ . It is important to note that, because  $\mu_1$  and  $\mu_2$  are the projects' *risk-neutral* or *risk-adjusted* drifts, they can be substantially lower than the drifts of

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<sup>8</sup>We calculate the asset risk by the formula  $\sigma_A = \sqrt{(\theta_D\sigma_D)^2 + (\theta_E\sigma_E)^2}$ , where  $\theta_D$  and  $\sigma_D$  are the fraction and risk of bank debt, and  $\theta_E$  and  $\sigma_E$  are the fraction and risk of equity. We assume that the risk of deposit debt is zero. The debt risk  $\sigma_D$  is estimated to be 0.08, the same as the long-term treasury bond, and the equity risk is estimated by monthly stock returns of U.S. banks over the period 1991–2010.

the projects under the *actual* or physical measure. In particular, they can be zero or even negative. The following table shows the calibrated parameters.<sup>9</sup>

|   | Parameter                       | Calibrated Value |
|---|---------------------------------|------------------|
| Risk-free rate  | $r$                             | 0.035            |
| Risk-neutral/Risk-adjusted drift of low-risk project  | $\mu_1$                         | -0.0002          |
| Volatility of low-risk project                        | $\sigma_1$                      | 0.126            |
| Risk-neutral/Risk-adjusted drift of high-risk project | $\mu_2$                         | -0.0062          |
| Volatility of high-risk project                       | $\sigma_2$                      | 0.148            |
| Excess Costs of Equity                                | $\lambda_1, \lambda_1^{social}$ | 0.288            |
| Private Distress Cost of Equity                       | $\lambda_2$                     | 0.442            |
| Social Distress Cost of Equity                        | $\lambda_2^{social}$            | 0.600            |
| Private Liquidation Cost                              | $\alpha$                        | 0.200            |
| Social Liquidation Cost                               | $\alpha^{social}$               | 0.272            |

### A. Impact of Regulation

In our first set of numerical experiments, we compare the fully regulated bank with the fully unregulated bank, where the model parameters take their baseline values. Table I displays the results.

First, we note that the optimal leverage of the regulated bank (77%) is much higher than that of the unregulated bank (65%), but its credit spread is lower (39 bp versus 67 bp), which is due to two factors. First, the regulator’s intervention in the bank’s project choices substantially reduces the risk of the bank. Second, the promise of capital injection by the regulator reduces the effective cost of debt. These factors enable the regulated bank to borrow much more than an unregulated bank, but with a lower credit risk. By the discussion in Section VI, the optimal leverage of the regulated bank implies an initial capital requirement of 23% imposed by the regulator, which is consistent with the Basel III committee’s proposal to require banks to hold much more equity relative to the Basel II requirements, and with the much higher capital ratios proposed by recent studies such as Admati et al (2011).

Second, the regulator prevents asset substitution by intervening in the bank’s management at the switching threshold of the bank, or at the intervention ratio of 6.9%. Interestingly, consistent with Theorem 5, the regulator also finds it optimal to engage in risk-shifting when the bank’s earnings are sufficiently low (0.327). In the baseline model, the risk-switching threshold of the regulator is less than the insolvency/bailout threshold (0.542). In other words, the regulator shifts risk when its equity stake in the bank is negative. The risk-shifting behavior of the regulator is

<sup>9</sup>With these parameters, the model is able to generate exactly the empirical leverage ratio 87% and the economic-fiscal cost ratio of 1.02. The model-generated credit spread is 97 bp, compared to the empirical value of 106 bp.

consistent with anecdotal evidence of an increase in the risk of Irish banks *after* regulators took over their operations.

Third, the *ex ante* benefit of optimal regulation is economically significant and equals 4% of the bank value. To get a rough ballpark estimate of the dollar value of the benefit, note that the total assets of the banking industry in the U.S. were \$7,390 billion in 2006, which implies that the social benefits of regulation were \$290 billion. Though we do not model the potential costs associated with monitoring and regulating the banks, the magnitude of social benefits likely exceed any such costs by far and thus optimal regulation is indeed socially desirable. Because there are additional deadweight social costs of capital injection, the bank's value (equity value plus debt value) is greater than the social (or social) value of the bank. Optimal regulation boosts the bank's value by 11%, implying huge benefits for the equity and debt holders of the bank. The benefits to the bank's claimholders are, however, partially offset by a significant external social cost of capital injection, equaling 7% of bank value.

Fourth, regulation has an even greater impact on the bank's value *ex post* at the insolvency boundary. The *ex post* debt value at insolvency increases from the case with no capital injection to the case where the regulator injects capital by 83%.<sup>10</sup> Even after excluding the external social cost of injection, the net benefit of the bank bailout is 29% of the value of the unbailed bank.

### *B. Impact of Individual Regulatory Policies*

We further explore the effects of different regulatory tools by analyzing environments in which the regulator has less than the full set of regulatory tools considered in the previous section. By considering individual regulatory policies, we isolate the effects of different components of bank regulation. Table II compares the solutions in several environments where the regulator only employ certain subsets of regulatory tools.

#### *Capital Injection*

We first study the scenario in which the regulator can promise to make capital injections but has no control over the bank's project choices or its leverage. Column 1 of Table II reports the solution to the regulator's problem in this environment. In this setting, the regulator injects capital after equity value drops to zero, thereby providing implicit insurance to debt holders. This allows the bank to choose a much higher initial leverage (87%) than in the unregulated environment (65%). Capital injection alone actually reduces the bank's social value by -1.9%. The *ex ante* costs of capital injection arises from the fact that the bank is able to engage in asset substitution without intervention by the regulator. Unconstrained asset substitution by the bank not only decreases social value, but also leads the regulator to inject less capital because the bank is more risky and

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<sup>10</sup>Note that at the insolvency boundary or later, the bank value is equal to the debt value of the bank.



less worthy of salvation.

### *Risk Intervention*

We now examine the environment in which the regulator can restrict the bank's risk choices, but does not inject capital into the bank. This scenario seems particularly relevant given the vigorous academic and political debate on the rationales for capital injection.

Column 3 of Table II reports the optimal policies in such an environment, and compares them with those in the unregulated environment. Because the regulator does not incur external capital injection costs in this setup, the bank value is equal to the social value. Therefore, the optimal initial leverage for the bank is the same as that for the regulator. This optimal leverage is 66%, slightly higher than that in the unregulated case (65%), because of the reduced asset substitution problem. Ex ante, capital regulation in this setup improves firm/social value by only 0.4%.

Comparing Column 2 and 3 in Table II, we see that capital injection should not be used alone as a regulatory tool, and prudential regulation that aims at reducing asset substitution is more effective. The combination of these policies together with the capital requirement (Column 6 of Table II), however, yields a much greater ex ante benefit (4%) than the benefit of capital regulation (0.4%) without injection. Therefore, our results suggest that there is a large impact of the interaction between capital regulation, risk intervention, and capital injection policies, when they are all optimally chosen.

### *Regulation without Initial Capital Requirement*

We next consider the environment where the regulator optimally intervenes in the bank's project choices, injects capital, and liquidates the bank, but does not impose an initial capital requirement. Studying such an environment allows us to understand the role of the initial capital requirement when other mechanisms such as investment intervention and capital injection are in place.

Column 4 of Table II compares the policies in such an environment with those in the unregulated environment. When the initial leverage is unrestricted, the optimal leverage chosen by the bank is 100% and the bank value increases by 31% compared to the unregulated case. Meanwhile, the social value of the bank decreases by 20% relative to the unregulated bank. The intuition behind these results is that because investment intervention and capital injection by the regulator protect debtholders, the bank exploits these protection mechanisms by increasing its debt as much as possible, thereby substantially increasing the expected social costs of capital injection. These results illustrate the necessity of imposing a minimum initial capital requirement, especially when other regulatory tools that aim at reducing bank risk are in place.

### *Regulation without Risk Intervention*

Finally, we consider the environment in which the regulator imposes an initial capital requirement, injects capital and liquidates the bank optimally, but does not intervene in the bank's project choices until it becomes insolvent.<sup>11</sup> Comparing this scenario with the full regulation case will allow us to evaluate the incremental value of risk intervention policy.

Column 5 of Table II presents the optimal policies in this environment. The optimal capital requirement is 28%, higher than that under full regulation. Without risk intervention, the bank's unconstrained asset substitution practices increase default risk and thus a higher capital requirement is necessary. The social benefit of regulation in this environment is 2.3%, or 1.6% less than that in the full regulation case. Therefore, risk intervention adds significant social benefit, even when the initial capital requirement is optimally imposed. We also note that the incremental value of risk intervention (1.6%) over other policies is substantially higher than the social benefit of risk intervention over the unregulated bank (0.4%). This indicates that when multiple regulatory policies are employed together, there can be significant interactive (in this case positive) effects on the social value of the bank.

### *C. Impact of Changes in Initial Capital Requirement*

We now analyze the scenario where the regulator imposes an initial capital requirement that may be suboptimal, but administers other regulatory policies optimally. Figure 2 presents the impact of the initial capital requirement in such an environment. Recall that the optimal leverage is equal to one minus the initial capital requirement. When initial capital requirement varies, the intervention ratio stays constant (Proposition 7 shows that the switching leverage and thus the intervention ratio is independent of initial capital structure). The ex ante social benefit of regulation decreases substantially when the initial capital requirement is relaxed. For example, when the initial capital requirement is equal to 8%, i.e., the Basel II requirement, the ex ante social benefit decreases by 3.5% of bank value and almost vanishes. Our result thus suggests that adopting the optimal capital requirement, which is much higher than those required by the Basel II accords, can improve social welfare substantially.

### *D. Comparative Statics*

We now examine how variations in the values of key underlying parameters affect the optimal regulatory policies and the associated benefits. The closed-form solution of the model greatly

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<sup>11</sup>The two remaining possible combinations of individual regulatory policies are identical to two scenarios we consider above. The case where the regulator imposes only capital requirement is the same as the case of the unregulated bank because without bailout, the bank optimizes capital structure in the same way as the regulator. Similarly, the other case where the regulator imposes capital requirement and intervenes in the bank's risk choices is the same as the case where the regulator implements only the risk intervention policy.

simplifies the computation of these “comparative static” relationships. Figure 3 shows the impact of changes in the parameters on optimal regulation.

### *The Social and Private Costs*

Panels A and B of Figure 3 display the effects of the social cost of equity issuance  $\lambda_1^{social}$  and the bank’s private cost  $\lambda_1$ . The initial capital requirement decreases with and is very sensitive to the social cost  $\lambda_1^{social}$ . When  $\lambda_1^{social}$  is high, the regulator has a strong preference for the bank to preform its welfare-improving functions, such as providing demand deposits, and therefore, imposes a less strict capital requirement on the bank. As a consequence, at times when the functions of banks are crucial, for example, during an economic depression or credit crunch, the regulator should optimally reduce the capital requirement of banks. Our model thus provides a rationale for lower capital requirements during recessions that is proposed by many recent commentators (see e.g. Drumond (2009) and Dewatripont, Rochet, and Tirole (2010)).

The initial capital requirement also depends negatively on the bank’s personal cost of equity  $\lambda_1$  (to a lesser extent than  $\lambda_1^{social}$ ), for a very different reason. The (unreported) optimal regulatory initial debt level is virtually independent of  $\lambda_1$ , which implies that the optimal regulatory initial leverage is higher (and capital requirement is lower) for higher  $\lambda_1$  since the bank’s equity value depends negatively on  $\lambda_1$ . The intervention ratio is relatively insensitive to changes in  $\lambda_1^{social}$  or  $\lambda_1$ . The intuition is that the incentives for risk-shifting, which determine the optimal switching trigger and, therefore, the capital adequacy ratio, mainly depend on the project characteristics, and not on the other parameters of the model. The ex ante social benefits of regulation increase sharply as  $\lambda_1^{social}$  increases due to the higher initial leverage which allows the bank to perform its functions better. In contrast, the social benefits are not sensitive to changes in  $\lambda_1$  as the regulator does not incorporate the costs that are only internal to the bank.

Here we can relate our findings to tax policy and capital regulation. When the government imposes a higher corporate tax rate on banks, it makes the bank’s private cost of equity  $\lambda_1$  higher, but should not affect the social cost  $\lambda_1^{social}$  as the tax policy is simply a wealth transfer.<sup>12</sup> Therefore, our comparative statics suggest that in a high-tax-rate regime, capital requirements should be less stringent, because higher tax rates decrease banks’ equity values, but do not affect debt values substantially. Importantly, our results suggest that bank capital regulation should not be independent of fiscal policy as suggested by Admati et al. (2011).

Panels C and D show the comparative statics for the systemic cost  $\lambda_2^{social}$  and the bank’s private distress cost  $\lambda_2$ . In contrast to the result with  $\lambda_1^{social}$ , when  $\lambda_2^{social}$  increases, the optimal initial capital requirement increases due to the higher systemic cost of distress leading to a lower optimal initial leverage. Since the regulator is concerned only with the systemic cost of distress, the optimal

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<sup>12</sup>This is a simplified analysis because tax policy may have social benefits or costs associated with redistribution of wealth, which is beyond the scope of this paper.

regulatory policies and social benefits are virtually independent of the bank's private distress cost  $\lambda_2$ .

Panels E and F show the effects of the social and personal costs of liquidation  $\alpha^{social}$  and  $\alpha$ . Surprisingly, the optimal initial capital requirement changes little with  $\alpha^{social}$  or  $\alpha$ , contrary to the usual intuition that leverage should decrease with default costs. The reason is that the regulator is able to inject capital, delay liquidation, and provide significant insurance to the debt holders, offsetting the negative impact on debt value and optimal leverage of high liquidation cost. Consistent with the above intuition, for higher liquidation cost  $\alpha^{social}$ , the ex ante benefits of regulation is larger.

### *The Risk-Free Rate*

Panel G of Figure 3 shows the impact of varying the risk-free interest rate  $r$ . The optimal initial capital requirement decreases with the risk-free rate. This implies that capital regulation should not be independent of monetary policy, and keeping a constant capital requirement as recommended by the Basel II accords could be sub-optimal. From the regulator's point of view, a looser monetary policy should be combined with a tighter capital requirement. Our findings also suggest that it could be sub-optimal to impose the same capital requirements in different countries with different monetary policies.<sup>13</sup>

In contrast with the effects of the risk-free rate on the initial capital requirement, the intervention ratio is relatively insensitive to the risk-free rate, for the same reason discussed above. The social benefit of regulation increases slightly with the risk-free rate, possibly due to the fact that the benefits of reducing asset substitution and capital injection are higher when the initial leverage of the bank is higher, i.e., when initial capital requirement is lower.

### *Project Risk-Neutral Drifts and Volatilities*

Panel H of Figure 3 shows the effects of varying the risk-neutral drift,  $\mu_1$ , of the low-risk project. The intervention ratio declines sharply when the risk-neutral drift  $\mu_1$  increases. The intuition is that, when the risk-neutral drift of the low-risk project increases, the bank's incentives for risk-shifting decline. The optimal initial capital requirement, however, increases with  $\mu_1$ , because a higher risk-neutral drift implies a lower debt capacity of the bank. The social benefit of regulation decreases with  $\mu_1$  due to the leverage effect.

Panel I shows the effects of varying the volatility of the low-risk project  $\sigma_1$ . When  $\sigma_1$  increases, the bank's average risk increases. Consequently, the optimal leverage declines, and the initial capital requirement increases. Perhaps surprisingly, the intervention ratio declines with  $\sigma_1$ . As  $\sigma_1$  increases, the incentives for choosing the high-risk project decline because the wedge between the

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<sup>13</sup>Acharya (2003) makes a similar point using a very different framework and for different reasons.

volatilities of the two projects declines. Consequently, the optimal intervention ratio is lower. In other words, even though the average risk of the bank is higher, the capital ratio at which the regulator intervenes is actually *lower*.

The above results suggest that prudential regulation should depend on the nature of particular operations of financial institutions. For example, larger commercial banks may regularly invest in projects that have higher NPVs and higher volatility than smaller community banks, which implies that they should be subject to higher capital requirements.

Finally, Panels J and K present the comparative statics for the risk-neutral drift,  $\mu_2$ , and volatility,  $\sigma_2$ , of the high-risk project. The intervention ratio is very sensitive to these parameters. The intuition is straightforward. An increase in the risk-neutral drift and/or the volatility of the high-risk project *aggravate* the asset substitution problem that leads to higher optimal capital requirement. The initial capital requirement, however, is largely stable with  $\mu_2$  and  $\sigma_2$ , which reflects the fact that the optimal leverage is primarily driven by the characteristics of the low-risk project because this is the project that the bank chooses in “normal” circumstances.

### *E. Dynamic Debt Restructuring*

In this section, we extend the basic model to incorporate the possibility that the bank or the regulator can dynamically adjust its leverage either upward or downward by issuing more debt when its leverage ratio falls sufficiently or by retiring debt when the leverage ratio is too high.

We follow Goldstein, Ju, and Leland (2001) in the treatment of dynamic upward debt restructuring. In particular, when the bank restructures, it first retires all existing debt at par value, and then issues new debt to increase leverage. Debt issuance is associated with a cost that is a proportion  $q$  of the amount of new debt issued. As the cost of issuing new debt is usually small in proportional terms, we set  $q = 1\%$  following Goldstein, Ju, and Leland (2001). We model dynamic downward debt restructuring similarly. To reduce debt levels, the bank first retires existing debt at par value and then issues a lower amount of new debt. As shown by Leland (1994) and Admati et al (2012), the bank never has the incentive to retire existing debt at par value because all the benefits of recapitalization would then accrue to debt holders. In our model, the regulator, however, may have the incentive to recapitalize a distressed bank. We assume that the regulator pays for costs of retiring existing debt at par value in a lump sum and incurs a social cost proportional to the amount of debt. We denote the coefficient of this social recapitalization cost by  $\beta^{social}$ . For brevity, we describe the details of the extended model and its solution in the online appendix.

Figure 4 presents the optimal regulatory policies and the associated social benefits for the extended model. The social restructuring cost parameter  $\beta^{social}$  is important in determining whether the regulator recapitalizes or liquidates when a bank is in deep distress. Therefore, we consider the optimal solution under different values of  $\beta^{social}$ .

Panel A of Figure 4 shows that for  $\beta^{social}$  sufficiently high ( $> 0.125$ ), the optimal regulatory pol-

icy is characterized by the thresholds  $(\theta, C_U, C_B, C_S^{bank}, C_S, C_L)$ , where  $C_U$  is the upward restructuring boundary. When the cash flow first reaches  $C_U$ , the bank’s leverage is sufficiently low and it is optimal to restructure debt (upward). When it is less costly to restructure the bank ( $\beta^{social} < 0.125$ ), the regulator chooses to recapitalize rather than liquidate a severely distressed bank. The optimal regulatory policy is then characterized by the thresholds  $(\theta, C_U, C_B, C_S^{bank}, C_S, C_D)$ , where  $C_D$  is the recapitalization threshold. We note that, even in the recapitalization case, the regulator may find it optimal to first inject capital (when  $C_D < C_t < C_B$ ) before retiring the bank’s debt at  $C_t = C_D$ . The reason is that recapitalization involves a lump sum cost and the regulator would like to delay it by using continuous capital injection when possible. This kind of policy is consistent with the fact that during the 2007–2009 crisis, the Federal Reserve typically bailed out financial institutions by promising to provide up to a certain amount of capital when necessary, but did not provide a lump sum immediately.<sup>14</sup>

The optimal capital requirement is 33% in the liquidation region (Panel B of Figure 4). The intuition for the higher capital requirement and lower leverage compared to the basic model is that the bank does not have to borrow as much initially when it anticipates future upward restructuring opportunities. In the restructuring region, the capital requirement is lower for a lower cost  $\beta^{social}$ . The intuition is that, as restructuring becomes less costly, the bank can start with a higher initial leverage in anticipation of a future downward restructuring (by the regulator). However, even when restructuring is *costless*, the capital requirement is still as high as 15%, suggesting that our finding that the optimal capital requirement is significantly higher than the one proposed in the Basel II accords is robust. Intuitively, the social benefit of regulation decreases with the cost of restructuring  $\beta^{social}$ , with a range between 3% and 10%.

Overall, the optimal regulatory policy and its social impact in the extended model with dynamic debt restructuring are quantitatively similar to its impact in the basic model (Table I).

## VIII. Conclusions

We investigate the optimal design of prudential bank regulation using a a continuous-time structural model. The optimal regulatory policy, which we characterize analytically, entails a combination of capital regulation, risk intervention, capital injection, and liquidation. The regulator intervenes to control a bank’s risk-taking when its capital ratio lies in an interval. We calibrate the model to data and, in particular, indirectly infer the private and social costs of banks’ financial distress. Our calibration suggests that the optimal capital requirement supports the substantially higher capital requirements being proposed by the Basel III committee. Prudential regulation can achieve substantial social benefits when it is appropriately designed, and there are significant interactive effects of different regulatory tools. Capital requirements should be higher during booms,

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<sup>14</sup>For example, see “U.S. Seizes Mortgage Giants – Government Ousts CEOs of Fannie, Freddie; Promises Up to \$200 Billion in Capital”, *The Wall Street Journal*, September 8, 2008.

large banks should be subject to stricter capital requirements, and optimal regulation should not be independent of fiscal and monetary policies.

For the sake of parsimony and tractability, the model that we develop in this study abstracts away from a number of important features that would be important to explore in future research. Subramanian and Yang (2012) analyze a more general model of dynamic, risk-taking with an arbitrary, finite number of available projects and derive generalizations of the analytical results we obtain in this study. Another interesting extension would be to build a model with multiple banks whose portfolios could be correlated, thereby giving rise to the possibility of a number of banks being in financial distress simultaneously. Such a framework would more fully endogenize the effects of systemic risk. We could also introduce competition among banks to explore how optimal regulatory policies are affected by competition. Another important and challenging extension would be to build a general equilibrium model that incorporates investors, banks and entrepreneurs to explore how the interplay between the demand and supply of capital affects regulation. Such a framework would allow us to endogenize the excess costs of equity issuance, and potentially facilitate the investigation of how liquidity risk affects regulation.

## Appendix. Proofs

### Proof of Proposition 1

By (16),

$$S_t = E_t \int_t^{\tau_B} e^{-r(u-t)} [1_{C_u \geq \theta} (1 - \lambda_1)(C_u - \theta) + 1_{C_u < \theta} (1 + \lambda_2)(C_u - \theta)] du. \quad (\text{A1})$$

From the above, the equity value must satisfy the following flow equation (expressed in infinitesimal form for convenience)

$$S_t = (1_{C_t \geq \theta} (1 - \lambda_1)(C_t - \theta) + 1_{C_t < \theta} (1 + \lambda_2)(C_t - \theta)) dt + E_t [e^{-rdt} S_{t+dt}]. \quad (\text{A2})$$

Subtracting the L.H.S. from the R.H.S., dividing throughout by  $dt$  and eliminating terms of  $o(dt)$ , we get

$$E_t \left[ \frac{S_{t+dt} - S_t}{dt} - rS_t + (1_{C_t \geq \theta} (1 - \lambda_1)(C_t - \theta) + 1_{C_t < \theta} (1 + \lambda_2)(C_t - \theta)) \right] = 0.$$

Applying Ito's lemma to the above, it follows from (3) and (4) that the equity value  $S(C)$  must satisfy the following system of ODEs

$$\begin{aligned} \frac{1}{2} \sigma_1^2 C^2 \frac{d^2 S}{dC^2} + \mu_1 C \frac{dS}{dC} - rS + (1_{C \geq \theta} (1 - \lambda_1)(C - \theta) + 1_{C < \theta} (1 + \lambda_2)(C - \theta)) &= 0 \text{ for } C > C_S \\ \frac{1}{2} \sigma_2^2 C^2 \frac{d^2 S}{dC^2} + \mu_2 C \frac{dS}{dC} - rS + (1_{C \geq \theta} (1 - \lambda_1)(C - \theta) + 1_{C < \theta} (1 + \lambda_2)(C - \theta)) &= 0 \text{ for } C < C_S. \end{aligned} \quad (\text{A3})$$

(i) Suppose that  $C_S > \theta$ . The solution of the above system of ODEs is as follows.

$$\begin{aligned}
S(C) &= A_1 C^{\gamma_1^-} + A_1' C^{\gamma_1^+} + \frac{(1-\lambda_1)C}{r-\mu_1} - \frac{(1-\lambda_1)\theta}{r} \quad \text{for } C > C_S \\
&= A_2 C^{\gamma_2^-} + A_3 C^{\gamma_2^+} + \frac{(1-\lambda_1)C}{r-\mu_2} - \frac{(1-\lambda_1)\theta}{r} \quad \text{for } C_S > C > \theta \\
&= A_4 C^{\gamma_2^-} + A_5 C^{\gamma_2^+} + \frac{(1+\lambda_2)C}{r-\mu_2} - \frac{(1+\lambda_2)\theta}{r} \quad \text{for } C > C_B,
\end{aligned} \tag{A4}$$

where  $\gamma_i^+, \gamma_i^-$  are the positive and negative root, respectively of (27). (It is easy to show that (27) has one positive and one negative root.) In the above, we have suppressed the dependence of the coefficients  $A_1, A_1', A_2, A_3, A_4, A_5$  on  $C_S$  and  $\theta$ .

Because  $S(C) \sim \frac{(1-\lambda_1)C}{r-\mu_1}$  as  $C \rightarrow \infty$ , we must have  $A_1' = 0$ . Further, because the equity value is zero at insolvency, and insolvency is optimally chosen by the bank to maximize its equity value, it follows from well-known arguments (e.g. see Leland (1994)) that the equity value must satisfy the value matching and smooth pasting conditions (25), which ensure that the equity value is differentiable at the insolvency threshold  $C_B$ . The coefficients  $A_1, \dots, A_5$  and the insolvency threshold  $C_B$  are determined by the conditions that the equity value must be differentiable at the switching threshold  $C_S$ , the debt level  $\theta$  and the insolvency threshold  $C_B$ , that is, we have 6 unknowns and 6 equations.

In the case of  $C_S < \theta$ , we can use similar arguments to those used above to show that the equity value is given by

$$S_t = S(C_t) = \begin{cases} \frac{(1-\lambda_1)C_t}{r-\mu_1} - \frac{(1-\lambda_1)\theta}{r} + A_1(C_S, \theta)C_t^{\gamma_1^-}, & \text{if } C_t \geq \theta, \\ \frac{(1+\lambda_2)C_t}{r-\mu_1} - \frac{(1+\lambda_2)\theta}{r} + A_2(C_S, \theta)C_t^{\gamma_1^-} + A_3(C_S, \theta)C_t^{\gamma_1^+}, & \text{if } \theta > C_t \geq C_S, \\ \frac{(1+\lambda_2)C_t}{r-\mu_2} - \frac{(1+\lambda_2)\theta}{r} + A_4(C_S, \theta)C_t^{\gamma_2^-} + A_5(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } C_S > C_t \geq C_B(C_S, \theta). \end{cases} \tag{A5}$$

(ii) Using arguments similar to those used for the equity value, the debt value satisfies the following system of ODEs:

$$\begin{aligned}
\frac{1}{2}\sigma_1^2 C^2 \frac{d^2 D}{dC^2} + \mu_1 C \frac{dD}{dC} - rD + \theta &= 0 \quad \text{for } C > C_S \\
\frac{1}{2}\sigma_2^2 C^2 \frac{d^2 D}{dC^2} + \mu_2 C \frac{dD}{dC} - rD + \theta &= 0 \quad \text{for } C < C_S.
\end{aligned} \tag{A6}$$

The general solution to the above system is

$$\begin{aligned}
D(C) &= B_1 C^{\gamma_1^-} + B_1' C^{\gamma_1^+} + \frac{\theta}{r} \quad \text{for } C > C_S \\
&= B_2 C^{\gamma_2^-} + B_3 C^{\gamma_2^+} + \frac{\theta}{r} \quad \text{for } C < C_S.
\end{aligned}$$



Since  $D(C) \sim \frac{\theta}{r}$  as  $C \rightarrow \infty$ ,  $B'_1 = 0$ . By (12), we must have

$$D(C_B) = \frac{(1 - \alpha)(1 - \lambda_1)C_B}{r - \mu_1}. \quad (\text{A7})$$

The coefficients  $B_1, B_2, B_3$  are determined by (A7) and the conditions that the debt value is differentiable at  $C_S$ , that is, we have 3 unknowns and 3 equations. Q.E.D.

### **Proof of Theorem 2**

We begin by stating the relevant dynamic programming verification theorem for our analysis.

**Proposition A1** [*Dynamic Programming Verification Theorem*] Let  $S_q(C)$  denote the equity value when the current earnings level is  $C$  if the bank follows a switching policy where it chooses project 1 when its earnings exceed  $q$  and project 2 when the earnings are below  $q$ . Suppose that  $S_q(C)$  satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} \max_{i \in \{1, 2\}} L_i S_q + (1_{C \geq \theta}(1 - \lambda_1)(C - \theta) + 1_{C < \theta}(1 + \lambda_2)(C - \theta)) &= 0; \\ S_q(C_B) = S'_q(C_B) &= 0, \end{aligned} \quad (\text{A8})$$

where

$$L_i S_q = \frac{1}{2} \sigma_i^2 C^2 \frac{d^2 S_q}{dC^2} + \mu_i C \frac{dS_q}{dC} - r S_q; i \in \{1, 2\} \quad (\text{A9})$$

Then  $S_q(C)$  is the optimal equity value function among all possible dynamic project choice policies (including non-stationary policies) and  $q$  is the optimal switching trigger.

Since the above follows from the general verification theorem for dynamic programming, we omit its proof for brevity and refer the reader to Fleming and Soner (1992).

We use the following lemma frequently in the proof.

**Lemma A2** *We have*

$$\gamma_i^+ > 1 \text{ for } i \in \{1, 2\} \quad (\text{A10})$$

$$\gamma_1^- < \gamma_2^- \quad (\text{A11})$$

$$r - \mu_i \gamma_i^- > 0 \text{ for } i \in \{1, 2\} \quad (\text{A12})$$

$$r - \mu_i \gamma_i^+ > 0 \text{ for } i \in \{1, 2\} \quad (\text{A13})$$

*Proof.* By (6), and since  $\gamma_i^+, \gamma_i^-$  are the roots of (27), we have  $0 > \mu_i - r = \frac{1}{2} \sigma_i^2 (1)^2 + (\mu_i - \frac{1}{2} \sigma_i^2)(1) - r = \frac{1}{2} \sigma_i^2 (1 - \gamma_i^+)(1 - \gamma_i^-)$ . Since  $\gamma_i^- < 0$ , we must have  $1 < \gamma_i^+$ .

Next, we note that, because  $\mu_2 < \mu_1$  and  $\sigma_1 < \sigma_2$ ,

$$\mu_2 - \frac{1}{2} \sigma_2^2 < \mu_1 - \frac{1}{2} \sigma_1^2$$

Because  $\gamma_1^- < 0$ , it follows that

$$\frac{1}{2} \sigma_2^2 (\gamma_1^-)^2 + (\mu_2 - \frac{1}{2} \sigma_2^2) (\gamma_1^-) - r > \frac{1}{2} \sigma_1^2 (\gamma_1^-)^2 + (\mu_1 - \frac{1}{2} \sigma_1^2) (\gamma_1^-) - r = 0,$$

Because  $\gamma_2^+, \gamma_2^-$  are the roots of  $\frac{1}{2}\sigma_2^2(x)^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)(x) - r = 0$ , we have

$$\frac{1}{2}\sigma_2^2(\gamma_1^-)^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)(\gamma_1^-) - r = \frac{1}{2}\sigma_2^2(\gamma_1^- - \gamma_2^-)(\gamma_1^- - \gamma_2^+) > 0.$$

Since  $\gamma_1^- < 0 < \gamma_2^+$ , it follows from the last inequality above that we must have  $\gamma_1^- < \gamma_2^-$ . To prove the third inequality, we proceed as follows.

$$0 = \frac{1}{2}\sigma_i^2(\gamma_i^-)^2 + (\mu_i - \frac{1}{2}\sigma_i^2)(\gamma_i^-) - r = \frac{1}{2}\sigma_i^2 \left( (\gamma_i^-)^2 - \gamma_i^- \right) + \mu_i(\gamma_i^-) - r$$

Because  $\gamma_1^- < 0$ ,  $\frac{1}{2}\sigma_i^2 \left( (\gamma_i^-)^2 - \gamma_i^- \right) > 0$ . Consequently, it follows from the above that (A12) must hold.

To prove the fourth inequality, observe that

$$0 = \frac{1}{2}\sigma_i^2(\gamma_i^+)^2 + (\mu_i - \frac{1}{2}\sigma_i^2)(\gamma_i^+) - r = \frac{1}{2}\sigma_i^2 \left( (\gamma_i^+)^2 - \gamma_i^+ \right) + \mu_i(\gamma_i^+) - r$$

By (A10),  $\frac{1}{2}\sigma_i^2 \left( (\gamma_i^+)^2 - \gamma_i^+ \right) > 0$  so that inequality (A13) follows from the above. Q.E.D.

We now proceed with the proof of the main theorem. Because there are different cases to consider, we prove the theorem by stating and proving propositions that deal with each case. The following proposition establishes a *necessary and sufficient* condition for the bank to optimally choose project 1 always, that is, engage in no risk-shifting.

**Proposition A3** [No Asset Substitution] *Suppose that*

$$L_2 S_0 + (1 + \lambda_2)(C - \theta)|_{C=C_B(0)^+} \leq 0, \quad (\text{A14})$$

where  $S_0$  is the equity value function when the bank always chooses project 1 and  $C_B(0)$  is the corresponding endogenous insolvency level. Then it is optimal for the bank to always choose project 1, that is, no asset substitution is optimal.

*Proof.* By Proposition 7, it suffices to show that

$$L_2 S_0 + (1_{C \geq \theta}(1 - \lambda_1)(C - \theta) + 1_{C < \theta}(1 + \lambda_2)(C - \theta)) \leq 0 \text{ for all } C \geq C_B(0). \quad (\text{A15})$$

By Proposition 1,

$$\begin{aligned} S_0(C) &= A_1 C^{\gamma_1^-} + \frac{(1 - \lambda_1)C}{r - \mu_1} - \frac{(1 - \lambda_1)\theta}{r} \text{ for } C \geq \theta \\ &= A_2 C^{\gamma_1^-} + A_3 C^{\gamma_1^+} + \frac{(1 + \lambda_2)C}{r - \mu_1} - \frac{(1 + \lambda_2)\theta}{r} \text{ for } C_B(0) < C < \theta. \end{aligned} \quad (\text{A16})$$

First, we note that

$$A_1 > 0 \quad (\text{A17})$$

because the equity value function must be greater than  $\frac{(1 - \lambda_1)C}{r - \mu_1} - \frac{(1 - \lambda_1)\theta}{r}$ , which is the equity value function in the hypothetical scenario where shareholders are not protected by limited liability.

Matching the value and derivative of the function  $S_0$  at  $C = \theta$ , we have

$$\begin{aligned} A_1 \theta^{\gamma_1^-} + \frac{(1 - \lambda_1) \theta}{r - \mu_1} - \frac{(1 - \lambda_1) \theta}{r} &= A_2 \theta^{\gamma_1^-} + A_3 \theta^{\gamma_1^+} + \frac{(1 + \lambda_2) \theta}{r - \mu_1} - \frac{(1 + \lambda_2) \theta}{r}, \\ \gamma_1^- A_1 \theta^{\gamma_1^-} + \frac{(1 - \lambda_1) \theta}{r - \mu_1} &= \gamma_1^- A_2 \theta^{\gamma_1^-} + \gamma_1^+ A_3 \theta^{\gamma_1^+} + \frac{(1 + \lambda_2) \theta}{r - \mu_1}. \end{aligned} \quad (\text{A18})$$

From the above, we obtain

$$(\gamma_1^+ - \gamma_1^-) A_3 \theta^{\gamma_1^+} = (\lambda_1 + \lambda_2) \theta \left[ \frac{\gamma_1^- - 1}{r - \mu_1} - \frac{\gamma_1^-}{r} \right] = (\lambda_1 + \lambda_2) \theta \frac{\mu_1 \gamma_1^- - r}{r(r - \mu_1)}.$$

By (A12), and the fact that  $\gamma_1^+ > \gamma_1^-$ , the above implies that

$$A_3 < 0. \quad (\text{A19})$$

Again, from (A18), we obtain

$$(\gamma_1^+ - \gamma_1^-) A_1 \theta^{\gamma_1^-} + (\lambda_1 + \lambda_2) \theta \left[ \frac{r - \mu_1 \gamma_1^+}{r(r - \mu_1)} \right] = (\gamma_1^+ - \gamma_1^-) A_2 \theta^{\gamma_1^-}.$$

By (A13) and (A17), the above implies that

$$A_2 > 0. \quad (\text{A20})$$

Next, we note that, for  $C < \theta$ ,

$$\begin{aligned} L_2 S_0 + 1_{C < \theta} (1 + \lambda_2) (C - \theta) &= A_2 \left[ \frac{1}{2} \sigma_2^2 (\gamma_1^-)^2 + (\mu_2 - \frac{1}{2} \sigma_2^2) \gamma_1^- - r \right] C^{\gamma_1^-} \\ &\quad + A_3 \left[ \frac{1}{2} \sigma_2^2 (\gamma_1^+)^2 + (\mu_2 - \frac{1}{2} \sigma_2^2) \gamma_1^+ - r \right] C^{\gamma_1^+} + \frac{(\mu_2 - \mu_1) C}{r - \mu_1} \\ &= A_2 \frac{1}{2} \sigma_2^2 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C^{\gamma_1^-} + A_3 \frac{1}{2} \sigma_2^2 (\gamma_1^+ - \gamma_2^-) (\gamma_1^+ - \gamma_2^+) C^{\gamma_1^+} \\ &\quad + \frac{(1 + \lambda_2) (\mu_2 - \mu_1) C}{r - \mu_1}, \end{aligned} \quad (\text{A21})$$

where the last equality follows from the fact that  $\gamma_2^+$  and  $\gamma_2^-$  are the roots of (27) for  $i = 2$ . Since  $\gamma_1^- < \gamma_2^- < \gamma_2^+$  by (A11), it follows from (A20) that

$$A_2 \frac{1}{2} \sigma_2^2 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C^{\gamma_1^-} > 0. \quad (\text{A22})$$

We need to consider two cases.

*Case 1:*  $\gamma_1^+ > \gamma_2^+$

By (A19),

$$A_3 \frac{1}{2} \sigma_2^2 (\gamma_1^+ - \gamma_2^-) (\gamma_1^+ - \gamma_2^+) C^{\gamma_1^+} \leq 0. \quad (\text{A23})$$

Since  $\mu_2 < \mu_1$ ,  $\frac{(1 + \lambda_2) (\mu_2 - \mu_1) C}{r - \mu_1} < 0$  and decreases with  $C$ . It then follows from (A21), (A22), and

(A23) that  $L_2S_0 + 1_{C < \theta}(1 + \lambda_2)(C - \theta)$  decreases with  $C$ . Since  $L_2S_0 + (1 + \lambda_2)(C - \theta)|_{C=C_B(0)+} \leq 0$  by (A14), we see that

$$L_2S_0 + (1 + \lambda_2)(C - \theta) \leq 0 \text{ for all } \theta > C \geq C_B(0).$$

*Case 2:*  $\gamma_1^+ \leq \gamma_2^+$

In this case,

$$A_3 \frac{1}{2} \sigma_2^2 (\gamma_1^+ - \gamma_2^-) (\gamma_1^+ - \gamma_2^+) C^{\gamma_1^+} \geq 0. \quad (\text{A24})$$

By (A21), (A22), (A24), and (A10), the function  $L_2S_0 + 1_{C < \theta}(1 + \lambda_2)(C - \theta)$  tends to  $\infty$  as  $C \rightarrow 0$  and as  $C \rightarrow \infty$  and has one unique global minimum that must be negative by (A14). We will show that the function is decreasing at  $C = \theta$  that, by condition (A14) would imply that it is negative in the region  $\theta > C \geq C_B(0)$ .

Observe that, for  $C > \theta$

$$\frac{d}{dC} [L_1S_0 + (1 - \lambda_1)(C - \theta)] = \frac{1}{2} \sigma_1^2 C^2 S_0''' + \sigma_1^2 C S_0'' + \mu_1 S_0' + \mu_1 C S_0'' - r S_0' + (1 - \lambda_1) = 0.$$

For  $C < \theta$

$$\frac{d}{dC} [L_1S_0 + (1 + \lambda_2)(C - \theta)] = \frac{1}{2} \sigma_1^2 C^2 S_0''' + \sigma_1^2 C S_0'' + \mu_1 S_0' + \mu_1 C S_0'' - r S_0' + (1 + \lambda_2) = 0.$$

Since  $S_0$  is twice differentiable at  $C = \theta$ , we have

$$\frac{1}{2} \sigma_1^2 C^2 S_0'''(\theta+) - \frac{1}{2} \sigma_1^2 C^2 S_0'''(\theta-) = \lambda_1 + \lambda_2 > 0.$$

As  $\sigma_1 < \sigma_2$ ,

$$\frac{1}{2} \sigma_2^2 C^2 S_0'''(\theta+) - \frac{1}{2} \sigma_2^2 C^2 S_0'''(\theta-) > \lambda_1 + \lambda_2. \quad (\text{A25})$$

Now note that, by (A25),

$$\begin{aligned} & \frac{d}{dC} [L_2S_0 + (1 - \lambda_1)(C - \theta)]|_{C=\theta+} - \frac{d}{dC} [L_2S_0 + (1 + \lambda_2)(C - \theta)]|_{C=\theta-} \\ &= \frac{1}{2} \sigma_2^2 C^2 S_0'''(\theta+) - \frac{1}{2} \sigma_2^2 C^2 S_0'''(\theta-) - (\lambda_1 + \lambda_2) > 0. \end{aligned} \quad (\text{A26})$$

By (A16), for  $C > \theta$ ,

$$L_2S_0 + (1 - \lambda_1)(C - \theta) = A_1 \frac{1}{2} \sigma_2^2 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C^{\gamma_1^-} + \frac{(1 - \lambda_1)(\mu_2 - \mu_1)C}{r - \mu_1}.$$

By (A11), (A17), and because  $\mu_2 < \mu_1$ , the R.H.S. above decreases with  $C$  and, in particular, is decreasing at  $C = \theta +$ . By (A26), therefore,  $\frac{d}{dC} [L_2S_0 + (1 + \lambda_2)(C - \theta)]|_{C=\theta-} < 0$ . Consequently,

$$L_2S_0 + (1 + \lambda_2)(C - \theta) \leq 0 \text{ for all } \theta > C \geq C_B(\theta).$$

It remains to show that

$$L_2 S_0 + (1 - \lambda_1)(C - \theta) < 0 \text{ for all } C > \theta.$$

Because  $S_0$  is twice differentiable at  $C = \theta$ , it follows from the above that

$$L_2 S_0 + (1 - \lambda_1)(C - \theta)|_{C=\theta+} < 0$$

We have already shown that  $L_2 S_0 + (1 - \lambda_1)(C - \theta)$  is decreasing for  $C > \theta$ . It follows from the above that it must be negative for  $C > \theta$ .

By Proposition A1, therefore, choosing project 1 throughout is optimal for the bank. Q.E.D.

The following proposition establishes a *necessary and sufficient* condition for the optimal switching trigger to exceed the debt level. The proofs of this and the next proposition involve similar techniques as the proof of Proposition A3 and are included in the online appendix.

**Proposition A4** [*Optimal Switching Trigger Greater than Debt Level*] Suppose that

$$L_1 S_\theta|_{C=\theta-} < 0, \tag{A27}$$

where  $S_\theta$  is the equity value function corresponding to the policy where the manager chooses project 2 for  $C < \theta$  and project 1 for  $C \geq \theta$ . There exists  $q^*$  with  $\theta < q^* < \infty$  such that the policy of switching projects at  $q^*$  is optimal for the bank and  $S_{q^*}$  is the corresponding equity value function.

The following proposition completes the remaining step in the proof of the theorem by establishing conditions under which the optimal switching trigger is less than the debt level.

**Proposition A5** [*Optimal Switching Trigger Less than Debt Level*] Suppose that

$$L_1 S_\theta|_{C=\theta-} \geq 0, L_2 S_0 + (1 + \lambda_2)(C - \theta)|_{C=C_B(0)+} > 0. \tag{A28}$$

There exists  $q^* \leq \theta$  such that the policy of switching projects at  $q^*$  is optimal for the bank and  $S_{q^*}$  is the corresponding equity value function.

The main theorem follows from Propositions A3, A4 and A5. Q.E.D.

### **Proof of Proposition 3**

By Proposition 1, the equity, debt and bank values for a given switching policy satisfy the following ODE:

$$\frac{1}{2}\sigma_i^2 C^2 \frac{d^2 Y}{dC^2} + \mu_i C \frac{dY}{dC} - rY + y(C, \theta) = 0 \text{ when project } i \in \{1, 2\} \text{ is chosen,}$$

where  $y(C, \theta)$  is the corresponding payout flow to equity, debt, or the bank. Note that  $y(C, \theta)$  is homogeneous of degree 1 in  $C$  and  $\theta$ , that is,

$$y(\xi C, \xi \theta) = \xi y(C, \theta).$$

Further, the boundary conditions that determine the solution to the corresponding equity, debt and firm value functions are also homogeneous of degree 1 in  $C$  and  $\theta$ . Consequently, the value

function must itself be homogeneous of degree 1 in  $C$  and  $\theta$ . It immediately follows from the proof of Theorem 2 that the optimal switching threshold and insolvency threshold must both scale linearly with  $\theta$  as stated in the proposition. The constancy of the switching leverage follows from the fact that the optimal switching trigger is proportional to  $\theta$ , and the debt value and bank value are homogeneous of degree 1 in current cash flow level and the debt level. Q.E.D.

#### **Proof of Proposition 4**

We state below the representation of the social value function in the case  $C_s < \theta$  (that is omitted in the statement of the Proposition).

$$F_t^{social} = F^{social}(C_t) = \begin{cases} \frac{(1-\lambda_1^{social})C_t}{r-\mu_1} + \frac{\lambda_1^{social}\theta}{r} + X_1(C_S, \theta)C_t^{\gamma_1^-}, & \text{if } C_t \geq \theta, \\ \frac{(1+\lambda_2^{social})C_t}{r-\mu_2} - \frac{\lambda_2^{social}\theta}{r} + X_2(C_S, \theta)C_t^{\gamma_1^-} + X_3(C_S, \theta)C_t^{\gamma_1^+}, & \text{if } \theta > C_t \geq C_S, \\ \frac{(1+\lambda_2^{social})C_t}{r-\mu_2} - \frac{\lambda_2^{social}\theta}{r} + X_4(C_S, \theta)C_t^{\gamma_2^-} + X_5(C_S, \theta)C_t^{\gamma_2^+}, & \text{if } C_S > C_t \geq C_L(C_S, \theta). \end{cases} \quad (\text{A29})$$

We omit the proof of the Proposition because it follows using exactly the same arguments used to prove Proposition 1. Q.E.D.

#### **Proof of Theorem 5**

As in the proof of Theorem 2, we establish the theorem by stating and proving intermediate propositions. The following proposition states the relevant dynamic programming verification theorem.

**Proposition A6** [*Verification Theorem for Regulator's Problem*] Let  $F_q^{social}(C)$  denote the social value when the current earnings level is  $C$  if the regulator follows a switching policy where it switches projects at the earnings level  $q$ . Suppose that  $F_q^{social}(C)$  satisfies the following HJB equation:

$$\begin{aligned} \max_{i \in \{1,2\}} L_i F_q^{social}(C) + \left[ 1_{C \geq \theta} \left\{ (1 - \lambda_1^{social})C + \lambda_1^{social}\theta \right\} + 1_{C < \theta} \left\{ (1 + \lambda_2^{social})C - \lambda_2^{social}\theta \right\} \right] &= 0; \\ F_q^{social}(C_L) = F_q^{social'}(C_L) &= 0, \end{aligned} \quad (\text{A30})$$

Then  $F_q^{social}(C)$  is the optimal social value function among all possible dynamic project choice policies (including non-stationary policies) and  $q$  is the optimal switching trigger.

The following proposition establishes the necessary and sufficient condition for the regulator to optimally choose project 1 throughout, that is, engage in no risk-shifting.

**Proposition A7** [*No Asset Substitution for Regulator*] Suppose that

$$L_2 F_0^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta|_{C=C_L(0)+} \leq 0, \quad (\text{A31})$$

where  $F_0^{social}$  is the social value function when the regulator always chooses project 1 and  $C_L(0)$  is the corresponding endogenous liquidation level. Then it is optimal for the regulator to always choose project 1, that is, no asset substitution is optimal.

*Proof.* By (A29), the social value when the regulator always chooses project 1 is given by

$$\begin{aligned} F_0^{social}(C) &= X_1 C \gamma_1^- + \frac{(1 - \lambda_1^{social})C}{r - \mu_1} + \frac{\lambda_1^{social} \theta}{r} \text{ for } C \geq \theta \\ &= X_2 C \gamma_1^- + X_3 C \gamma_1^+ + \frac{(1 + \lambda_2^{social})C}{r - \mu_1} - \frac{\lambda_2^{social} \theta}{r} \text{ for } C < \theta. \end{aligned} \quad (\text{A32})$$

The social value function in the hypothetical scenario where the bank's assets are never liquidated and/or debt is completely risk-free is  $\frac{(1 - \lambda_1^{social})C}{r - \mu_1} + \frac{\lambda_1^{social} \theta}{r}$ . Since  $F_0^{social}(C)$  must clearly be less, we must have

$$X_1 < 0. \quad (\text{A33})$$

Next, we note that, for  $C > \theta$ ,

$$L_2 F_0^{social} + (1 - \lambda_1^{social})C + \lambda_1^{social} \theta = \frac{1}{2} \sigma_2^2 X_1 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C \gamma_1^- + \frac{(1 - \lambda_1^{social})C(\mu_2 - \mu_1)}{r - \mu_1}.$$

Because  $\gamma_1^- < \gamma_2^-$  by (A11), it follows from (A33) that the first term on the R.H.S. above is strictly negative. The second term is also strictly negative because  $\mu_2 < \mu_1$ . Consequently,  $L_2 F_0^{social} + (1 - \lambda_1^{social})C + \lambda_1^{social} \theta < 0$ , which establishes the verification condition for  $C > \theta$ .

Since the social value function is differentiable at  $\theta$ ,

$$\begin{aligned} X_1 \theta \gamma_1^- + \frac{(1 - \lambda_1^{social})\theta}{r - \mu_1} + \frac{\lambda_1^{social} \theta}{r} &= X_2 \theta \gamma_1^- + X_3 \theta \gamma_1^+ + \frac{(1 + \lambda_2^{social})\theta}{r - \mu_1} - \frac{\lambda_2^{social} \theta}{r} \\ \gamma_1^- X_1 \theta \gamma_1^- + \frac{(1 - \lambda_1^{social})\theta}{r - \mu_1} &= \gamma_1^- X_2 \theta \gamma_1^- + \gamma_1^+ X_3 \theta \gamma_1^+ + \frac{(1 + \lambda_2^{social})\theta}{r - \mu_1} \end{aligned}$$

From the above, we obtain

$$(\gamma_1^+ - \gamma_1^-) X_3 \theta \gamma_1^+ = (\lambda_1^{social} + \lambda_2^{social}) \theta \left[ \frac{\gamma_1^- - 1}{r - \mu_1} - \frac{\gamma_1^-}{r} \right] = (\lambda_1^{social} + \lambda_2^{social}) \theta \frac{\mu_1 \gamma_1^- - r}{r(r - \mu_1)}.$$

By (A12), and the fact that  $\gamma_1^+ > \gamma_1^-$ , the above implies that

$$X_3 < 0. \quad (\text{A34})$$

Next, note that, for  $C < \theta$ ,

$$\begin{aligned} L_2 F_0^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social} \theta &= \frac{1}{2} \sigma_2^2 X_2 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C \gamma_1^- + \frac{1}{2} \sigma_2^2 X_3 (\gamma_1^+ - \gamma_2^-) (\gamma_1^+ - \gamma_2^+) C \gamma_1^+ \\ &\quad + \frac{(1 - \lambda_1^{social})C(\mu_2 - \mu_1)}{r - \mu_1}. \end{aligned} \quad (\text{A35})$$

Because  $F_0^{social}$  is twice differentiable at  $C = \theta$ , it follows from our earlier result that  $L_2 F_0^{social} + (1 - \lambda_1^{social})C + \lambda_1^{social} \theta < 0$  for  $C > 0$  that

$$L_2 F_0^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social} \theta|_{C=\theta-} = L_2 F_0^{social} + (1 - \lambda_1^{social})C + \lambda_1^{social} \theta|_{C=\theta+} < 0 \quad (\text{A36})$$

We need to consider two cases.

*Case 1:*  $\gamma_1^+ \geq \gamma_2^+$ .

In this case, the second term on the right hand side of (A35) is non-positive by (A34). The third term is negative because  $\mu_2 < \mu_1$ . If  $X_2 \leq 0$ , then the first term is non-positive by (A11) so that the entire expression on the R.H.S. of (A35) is negative for  $C < \theta$ . If  $X_2 > 0$ , then the expression on the R.H.S. of (A35) is decreasing with  $C$ . In this case, condition (A31) implies that it is again negative for  $C_L(0) < C < \theta$ .

*Case 2:*  $\gamma_1^+ < \gamma_2^+$

In this case, the second term on the right hand side of (A35) is positive by (A34). If  $X_2 < 0$ , then the first term is positive by (A11). The function on the R.H.S. of (A35) tends to  $\infty$  as  $C \rightarrow 0$  and as  $C \rightarrow \infty$  and has a unique local (and global) minimum that is strictly negative by (A31). Condition (A31) and (A36) together imply that  $L_2 F_0^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta$  must, in fact, be negative for  $C_L(0) < C < \theta$ .

In summary, we have shown that  $F_0^{social}$  satisfies the HJB equation. Moreover,  $L_2 F_q^{social}(C) + [1_{C \geq \theta} \{(1 - \lambda_1^{social})C + \lambda_1^{social}\theta\} + 1_{C < \theta} \{(1 + \lambda_2^{social})C - \lambda_2^{social}\theta\}] < 0$  for  $C_L(0) < C < \infty$ , which implies that choosing project 1 always is the unique optimal policy for the regulator. Q.E.D.

The following proposition shows that, if condition (A31) does not hold, the regulator optimally switches projects at a trigger that is below the debt level.

**Proposition A8** [*Optimal Switching Trigger*] Suppose that

$$L_2 F_0^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta|_{C=C_L(0)+} > 0. \quad (\text{A37})$$

There exists  $q^* < \theta$  such that it is optimal for the regulator to choose project 2 for  $C < q^*$ , and project 1 for  $C \geq q^*$ .

*Proof.* We split the proof into several steps.

*Step 1.* Consider the policy where the switching trigger is equal to  $\theta$ . By (33), the social value function for  $C > \theta$  has the form

$$F_\theta^{social}(C) = X_1 C \gamma_1^- + \frac{(1 - \lambda_1^{social})C}{r - \mu_1} + \frac{\lambda_1^{social}\theta}{r}.$$

By the same argument used in the proof of Proposition A7,

$$X_1 < 0. \quad (\text{A38})$$

Consequently, for  $C > \theta$ ,

$$L_2 F_\theta^{social} + (1 - \lambda_1^{social})C + \lambda_1^{social}\theta = \frac{1}{2} \sigma_2^2 X_1 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C \gamma_1^- + \frac{(1 - \lambda_1^{social})C(\mu_2 - \mu_1)}{r - \mu_1} < 0, \quad (\text{A39})$$

so that

$$L_2 F_\theta^{social} + (1 - \lambda_1^{social})C + \lambda_1^{social}\theta|_{C=\theta+} < 0. \quad (\text{A40})$$

We know that, because project 2 is chosen for  $C < \theta$ , and project 1 for  $C > \theta$ ,

$$\begin{aligned} L_2 F_\theta^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta|_{C=\theta-} &= 0, \\ L_1 F_\theta^{social} + (1 - \lambda_1)C + \lambda_1\theta|_{C=\theta+} &= 0. \end{aligned}$$



From the above equations and (A40), we obtain

$$L_1 F_\theta^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta|_{C=\theta-} > 0. \quad (\text{A41})$$

By (A37) and (A41), the policy of always choosing project 1, and the policy of switching projects at  $\theta$  are both sub-optimal within the restricted class of “single switching trigger” policies where the switching trigger is less than or equal to  $\theta$ . By the continuity of the regulator’s objective function, there exists  $q^* < \theta$  such that the policy of switching projects at  $q^*$  is optimal within the restricted class of “single switching trigger” policies. We will show that the policy is, in fact, *globally optimal* as in the proof of Proposition A5.

*Step 2.* By arguments similar to those used in Step 2 of the proof of Proposition A5, we must have

$$L_2 F_{q^*}^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta|_{C=q^*+} = 0, \quad (\text{A42})$$

$$L_1 F_{q^*}^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta|_{C=q^*-} = 0. \quad (\text{A43})$$

Further, the above imply that  $F_{q^*}^{social}$  is twice differentiable at  $q^*$ .

*Step 3.* Using arguments similar to those used in the proof of Proposition A7, we can show that, condition (A42) implies that  $L_2 F_{q^*}^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta < 0$  for  $q^* < C < \theta$ .

*Step 4.* It remains to show that  $L_1 F_{q^*}^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta < 0$  for  $C_L(q^*) < C < q^*$ . By arguments similar to those used in Step 4 of the proof of Proposition A5, we can show that  $L_1 F_{q^*}^{social} + (1 + \lambda_2^{social})C - \lambda_2^{social}\theta$  is increasing for  $C < q^*$ . Condition (A43), therefore, implies that it is negative for  $C < q^*$ .

Hence, the function  $F_{q^*}^{social}$  satisfies all the conditions of the verification theorem. Moreover, the results that

$$L_2 F_{q^*}^{social}(C) + \left[ 1_{C \geq \theta} \left\{ (1 - \lambda_1^{social})C + \lambda_1^{social}\theta \right\} + 1_{C < \theta} \left\{ (1 + \lambda_2^{social})C - \lambda_2^{social}\theta \right\} \right] < 0 \text{ for } C > q^*,$$

$$L_1 F_{q^*}^{social}(C) + \left[ 1_{C \geq \theta} \left\{ (1 - \lambda_1^{social})C + \lambda_1^{social}\theta \right\} + 1_{C < \theta} \left\{ (1 + \lambda_2^{social})C - \lambda_2^{social}\theta \right\} \right] < 0 \text{ for } C < q^*,$$

together imply that switching projects at  $q^*$  is the unique optimal policy for the regulator. Q.E.D.

## **Proof of Proposition 6**

To simplify the notation, let  $q^*$  and  $\tilde{q}^*$  denote the optimal switching triggers in the bank’s and regulator’s problems, respectively. We need to show that  $\tilde{q}^* \leq q^*$ . Since the regulator’s optimal switching trigger is less than  $\theta$  by Theorem 5, it suffices to consider the case where  $q^* < \theta$ . We prove the proposition in two main parts.

*Part 1.* Let  $S^{social}$  be the social equity value function with cost parameters  $(\lambda_1^{social}, \lambda_2^{social})$ . We will show that the optimal switching point  $q_1^*$  defined for the “social shareholders” is lower than  $q^*$ .

Because the problem is scaling invariant in  $(1 - \lambda_1, 1 + \lambda_2)$ , we can rescale the pair  $(1 - \lambda_1^{social}, 1 + \lambda_2^{social})$  to  $\left( (1 - \lambda_1^{social}) \frac{1 + \lambda_2}{1 + \lambda_2^{social}}, 1 + \lambda_2 \right)$  and the switching trigger for the social shareholders will not change. Since  $\frac{1 + \lambda_2^{social}}{1 + \lambda_2} \geq \frac{1 - \lambda_1^{social}}{1 - \lambda_1}$ , without loss of generality, we may thus assume  $\lambda_1 \geq \lambda_1^{social}$  and  $\lambda_2 = \lambda_2^{social}$ . Consider a switching policy described by a trigger  $q$ , where  $q \geq C_B(q)$ , where  $C_B(q)$

is the endogenous insolvency threshold corresponding to the switching policy. Define the function

$$H_q = S_q^{social} - S_q \quad (\text{A44})$$

By (16),  $H_q$  satisfies the following system of ODEs:

$$\begin{aligned} L_1 H_q + (\lambda_1 - \lambda_1^{social})(C - \theta) &= 0 \text{ for } C \geq \theta \\ L_1 H_q &= 0 \text{ for } q \leq C < \theta \\ L_2 H_q &= 0 \text{ for } C_B(q) \leq C < q \end{aligned} \quad (\text{A45})$$

We also have  $H_q \geq 0$  since the social cost is less than the private cost of equity ( $\lambda_1^{social} \leq \lambda_1$ ). By arguments similar to those used in the proof of Proposition 1,  $H_q$  has the following form:

$$H_q(C) = \begin{cases} W_1 C^{\gamma_1^-} + (\lambda_1 - \lambda_1^{social})\left(\frac{C}{r - \mu_1} - \frac{\theta}{r}\right), & \text{for } \theta \leq C, \\ W_2 C^{\gamma_1^-} + W_3 C^{\gamma_1^+}, & \text{for } q \leq C < \theta, \\ W_4 C^{\gamma_2^-} + W_5 C^{\gamma_2^+}, & \text{for } C_B(q) \leq C < q. \end{cases} \quad (\text{A46})$$

From the value matching condition at  $C = C_B(q)$  and that  $H_q \geq 0$ , we solve

$$W_4 = -W C_B^{-\gamma_2^-}, W_5 = W C_B^{-\gamma_2^+}, \text{ for some } W > 0 \quad (\text{A47})$$

An examination of the proof of Theorem 5 reveals that it suffices to show that

$$L_2 H_q|_{C=q^+} \leq 0 \quad (\text{A48})$$

to ensure the ‘‘social shareholders’’ always choose project 1 when  $C > q$ . It is also straightforward to verify that (A48) is equivalent to

$$L_1 H_q|_{C=q^-} \geq 0. \quad (\text{A49})$$

Plugging (A47) into (A49) yields the equivalent condition

$$-(\gamma_2^- - \gamma_1^+)(\gamma_2^- - \gamma_1^-) \left(\frac{q}{C_B}\right)^{\gamma_2^-} + (\gamma_2^+ - \gamma_1^+)(\gamma_2^+ - \gamma_1^-) \left(\frac{q}{C_B}\right)^{\gamma_2^+} \geq 0. \quad (\text{A50})$$

Since  $\gamma_2^+ > \gamma_1^+ > \gamma_2^- > \gamma_1^-$ , the left hand side of (A50) is always positive and thus the condition holds.

*Part 2.* We now show that the optimal switching trigger in the regulator’s problem is lower than in the ‘‘social’’ shareholders’ problem, i.e.,  $\tilde{q}^* \leq q_1^*$ . Equivalently, we assume below  $\lambda_1 = \lambda_1^{social}$  and  $\lambda_2 = \lambda_2^{social}$  and prove that  $\tilde{q}^* \leq q^*$ .

*Step 1.* Consider a switching policy described by a trigger  $q$ , where  $q \geq C_B(q)$ , where  $C_B(q)$  is the endogenous insolvency threshold corresponding to the switching policy. Define the function

$$G_q = F_q^{social} - S_q \quad (\text{A51})$$

By (16) and (18),  $G$  satisfies the following system of ODEs:

$$\begin{aligned} L_1 G_q + \theta &= 0 \text{ for } C > q \\ L_2 G_q + \theta &= 0 \text{ for } C_B(q) < C < q \\ L_2 G_q + (1 - \lambda_1)C + \lambda_1 \theta &= 0 \text{ for } C_L(q) < C < C_B(q), \end{aligned} \quad (\text{A52})$$

where  $C_L(q)$  is the endogenous liquidation threshold corresponding to the switching policy characterized by the trigger  $q$ . By arguments similar to those used in the proof of Proposition 1,  $G$  has the following form for  $C > q$ :

$$G_q(C) = Y_1 C^{\gamma_1^-} + \frac{\theta}{r}. \quad (\text{A53})$$

Since  $\frac{\theta}{r}$  is the value of  $G_q(C)$  in the hypothetical scenario where debt is completely risk-free and liquidation never occurs, we must have

$$Y_1 < 0. \quad (\text{A54})$$

Next, we note that

$$L_2(G_q) + \theta = \frac{1}{2} Y_1 \sigma_2^2 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C^{\gamma_1^-} < 0 \quad (\text{A55})$$

because  $Y_1 < 0$  and  $\gamma_1^- < \gamma_2^-$  by (A11).

*Step 2.* Suppose that  $q^* = C_B(q^*)$ , which implies that project 1 is always chosen. By (A14),

$$L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{C=C_B(q^*)+} \leq 0. \quad (\text{A56})$$

An examination of the proof of Theorem 5 reveals that it suffices to show that

$$L_2 F_{q^*}^{social} + (1 + \lambda_2)C - \lambda_2 \theta|_{C=C_B(q^*)+} \leq 0$$

to ensure that the regulator also optimally chooses project 1 for all  $C \geq C_B(q^*)$ , that is, the regulator's optimal switching trigger (if it exists) is less than  $C_B(q^*)$ . By (A51) and (A55),

$$\begin{aligned} L_2 F_{q^*}^{social} + (1 + \lambda_2)C - \lambda_2 \theta &= L_2 S_{q^*} + (1 + \lambda_2)(C - \theta) + \\ &L_2(G_{q^*}) + \theta < 0, \end{aligned}$$

where the last inequality above follows from (A56) and (A55).

*Step 3.* Suppose that  $q^* > C_B(q^*)$ .

By the proof of Theorem 2,

$$L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{C=q^*+} = 0 \quad (\text{A57})$$

By the proof of Theorem 5, it suffices to show that

$$L_2 F_{q^*}^{social} + (1 + \lambda_2)C - \lambda_2 \theta|_{C=q^*+} \leq 0.$$

The above, however, immediately follows from (A55) and (A57). Q.E.D.

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Table I: The Impact of Optimal Prudential Regulation

This table reports the solutions to the case of the unregulated bank and the case where the regulator optimally implements the full set of regulatory tools.

|                                       | Unregulated<br>Bank | Full<br>Regulation |
|---------------------------------------|---------------------|--------------------|
| <i>Debt Level and Thresholds</i>      |                     |                    |
| Initial Coupon Level                  | 0.632               | 0.776              |
| Switching Boundary (Bank)             | 0.576               | 0.708              |
| Switching Boundary (Regulator)        |                     | 0.327              |
| Insolvency Boundary                   | 0.426               | 0.542              |
| Liquidation Boundary                  |                     | 0.283              |
| <i>Leverage and Regulatory Ratios</i> |                     |                    |
| Initial Leverage                      | 64.5%               | 76.8%              |
| Initial Capital Requirement           |                     | 23.2%              |
| Switching/Intervention Leverage       | 90.2%               | 93.1%              |
| Intervention Ratio                    |                     | 6.9%               |
| Switching Leverage (Regulator)        |                     | 161.5%             |
| Spread (bp)                           | 67.2                | 39.1               |
| <i>Ex Ante Values</i>                 |                     |                    |
| Bank Value                            | 23.490              | 25.990             |
| Social Value                          | 23.308              | 24.230             |
| Ex Ante Benefit (Bank Value, %)       |                     | 10.6%              |
| Ex Ante Benefit (Social Value, %)     |                     | 3.9%               |
| <i>Ex Post Values at Insolvency</i>   |                     |                    |
| Ex Post Benefit (Bank Value, %)       |                     | 82.8%              |
| Ex Post Benefit (Social Value, %)     |                     | 28.9%              |

Table II: The Impact of Individual Components of Prudential Regulation

This table reports the key regulatory ratios and ex ante benefits in several cases where different sets of regulatory tools are available. Column 1 reports the case of the unregulated bank. Column 2 reports the case where the regulator injects capital when the bank is insolvent and optimally liquidates the bank, but does not intervenes in project choices or impose initial capital requirement. Column 3 reports the case where the regulator's only action is intervention in the bank's project choices. Column 4 reports the case where the regulator employs all regulatory tools (investment intervention, capital injection, and liquidation) except the initial capital requirement. Column 5 reports the case where the regulator employs all regulatory tools (initial capital requirement, capital injection, and liquidation) except risk intervention. Column 6 reports the case where the regulator optimally employ all regulatory tools.

|                                       | Unregulated<br>Bank | Capital<br>Injection<br>Only | Risk<br>Intervention<br>Only | Capital<br>Injection<br>and Risk<br>Intervention | Capital<br>Injection<br>and Capital<br>Requirement | Full<br>Regulation |
|---------------------------------------|---------------------|------------------------------|------------------------------|--|--|--------------------|
| <i>Leverage and Regulatory Ratios</i> |                     |                              |                              |  |  |                    |
| Initial Leverage                      | 64.5%               | 87%                          | 66.0%                        | 100%   | 71.9%  | 76.8%              |
| Initial Capital Requirement           |                     |                              |                              |  | 28.1%  | 23.2%              |
| Intervention Ratio                    |                     | 8.1%                         | 9.2%                         | 6.9%   | 0%   | 6.9%               |
| <i>Ex Ante Values</i>                 |                     |                              |                              |  |  |                    |
| Bank Value                            | 23.490              | 25.298                       | 23.588                       | 30.777   | 25.270   | 25.990             |
| Social Value                          | 23.308              | 22.871                       | 23.393                       | 18.659   | 23.886   | 24.230             |
| Ex Ante Benefit (Bank Value, %)       |                     | 7.7%                         | 0.4%                         | 31.0%  | 7.0%   | 10.6%              |
| Ex Ante Benefit (Social Value, %)     |                     | -1.9%                        | 0.4%                         | -19.8%   | 2.3%   | 3.9%               |

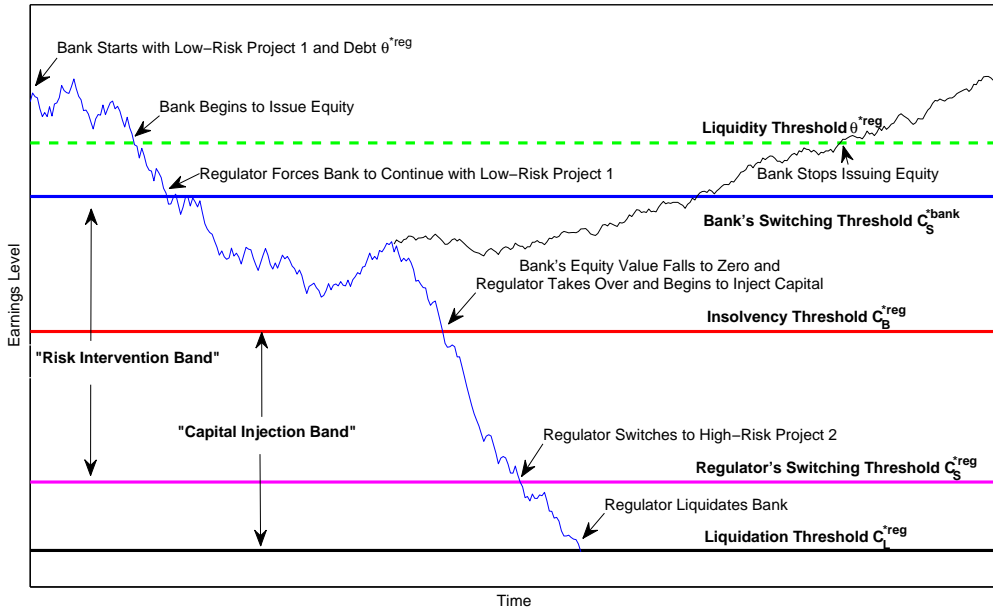


Figure 1: **Optimal Prudential Regulation: An Illustration** This figure plots sample earnings paths and project choices of a fully regulated bank.

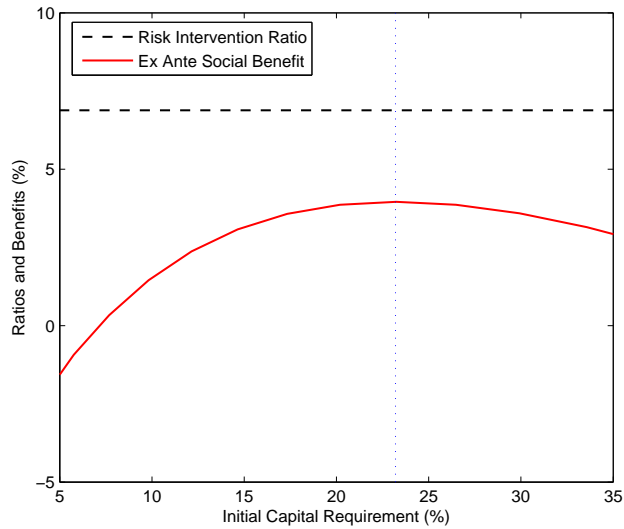


Figure 2: **Impact of Changes in Initial Capital Requirement** This figure plots the changes in optimal regulatory policy and the associated benefits when the initial capital requirement deviates from the optimal level. The model parameters are the baseline parameters. The vertical dotted line corresponds to the optimal regulatory policy.



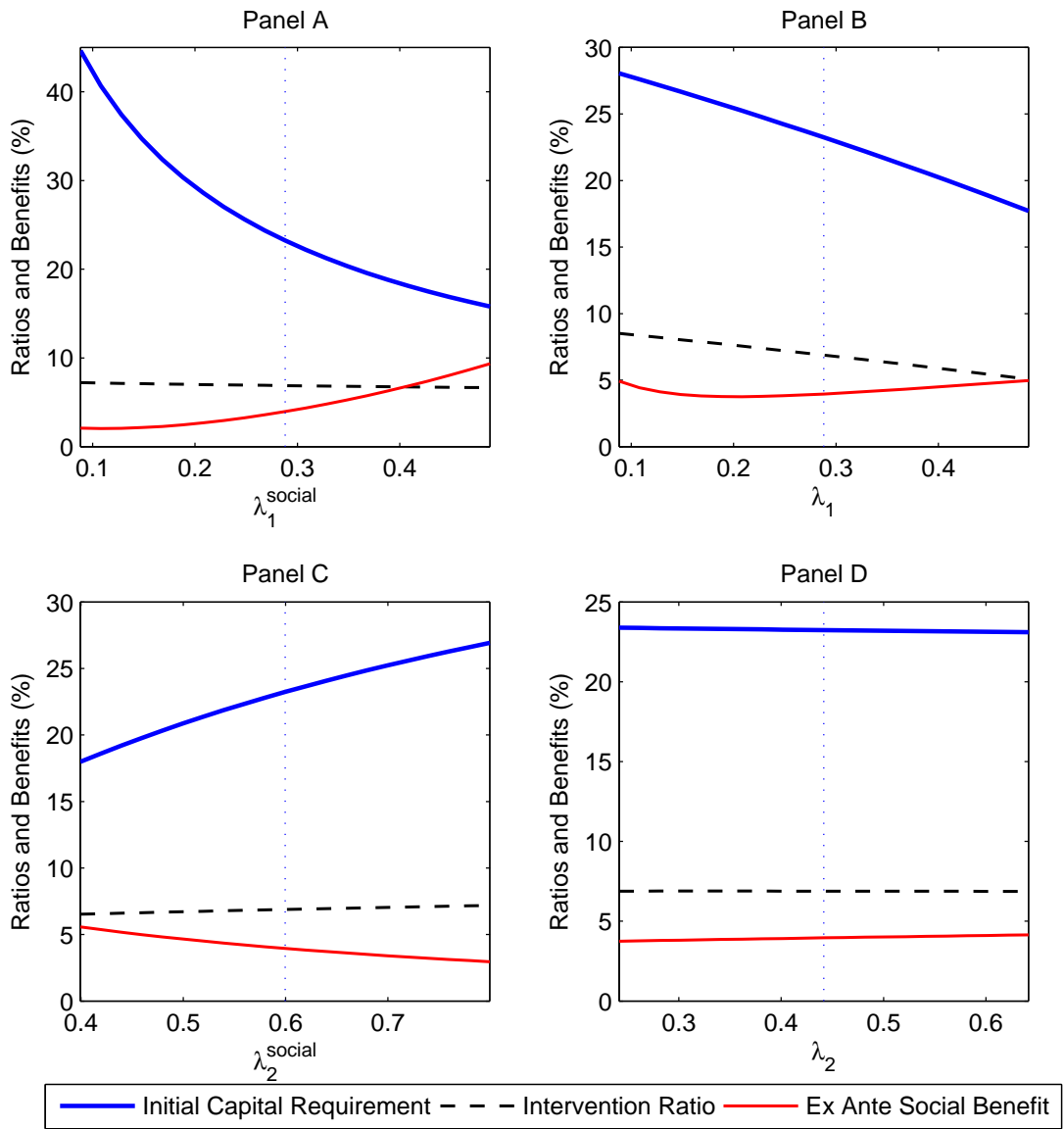


Figure 3: **Optimal Prudential Regulation: Comparative Statics** This figure reports the optimal regulatory ratios and the benefits to the bank and social welfare of optimal regulation under different values of the model's parameters. The vertical dotted lines correspond to the baseline case.

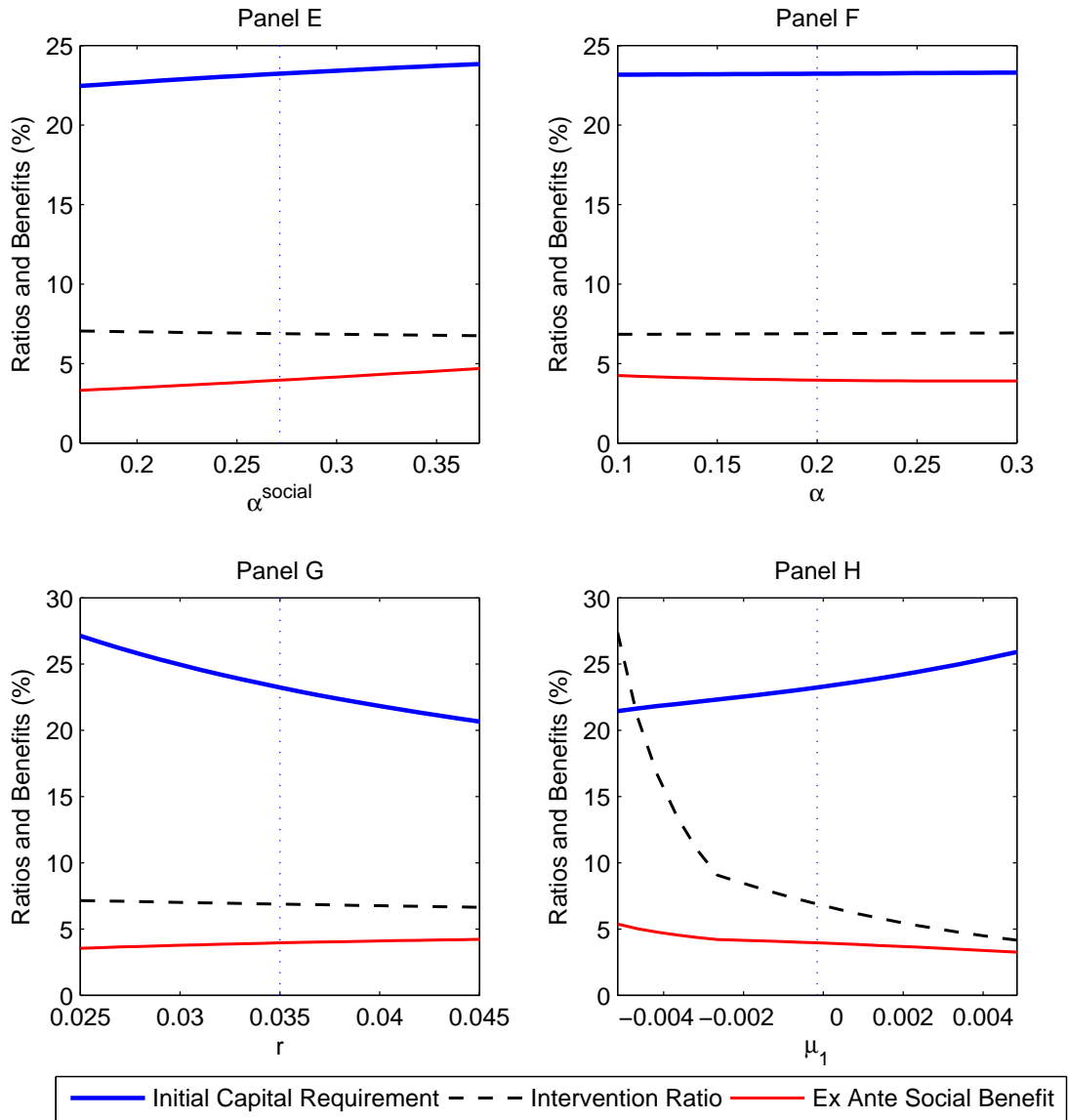


Figure 3: Optimal Prudential Regulation: Comparative Statics (Continued)

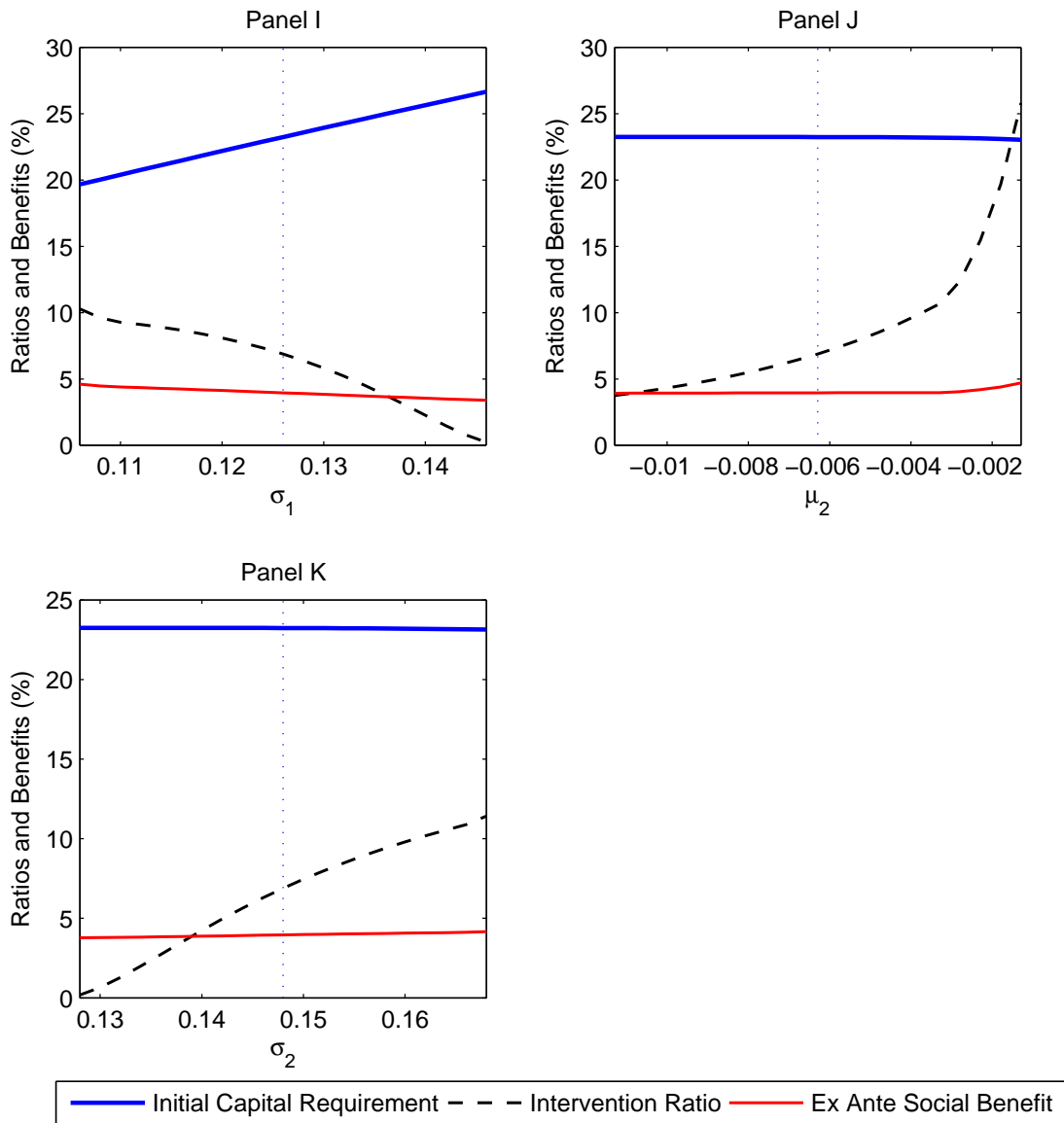
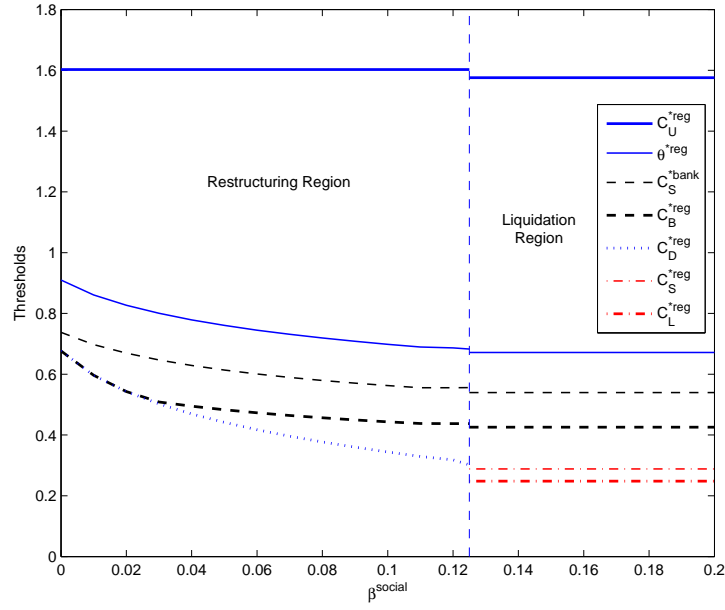
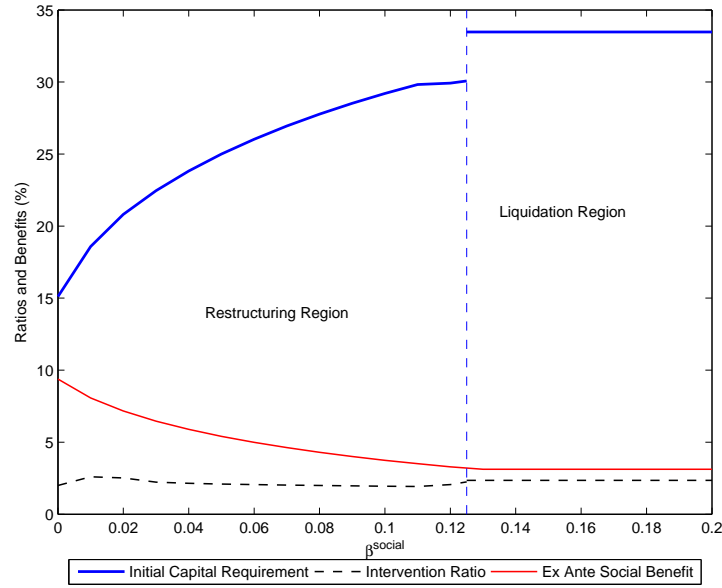


Figure 3: Optimal Prudential Regulation: Comparative Statics (Continued)



Panel A. Optimal Regulatory Thresholds



Panel B. Optimal Regulatory Ratios and Social Benefits

Figure 4: **Dynamic Debt Restructuring** This figure plots the optimal regulatory policies for the extended model where the bank can dynamically adjust capital structure upward or downward as functions of costs of recapitalization  $\beta^{\text{social}}$ . Panel A plots the thresholds that characterizes the optimal regulatory policies. Panel B plots the optimal regulatory ratios and the ex ante social benefits of regulation.

# Internet Appendix to “Dynamic Prudential Regulation”

Ajay Subramanian and Baozhong Yang

## IA1. Explicit Deposit Insurance

Assume a fraction  $\beta$  of all debt is insured by the regulator (deposit insurance). We assume the regulator embodies the role of the insurer so that there is no need to model the insurer separately. Since cross-holdings of bank deposits or large deposits from customers may not be insured, the fraction  $\beta$  of insured bank debt will be typically smaller than the fraction of total deposits in bank debt. The shareholders pay an insurance premium ex ante when debt is first issued.

When the bank is liquidated, debt holders receive the par value of fraction  $\beta$  of their holdings, paid by the regulator/insurer, and receive only the liquidation value on the remaining  $1 - \beta$  of the debt. So if deposit insurance is fairly priced, the ex ante premium should be

$$InsPr_0 = E_0 [e^{-r\tau_L} \beta (D_0(C_0) - D_{\tau_L})] \quad (\text{IA1})$$

The following proposition shows that deposit insurance (whether or not it is fairly priced ex ante) does not affect the optimal regulation.

**Proposition IA1** *The optimal regulatory policies and the bank’s optimal policy under full regulation when there is deposit insurance (not necessarily fairly priced) are the same as those when deposit insurance is absent.*

*Proof.* When the bank is optimally liquidated, the regulator pays off the insured depositors  $\beta D_0(C_0)$  and receives the liquidation payoff  $\beta D_{\tau_L}$ . The above represents wealth transfers between tax payers and depositors and thus does not affect the social welfare. By the same token, the ex ante insurance premium is a wealth transfer from shareholders to tax payers and thus also does not affect the regulator’s objective function. Therefore, the optimal regulatory policies do not depend on deposit insurance. When the bank is fully regulated, the ex ante leverage choice is restricted by the regulator. Because the deposit insurance premium is paid ex ante, deposit insurance does not affect the shareholders’ ex post values and thus their optimal policies. Q.E.D.

## IA2. Distress and Injection Costs

In this section we provide the closed-form representations of the social distress cost and injection cost defined in (43) and (42). We define expected social distress cost and injection cost at any time by

$$InjCost_t = E_t \left[ \int_{\tau_B}^{\tau_L} e^{-r(s-t)} (C_s - \theta) ds \right], \quad (\text{IA2})$$

$$DistressCost_t^{social} = E_t \left[ \int_0^{\tau_L} e^{-r(s-t)} \lambda_2^{social} (C_s - \theta) 1_{C_s \leq \theta} ds \right]. \quad (\text{IA3})$$

We only provide the formulae for  $InjCost_t$  and  $DistressCost_t^{social}$  in the case  $C_S^{reg} > \theta > C_B^*(\theta) > C_L^{reg}$  and leave the other cases to the reader.

$$InjCost_t = InjCost(C_t) = \begin{cases} L_1 C_t^{\gamma_1^-}, & \text{if } C_t \geq C_S^{reg}, \\ L_2 C_t^{\gamma_2^-} + L_3 C_t^{\gamma_2^+}, & \text{if } C_S^{reg} > C_t \geq C_B^*(\theta), \\ L_4 C_t^{\gamma_2^-} + L_5 C_t^{\gamma_2^+} + \frac{C_t}{r-\mu} - \frac{\theta}{r}, & \text{if } C_B^*(\theta) > C_t \geq C_L^{reg}. \end{cases} \quad (\text{IA4})$$

$$DistressCost^{social}(C_t) = \begin{cases} M_1 C_t^{\gamma_1^-}, & \text{if } C_t \geq C_S^{reg}, \\ M_2 C_t^{\gamma_2^-} + M_3 C_t^{\gamma_2^+}, & \text{if } C_S^{reg} > C_t \geq \theta, \\ M_4 C_t^{\gamma_2^-} + M_5 C_t^{\gamma_2^+} + \lambda_2^{social} \left( \frac{C_t}{r-\mu} - \frac{\theta}{r} \right), & \text{if } \theta > C_t \geq C_L^{reg}. \end{cases} \quad (\text{IA5})$$

### IA3. Dynamic Debt Restructuring Model

In this section we provide details for the dynamic restructuring model introduced in Section VII.E.

#### A. Unregulated Bank

We first consider the case of an unregulated bank. We follow Goldstein, Ju, and Leland (2001) in the treatment of dynamic upward debt restructuring. In particular, when the bank restructures, it first retires all existing debt at par value, and then issues new debt to increase leverage. Debt issuance carries a linear cost proportional to the amount of new debt issued. We denote the coefficient of the proportional cost by  $q$ . As it is known that the bank will optimally choose not to adjust leverage downward by retiring debt at par value (Leland (1994) and Admati et al (2012)), we don't have to model downward debt restructuring for the unregulated bank (we do, however, model downward debt restructuring for the fully regulated bank in the next subsection).

The optimal upward restructuring policy is determined by a cash flow threshold  $C_U^*$ . When cash flows  $C_t$  first equal  $C_U^*$ , the bank calls existing debt at par, and issues new debt with a coupon level  $\theta^{1,*}$ . Because the model is homogeneous of degree 1 in the cash flows, if  $C_U^* = k_U C_0$ , then the new optimal debt level satisfies  $\theta^{1,*} = k_U \theta^*$ , where  $\theta^*$  is the initial optimal debt level. Furthermore, if the optimal switching and insolvency (bankruptcy) thresholds are  $C_S^*$  and  $C_B^*$ , respectively, before the restructuring, then after the leverage adjustment the new optimal switching and insolvency thresholds are  $C_S^{1,*} = k_U C_S^*$ ,  $C_B^{1,*} = k_U C_B^*$ . In general, the optimal policy is prescribed by  $(\theta^{n,*}, C_U^{n,*}, C_S^{n,*}, C_B^{n,*}) = (k_U^n \theta^*, k_U^n C_U^*, k_U^n C_S^*, k_U^n C_B^*)$  after the  $n$ -th restructuring time but before the  $(n+1)$ -th restructuring time.

Let  $S(C_t)$  and  $D(C_t)$  be the value of equity and debt before the first restructuring time. The following proposition characterizes the debt and equity values in closed-form, given a policy  $(\theta, C_U, C_S, C_B)$ .

**Proposition IA2** *Let  $C_U = k_U C_0$  and assume  $C_B \leq \theta \leq C_S \leq C_U$ .<sup>15</sup>*

*i) The equity value satisfies*

$$S(C_t) = \begin{cases} A_1 C_t^{\gamma_1^-} + A_2 C_t^{\gamma_1^+} + \frac{(1-\lambda_1)C_t}{r-\mu_1} - \frac{(1-\lambda_1)\theta}{r}, & C_S < C_t \leq C_U \\ A_3 C_t^{\gamma_2^-} + A_4 C_t^{\gamma_2^+} + \frac{(1-\lambda_1)C_t}{r-\mu_2} - \frac{(1-\lambda_1)\theta}{r}, & \theta < C_t \leq C_S \\ A_5 C_t^{\gamma_2^-} + A_6 C_t^{\gamma_2^+} + \frac{(1+\lambda_2)C_t}{r-\mu_2} - \frac{(1+\lambda_2)\theta}{r}, & C_B < C_t \leq \theta \end{cases} \quad (\text{IA6})$$

<sup>15</sup>For brevity, we omit other scenarios with different orderings of the thresholds  $(\theta, C_U, C_S, C_B)$  under which the debt and equity values have different but analogous closed-form representations.

and the boundary conditions

$$S(C_B) = 0, \quad (\text{IA7})$$

$$S(C_U) = k_U S(C_0) + (1 - q)k_U D(C_0) - D(C_0). \quad (\text{IA8})$$

ii) The debt value satisfies

$$D(C_t) = \begin{cases} B_1 C_t^{\gamma_1^-} + B_2 C_t^{\gamma_1^+} + \frac{\theta}{r}, & C_S < C_t \leq C_U \\ B_3 C_t^{\gamma_2^-} + B_4 C_t^{\gamma_2^+} + \frac{\theta}{r}, & C_B < C_t \leq C_S \end{cases} \quad (\text{IA9})$$

and the boundary conditions

$$D(C_U) = D(C_0), \quad (\text{IA10})$$

$$D(C_B) = \frac{(1 - \lambda_1)(1 - \alpha)C_B}{r - \mu_1}. \quad (\text{IA11})$$

The proof is omitted due to similarity to propositions for the basic model in the paper. We note that the representations of equity and debt values are similar to those for the main model without debt restructuring, with the addition of boundary conditions (IA8) and (IA10) at the restructuring point  $C_t = C_U$ . Condition (IA8) reflects the fact that the equity value just before restructuring is equal to the firm value  $k_U S(C_0) + k_U D(C_0)$  minus restructuring cost  $qk_U D(C_0)$  and the cost of retiring outstanding debt  $D(C_0)$ . Condition (IA10) reflects that existing debt is called at par at the restructuring point. The coefficients  $A_i$  and  $B_i$  are determined by the condition that equity and debt values are continuously differentiable. Using the analytical representations given in Proposition IA2, the bank's optimal bankruptcy and switching boundaries  $C_B^*$  and  $C_S^*$  are solved as before by the smooth-pasting and super-contact conditions  $S'(C_B^*) = 0$  and  $S''(C_S^*) = 0$ . The optimal restructuring point  $C_U^*$  is determined by the smooth-pasting condition  $S'(C_U^*) = 0$ . Finally, the bank determines the optimal initial debt level  $\theta^*$  by maximizing the initial firm value  $F(C_0) = S(C_0) + D(C_0)$ .

## B. Full Regulation

The regulator can impose the first-best policy through various regulatory tools. In order to maximize social value, the regulator may be willing to provide the capital to reduce leverage of distressed banks, even when there are accompanied social costs of recapitalization. Therefore, there may exist a threshold  $C_D^{*reg}$  such that when  $C_t = C_D^*$ , the regulator recapitalizes the bank by retiring the existing debt at par value (incurring a proportional cost with coefficient  $\beta^{social}$ ) and lets the bank issue a smaller amount of debt (with the same linear issuance cost  $q$  defined above). Assume the new debt level is  $\theta^{1,*reg} = k_D \theta^{*reg}$  with  $k_D < 1$ , then after the restructuring, all the optimal thresholds are rescaled down by the same factor  $k_D$ ; for example, the new optimal upward restructuring and switching thresholds are given by  $C_U^{1,*reg} = k_D C_U^{*reg}$ ,  $C_B^{1,*reg} = k_D C_B^{*reg}$ . If such a threshold  $C_D^{*reg} > 0$  exists, then the bank will never be liquidated because the regulator can always bail out the bank by recapitalization. Therefore, there may exist two different kinds of optimal policies, depending on the parameters of the model. The first type is given by optimal thresholds  $(\theta^{*reg}, C_U^{*reg}, C_S^{*reg}, C_L^{*reg})$  and the regulator liquidates a sufficiently distressed bank at  $C_t = C_L^{*reg}$  and does not do downward debt adjustment. The second type of policies is given by optimal

thresholds  $(\theta^{*reg}, C_U^{*reg}, C_S^{*reg}, C_D^{*reg})$  and the regulator recapitalizes a sufficiently distressed bank at  $C_t = C_D^{*reg}$  and does not liquidate it. In both types of solutions, the bank determines the optimal time  $C_B^*$  to declare bankruptcy in response to the regulator's policy. The following proposition characterizes the social value  $F^{social}(C_t)$  before the restructuring point, given a set of the regulator's policies  $(\theta, C_U, C_S, C_L)$  or  $(\theta, C_U, C_S, C_D)$ . Using these analytical representations, the optimal thresholds  $(\theta^{*reg}, C_U^{*reg}, C_S^{*reg}, C_L^{*reg})$  or  $(\theta^{*reg}, C_U^{*reg}, C_S^{*reg}, C_D^{*reg})$  are determined similarly to the case of unregulated bank. Finally, one can solve the optimal insolvency threshold  $C_B^*$  by the smooth-pasting condition, given the regulatory policies.

**Proposition IA3** 1) Suppose that the regulator adopts a liquidation-based policy and  $C_L \leq \theta \leq C_S \leq C_U$ .<sup>16</sup> Let  $k_U = C_U/C_0$ . The bank's social value at any date  $t$  is given by

$$F^{social}(C_t) = \begin{cases} \frac{(1-\lambda_1^{social})C_t}{r-\mu_1} + \frac{\lambda_1^{social}\theta}{r} + X_1C_t^{\gamma_1^-} + X_2C_t^{\gamma_2^+}, & \text{if } C_S \leq C_t \leq C_U, \\ \frac{(1-\lambda_1^{social})C_t}{r-\mu_2} + \frac{\lambda_1^{social}\theta}{r} + X_3C_t^{\gamma_2^-} + X_4C_t^{\gamma_2^+}, & \text{if } \theta \leq C_t < C_S \\ \frac{(1+\lambda_2^{social})C_t}{r-\mu_2} - \frac{\lambda_2^{social}\theta}{r} + X_5C_t^{\gamma_2^-} + X_6C_t^{\gamma_2^+}, & \text{if } C_L \leq C_t < \theta, \end{cases} \quad (\text{IA12})$$

and satisfies the boundary conditions

$$F^{social}(C_L) = \frac{(1-\alpha^{social})(1-\lambda_1^{social})C_L}{r-\mu_1}, \quad (\text{IA13})$$

$$F^{social}(C_U) = k_U F^{social}(C_0) - qk_U D(C_0). \quad (\text{IA14})$$

2) Suppose that the regulator adopts a recapitalization-based policy and  $C_D \leq \theta \leq C_S \leq C_U$ . Let  $k_U = C_U/C_0$  and  $k_D = C_D/C_0$ . Then the bank's social value at any date  $t$  is given by

$$F^{social}(C_t) = \begin{cases} \frac{(1-\lambda_1^{social})C_t}{r-\mu_1} + \frac{\lambda_1^{social}\theta}{r} + Y_1C_t^{\gamma_1^-} + Y_2C_t^{\gamma_2^+}, & \text{if } C_S \leq C_t \leq C_U, \\ \frac{(1-\lambda_1^{social})C_t}{r-\mu_2} + \frac{\lambda_1^{social}\theta}{r} + Y_3C_t^{\gamma_2^-} + Y_4C_t^{\gamma_2^+}, & \text{if } \theta \leq C_t < C_S \\ \frac{(1+\lambda_2^{social})C_t}{r-\mu_2} - \frac{\lambda_2^{social}\theta}{r} + Y_5C_t^{\gamma_2^-} + Y_6C_t^{\gamma_2^+}, & \text{if } C_D \leq C_t < \theta, \end{cases} \quad (\text{IA15})$$

and satisfies the boundary conditions

$$F^{social}(C_D) = k_D F^{social}(C_0) - (qk_D + \beta^{social})D(C_0), \quad (\text{IA16})$$

$$F^{social}(C_U) = k_U F^{social}(C_0) - qk_U D(C_0). \quad (\text{IA17})$$

The boundary condition (IA14) means that the social value at the upward restructuring point is the scaled-up value of the initial social value  $k_U F^{social}(C_0)$  (because  $C_U = k_U C_0$ ) minus the restructuring cost  $qk_U D(C_0)$ . Similarly, The boundary condition (IA16) means that the social value at the downward restructuring point is the scaled-down value of the initial social value  $k_D F^{social}(C_0)$  minus the recapitalization and issuance cost  $(qk_D + \beta^{social})D(C_0)$ .

<sup>16</sup>We again omit other possible scenarios of the orders of these thresholds for the sake of space.



## IA4. Additional Proofs

In this section we provide the details of proofs omitted from the main text of the paper.

*Proof of Proposition A4.* We split the proof into several steps.

*Step 1.* Let  $S_\infty$  be the policy of always choosing project 2. It follows using arguments similar to those used to prove Proposition 1 that

$$S_\infty(C) = A_\infty C^{\gamma_2^-} + \frac{(1-\lambda_1)C}{r-\mu_2} - \frac{\theta}{r} \text{ for } C > \theta. \quad (\text{IA18})$$

Further, because equity holders are protected by limited liability,  $S_\infty(C) > \frac{(1-\lambda_1)C}{r-\mu_2} - \frac{\theta}{r}$  so that  $A_\infty > 0$ . Next,

$$L_1 S_\infty(C) + (1-\lambda_1)(C-\theta) = \frac{1}{2}\sigma_1^2 A_\infty (\gamma_2^- - \gamma_1^-)(\gamma_2^- - \gamma_1^+) C^{\gamma_2^-} + \frac{(1-\lambda_1)C(\mu_1 - \mu_2)}{r - \mu_2}$$

Because,  $\mu_1 > \mu_2$  and  $\gamma_2^- < 0$ ,

$$L_1 S_\infty(C) + (1-\lambda_1)(C-\theta) \rightarrow \infty \text{ as } C \rightarrow \infty. \quad (\text{IA19})$$

*Step 2.* The function

$$\Gamma(q) = L_1 S_q + (1_{C \geq \theta}(1-\lambda_1)(C-\theta) + 1_{C < \theta}(1+\lambda_2)(C-\theta))|_{C=q-}$$

is a continuous function of  $q$ . By (IA19),  $\Gamma(q) \rightarrow \infty$  as  $q \rightarrow \infty$ . By (A27), there exists  $q^* \in (\theta, \infty)$  such that

$$L_1 S_{q^*} + (1-\lambda_1)(C-\theta)|_{q^*-} = 0. \quad (\text{IA20})$$

Since  $S_{q^*}$  is the equity value function corresponding to the policy of choosing project 2 for  $C < q^*$  and project 1 for  $C > q^*$ ,

$$L_1 S_{q^*} + (1-\lambda_1)(C-\theta)|_{q^*+} = 0, \quad (\text{IA21})$$

$$L_2 S_{q^*} + (1-\lambda_1)(C-\theta)|_{q^*-} = 0. \quad (\text{IA22})$$

Subtracting (IA20) from (IA21), and using the fact that  $S_{q^*}$  is differentiable at  $q^*$ , we see that

$$\frac{d^2 S_{q^*}}{dC^2}|_{q=q^*+} = \frac{d^2 S_{q^*}}{dC^2}|_{q=q^*-}. \quad (\text{IA23})$$

By (IA22), (IA23), and the fact that  $S_{q^*}$  is differentiable at  $q^*$ , see that

$$L_2 S_{q^*} + (1-\lambda_1)(C-\theta)|_{q^*+} = 0. \quad (\text{IA24})$$

*Step 3.* We show that  $q^*$  is the optimal switching trigger. By Proposition A1, we need to show that

$$\begin{aligned} L_2 S_{q^*} + (1-\lambda_1)(C-\theta) &\leq 0 \text{ for } C > q^* \\ L_1 S_{q^*} + (1-\lambda_1)(C-\theta) &\leq 0 \text{ for } q^* \geq C > \theta \\ L_1 S_{q^*} + (1+\lambda_2)(C-\theta) &\leq 0 \text{ for } \theta \geq C > C_B(q^*). \end{aligned} \quad (\text{IA25})$$

By (24), for  $C > q^*$ ,

$$L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) = \frac{1}{2} \sigma_2^2 A_1 C^{\gamma_1^-} (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) + \frac{(1 - \lambda_1)(\mu_2 - \mu_1)C}{r - \mu_1}.$$

Since  $\mu_2 < \mu_1$ , the second term on the R.H.S. above is negative. By (A11), we must have  $A_1 > 0$  for (IA21) to hold. In this case, however, the R.H.S. of the above is a strictly decreasing function of  $C$ . The first condition in (IA25) then follows from (IA21).

*Step 4.* We now show that the second condition in (IA25) holds. Since  $S_{q^*}$  is differentiable at  $q^*$ ,

$$\begin{aligned} A_1 (q^*)^{\gamma_1^-} + \frac{(1 - \lambda_1)q^*}{r - \mu_1} - \frac{(1 - \lambda_1)\theta}{r} &= A_2 (q^*)^{\gamma_2^-} + A_3 (q^*)^{\gamma_2^+} + \frac{(1 - \lambda_1)q^*}{r - \mu_2} - \frac{(1 - \lambda_1)\theta}{r}, \\ \gamma_1^- A_1 (q^*)^{\gamma_1^-} + \frac{(1 - \lambda_1)\theta}{r - \mu_1} &= \gamma_1^- A_2 (q^*)^{\gamma_2^-} + \gamma_1^+ A_3 (q^*)^{\gamma_2^+} + \frac{(1 - \lambda_1)q^*}{r - \mu_2}. \end{aligned} \quad (\text{IA26})$$

After some algebra, we obtain

$$(\gamma_2^+ - \gamma_2^-) A_1 (q^*)^{\gamma_1^-} + \frac{(\gamma_2^+ - 1)(1 - \lambda_1)C(\mu_1 - \mu_2)}{(r - \mu_1)(r - \mu_2)} = (\gamma_2^+ - \gamma_2^-) A_2 (q^*)^{\gamma_2^-}.$$

Since  $\gamma_2^+ - 1 > 0$  by (A10) and  $\mu_1 > \mu_2$ , it follows from our earlier results that  $A_1 > 0$  that

$$A_2 > 0. \quad (\text{IA27})$$

For  $\theta < C < q^*$ ,

$$\begin{aligned} L_1 S_{q^*} + (1 - \lambda_1)(C - \theta) &= \overbrace{A_2 \frac{1}{2} \sigma_1^2 (\gamma_2^- - \gamma_1^-) (\gamma_2^- - \gamma_1^+) C^{\gamma_2^-}}^{<0} + A_3 \frac{1}{2} \sigma_2^2 (\gamma_2^+ - \gamma_1^-) (\gamma_2^+ - \gamma_1^+) C^{\gamma_1^+} \\ &\quad + \frac{(1 - \lambda_1)C(\mu_1 - \mu_2)}{r - \mu_2} \end{aligned}$$

The first term on the R.H.S. above is negative by (A11) and (IA27). The third term is positive because  $\mu_1 > \mu_2$ . There are two cases to consider.

*Case 1.* Suppose that the second term on the R.H.S. above is positive.

It is then easy to see that the entire expression on the R.H.S. is increasing. It follows from (IA20) that the second condition in (IA25) holds.

*Case 2.* Suppose the second term on the on the R.H.S. above is negative. In this case, it follows from (A10) that the expression tends to  $-\infty$  as  $C \rightarrow 0$ , to  $-\infty$  as  $C \rightarrow \infty$ , and has a unique local (and global) maximum. If we show that  $L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)$  is increasing to the left of  $q^*$ , it will follow that it is negative for  $\theta < C < q^*$ . We proceed as follows.

By the arguments in *Step 3*,

$$\frac{d}{dC} [L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*+} < 0 \quad (\text{IA28})$$

Since  $L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) = 0$  for  $\theta < C < q^*$ ,

$$\frac{d}{dC} [L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*-} = 0. \quad (\text{IA29})$$

Subtracting (IA29) from (IA28), evaluating the derivatives, and using the fact that  $S_{q^*}$  is twice differentiable at  $q^*$  by (IA23), we conclude that

$$\frac{d^3}{dC^3} (S_{q^*})|_{C=q^*+} - \frac{d^3}{dC^3} (S_{q^*})|_{C=q^*-} < 0. \quad (\text{IA30})$$

Next, we note that, because  $L_1 S_{q^*} + (1 - \lambda_1)(C - \theta) = 0$  for  $C > q^*$ ,

$$\frac{d}{dC} [L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*+} = 0. \quad (\text{IA31})$$

From the above, (IA30), and the twice differentiability of  $S_{q^*}$  at  $q^*$ , we see that

$$\frac{d}{dC} [L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*-} > 0,$$

which is what we wanted to prove. It follows from (IA20) that  $L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)$  is negative and increasing for  $\theta < C < q^*$ . This establishes the second condition in (IA25).

*Step 5.* We now show that the third condition in (IA25) holds.

Since  $S_{q^*}$  is differentiable at  $\theta$ ,

$$\begin{aligned} A_2 \theta^{\gamma_2^-} + A_3 \theta^{\gamma_2^+} + \frac{(1 - \lambda_1)\theta}{r - \mu_2} - \frac{(1 - \lambda_1)\theta}{r} &= A_4 \theta^{\gamma_2^-} + A_5 \theta^{\gamma_2^+} + \frac{(1 + \lambda_2)\theta}{r - \mu_2} - \frac{(1 + \lambda_2)\theta}{r}, \\ \gamma_2^- A_2 \theta^{\gamma_2^-} + \gamma_2^+ A_3 \theta^{\gamma_2^+} + \frac{(1 - \lambda_1)\theta}{r - \mu_2} &= \gamma_2^- A_4 \theta^{\gamma_2^-} + \gamma_2^+ A_5 \theta^{\gamma_2^+} + \frac{(1 + \lambda_2)\theta}{r - \mu_2}. \end{aligned} \quad (\text{IA32})$$

After some algebra, we can show that

$$(\gamma_2^+ - \gamma_2^-) A_2 \theta^{\gamma_2^-} + (\lambda_1 + \lambda_2)\theta \left[ \frac{r - \gamma_2^+ \mu_2}{r(r - \mu_2)} \right] = (\gamma_2^+ - \gamma_2^-) A_4 \theta^{\gamma_2^-}.$$

The first term on the L.H.S. above is positive by (IA27). The second term is also positive by (A13). Consequently,

$$A_4 > 0. \quad (\text{IA33})$$

For  $C < \theta$ ,

$$\begin{aligned} L_1 S_{q^*} + (1 + \lambda_2)(C - \theta) &= \overbrace{A_4 \frac{1}{2} \sigma_1^2 (\gamma_2^- - \gamma_1^-) (\gamma_2^- - \gamma_1^+) C^{\gamma_2^-}}^{<0} + A_5 \frac{1}{2} \sigma_2^2 (\gamma_2^+ - \gamma_1^-) (\gamma_2^+ - \gamma_1^+) C^{\gamma_1^+} \\ &\quad + \frac{(1 + \lambda_2)C(\mu_1 - \mu_2)}{r - \mu_2} \end{aligned}$$

The first term on the R.H.S. above is negative by (A11) and (IA33). The third term is positive because  $\mu_1 > \mu_2$ . There are again two cases to consider.

*Case 1.* Suppose that the second term on the R.H.S. above is positive.

It is then easy to see that the entire expression on the R.H.S. is increasing. By the arguments in Step 4,

$$L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{C=\theta+} < 0.$$

Since  $S_{q^*}$  is twice differentiable at  $C = \theta$ ,

$$L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{C=\theta-} < 0. \quad (\text{IA34})$$

Since  $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)$  is increasing for  $C < \theta$ , it is negative for  $C < \theta$ .

*Case 2.* Suppose the second term on the on the R.H.S. above is negative. In this case, it follows from (A10) that the expression tends to  $-\infty$  as  $C \rightarrow 0$ , to  $-\infty$  as  $C \rightarrow \infty$ , and has a unique local (and global) maximum. If we show that  $L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)$  is increasing to the left of  $\theta$ , it will follow that it is negative for  $C < \theta$ . We proceed as follows.

First, note that, because

$$\begin{aligned} L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) &= 0 \text{ for } C > \theta, \\ L_2 S_{q^*} + (1 + \lambda_2)(C - \theta) &= 0 \text{ for } C < \theta, \end{aligned}$$

$$\frac{d}{dC} L_2 S_{q^*}|_{C=\theta+} - \frac{d}{dC} L_2 S_{q^*}|_{C=\theta-} = \lambda_1 + \lambda_2 > 0.$$

Since  $S_{q^*}$  is twice differentiable at  $\theta$ ,

$$\frac{d}{dC} L_2 S_{q^*}|_{C=\theta+} - \frac{d}{dC} L_2 S_{q^*}|_{C=\theta-} = \frac{1}{2} \sigma_2^2 \left[ \frac{d^3}{dC^3} (S_{q^*})|_{C=\theta+} - \frac{d^3}{dC^3} (S_{q^*})|_{C=\theta-} \right] > 0$$

Since  $\sigma_2 > \sigma_1$

$$\begin{aligned} \frac{1}{2} \sigma_2^2 \left[ \frac{d^3}{dC^3} (S_{q^*})|_{C=\theta+} - \frac{d^3}{dC^3} (S_{q^*})|_{C=\theta-} \right] &> \frac{1}{2} \sigma_1^2 \left[ \frac{d^3}{dC^3} (S_{q^*})|_{C=\theta+} - \frac{d^3}{dC^3} (S_{q^*})|_{C=\theta-} \right] \\ &= \frac{d}{dC} L_1 S_{q^*}|_{C=\theta+} - \frac{d}{dC} L_1 S_{q^*}|_{C=\theta-} \end{aligned}$$

It follows from the above that

$$\frac{d}{dC} L_1 S_{q^*}|_{C=\theta+} - \frac{d}{dC} L_1 S_{q^*}|_{C=\theta-} - (\lambda_1 + \lambda_2) < 0$$

By the results of Step 4,  $\frac{d}{dC} [L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=\theta+} > 0$ . It follows from the above that

$$\frac{d}{dC} [L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)]|_{C=\theta-} > 0,$$

which is exactly what we wanted to prove. It follows from (IA34) that  $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta) < 0$  for  $C < \theta$ .

In conclusion, we have shown that all three conditions of (IA25) hold and, moreover, the inequalities are strict. Consequently, by Proposition A1, the policy of switching projects at  $q^*$  is the unique optimal policy for the bank. Q.E.D.

*Proof of Proposition A5* We again split the proof up into several steps.

*Step 1.* Suppose first that  $L_1 S_\theta|_{C=\theta-} > 0$ . By (A28), It follows as a special case of Proposition A1 that, within the sub-class of policies characterized by a single switching trigger that is less than  $\theta$ , the policy of always choosing project 1 as well as the policy of switching at  $C = \theta$  are both sub-optimal. By the continuity of the bank's objective function in the switching trigger, it follows that there exists  $q^* < \theta$  such that the policy of switching projects at  $q^*$  is optimal within the restricted sub-class of policies characterized by a single switching trigger. We will show that the policy of switching projects at  $q^*$  is, in fact, *globally optimal* among all possible dynamic project choice policies.

*Step 2.* We now show that

$$L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*-} = L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*+} = 0. \quad (\text{IA35})$$

Suppose to the contrary that  $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*-} < 0$ . We can use Ito's lemma to show that there exists  $q^{**} > q^*$  such that the value of the policy of switching projects at  $q^{**}$  is greater than the corresponding value for switching projects at  $q^*$ . Suppose, on the other hand, that  $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*-} > 0$ . In this case, we can use Ito's lemma to show that there exists  $q^{**} < q^*$  such that the value of the policy of switching projects at  $q^{**}$  is greater than the corresponding value for switching projects at  $q^*$ . In either case, the policy of switching projects at  $q^*$  is suboptimal within the sub-class of policies characterized by a single switching trigger, which is a contradiction. Hence,  $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*-} = 0$ . Since the bank chooses project 1 for  $C > q^*$ ,  $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*+} = 0$ . It then immediately follows that  $L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*+} = 0$ . Moreover, the super contact condition (IA23) holds at  $q^*$ , that is, the value function is twice differentiable at  $q^*$ .

*Step 3.* By Proposition A1, we need to show that

$$\begin{aligned} L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) &\leq 0 \text{ for } C > \theta \\ L_2 S_{q^*} + (1 + \lambda_2)(C - \theta) &\leq 0 \text{ for } \theta \geq C > q^* \\ L_1 S_{q^*} + (1 + \lambda_2)(C - \theta) &\leq 0 \text{ for } q^* \geq C > C_B(q^*). \end{aligned} \quad (\text{IA36})$$

By (A5),

$$L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) = \frac{1}{2} \sigma_2^2 A_1 C^{\gamma_1^-} (\gamma_1^- - \gamma_2^-)(\gamma_1^- - \gamma_2^+) + \frac{(1 - \lambda_1)(\mu_2 - \mu_1)C}{r - \mu_1} \quad (\text{IA37})$$

By the arguments in Step 1,  $S_{q^*}$  is strictly greater than the equity value from the policy of always choosing project 1. The latter, in turn, exceeds  $\frac{(1 - \lambda_1)C}{r - \mu_1}$  because equity is protected by limited liability. Consequently,

$$A_1 > 0. \quad (\text{IA38})$$

It follows from (A11) and the fact that  $\mu_1 > \mu_2$  that both terms on the R.H.S. of (IA37) are decreasing. Hence,  $L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)$  decreases for  $C > \theta$ .

Next, observe that, for  $C > \theta$

$$\frac{d}{dC}[L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)] = \frac{1}{2} \sigma_1^2 C^2 S_{q^*}''' + \sigma_1^2 C S_{q^*}'' + \mu_1 S_{q^*}' + \mu_1 C S_{q^*}'' - r S_{q^*}' + (1 - \lambda_1) = 0.$$

For  $q^* < C < \theta$

$$\frac{d}{dC}[L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)] = \frac{1}{2}\sigma_1^2 C^2 S_{q^*}''' + \sigma_1^2 C S_{q^*}'' + \mu_1 S_{q^*}' + \mu_1 C S_{q^*}'' - r S_{q^*}' + (1 + \lambda_2) = 0.$$

Since  $S_{q^*}$  is twice differentiable at  $C = \theta$ , we have

$$\frac{1}{2}\sigma_1^2 C^2 S_{q^*}'''(\theta+) - \frac{1}{2}\sigma_1^2 C^2 S_{q^*}'''(\theta-) = \lambda_1 + \lambda_2 > 0.$$

As  $\sigma_1 < \sigma_2$ ,

$$\frac{1}{2}\sigma_2^2 C^2 S_{q^*}'''(\theta+) - \frac{1}{2}\sigma_2^2 C^2 S_{q^*}'''(\theta-) > \lambda_1 + \lambda_2. \quad (\text{IA39})$$

Now note that, by (A25),

$$\begin{aligned} & \frac{d}{dC}[L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=\theta+} - \frac{d}{dC}[L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)]|_{C=\theta-} \\ &= \frac{1}{2}\sigma_2^2 C^2 S_{q^*}'''(\theta+) - \frac{1}{2}\sigma_2^2 C^2 S_{q^*}'''(\theta-) > 0. \end{aligned} \quad (\text{IA40})$$

We have already shown earlier that  $\frac{d}{dC}[L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=\theta+} < 0$ . Consequently,  $\frac{d}{dC}[L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)]|_{C=\theta-} < 0$ .

Therefore, we have shown that  $L_2 S_{q^*} + 1_{C < \theta}(1 + \lambda_2)(C - \theta) + 1_{C \geq \theta}(1 + \lambda_2)(C - \theta)$  is strictly decreasing for all  $C > q^*$ . It then follows from (IA35) that it must be negative for all  $C > q^*$ . Hence, we have established the first two conditions in (IA36). It remains to establish the third condition.

*Step 4.* By our previous arguments,

$$\frac{d}{dC}[L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)]|_{C=q^*+} < 0. \quad (\text{IA41})$$

Since the bank chooses project 2 for  $C < q^*$ ,

$$\frac{d}{dC}[L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)]|_{C=q^*-} = 0. \quad (\text{IA42})$$

Subtracting (IA42) from (IA41), we get

$$S_{q^*}'''(q^*+) - S_{q^*}'''(q^*-) > 0. \quad (\text{IA43})$$

Next, because the bank chooses project 1 for  $C > q^*$ ,

$$\frac{d}{dC}[L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)]|_{C=q^*+} = 0. \quad (\text{IA44})$$

By (IA43) and (IA44),

$$\frac{d}{dC}[L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)]|_{C=q^*-} > 0.$$

We can use arguments similar to those we have used previously in the proof to show that the above condition implies that  $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)$  is strictly increasing for  $C < q^*$ . It then follows from (IA35) that it is strictly negative for  $C < q^*$ , which establishes the third condition in (IA36).

*Step 5.* Suppose first that  $L_1 S_{\theta}|_{C=\theta-} = 0$ . We can use arguments very similar to those used above to

show that the policy of switching projects at  $\theta$  is optimal. We omit the arguments for brevity. Q.E.D.