Smooth Trading with Overconfidence and Market Power

Albert S. Kyle, Anna A. Obizhaeva, and Yajun Wang*

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This paper presents a continuous time model of oligopolistic trading among symmetric traders who agree to disagree concerning the precision of continuous flows of private information. Although traders do not share a common prior, they apply Bayes law consistently. If there is enough disagreement among traders, an equilibrium exists in which prices reveal the average of all traders' signals immediately, but traders continue to trade on their information slowly. The speed with which traders adjust inventories reflects a trade-off between incentives to slow down to reduce market impact costs and incentives to speed up to profit from perishable information. The price impact depends linearly on the changes in the inventories and the changes in the speed of trading. Trading modest quantities much faster than consistent with equilibrium strategies results in flash-crash patterns with price spikes followed by price reversals.

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* Kyle: Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, akyle@rhsmith.umd.edu. Obizhaeva: Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, obizhaeva@rhsmith.umd.edu. Wang: Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, ywang22@rhsmith.umd.edu. We thank Vincent Fardeau, Jeongmin Lee, Wei Li, Hong Liu, Mark Loewenstein, Ioanid Rosu, Dimitri Vayanos, Bin Wei, Liyan Yang, and seminar participants at Michigan State University, University of Maryland, 2013 CICF and 2013 SIF for their comments.

When large traders in financial markets seek to profit from private information while keeping transactions cost low, they face a fundamental tradeoff between slowing down order execution to reduce temporary price impact costs resulting from adverse selection and speeding up order execution to profit from perishable private information before others can do likewise. As a trader spreads order execution out over time, large orders are often split orders into many small trades. In this paper, we describe a model of continuous information-based trading which explains why, in equilibrium, orders must be broken into small pieces and executed gradually over time to avoid large price impact. We show that traders execute orders at an endogenously derived rate which optimizes the trade-off between trading slowly to minimize transitory price impact costs and trading rapidly to profit from perishable private information.

By smoothing their trading, traders may lower execution prices by "walking the demand schedules" of other traders. For example, in Kyle (1985), the informed trader smooths his trading but the noise traders and market makers do not. Unlike Kyle (1985), this paper describes a model in which all traders smooth their trading, and no traders are willing to provide instantaneous liquidity to large blocks. When all traders break orders into tiny pieces and spread the execution out over time, the nature of the equilibrium changes.

We model the equilibrium as a continuous version of the auction mechanism in Kyle (1989). Traders continuously submit demand schedules, but the schedules are for flows—derivatives of inventories which define the speed of trading—rather than for shares. Prices change continuously and inventories have continuous first derivatives. Both the level of prices and the derivatives of inventories follow diffusions.[1]

Each trader correctly believes that the price *level* is a linear function of level of his inventory and also the *level of its derivative*, which measures the speed of trading. Let $P_0(t)$ denote what the market price would be if a trader had no inventories and did not trade, and let $P(t)$ denote the price if the trader holds inventories $S(t)$ and trades at the rate $S'(t)$. Then the price $P(t)$ is given by

$$P(t) = P_0(t) + \lambda_S \cdot S(t) + \lambda_x \cdot S'(t). \tag{1}$$

The λ_S term is permanent linear price impact, similar to Kyle (1985). In Kyle (1985), the informed trader's private information does not decay, the cost of trading does not depend on the speed of inventory adjustment, and the equilibrium speed of inventory adjustment is not derived from the informed trader's maximization problem. The λ_x term represents temporary price impact, which generates trading costs and price effects absent from Kyle (1985).

In our continuous-time model, where all traders smooth out inventory adjustment, trading a nontrivial quantity over a very short period of time can result in large temporary price impact, consistent with a flash crash for rapidly-executed large sales.

[1] Our informal use the term "smooth trading" is different from the mathematical sense that implies derivatives of all order exist. Sine the first derivatives of a traders inventories follow diffusions, higher order derivatives do not exist.

Each trader calculates a target inventory proportional to the difference between his own valuation and the average valuation of others. Each trader optimally adjusts his inventory towards his target level gradually. The speed of partial adjustment—which Black (1995) calls "urgency"— arises endogenously from the trade-off between the half-life of the private information and the resiliency of prices. In the continuous model, quantities adjust slowly, even though information is incorporated into prices immediately. In sharp contrast to the competitive model of Milgrom and Stokey (1982), our imperfectly competitive traders continue to trade on information after it has been incorporated into prices.

We develop a dynamic continuous-time infinite-horizon model of information-based trading with overconfidence and market power. In continuous time, "smooth trading" means that traders' inventories are differentiable functions of time.An infinite-time horizon leads to a steady-state equilibrium with meaningful concepts of depth and returns volatility. To model information decay, each trader is assumed to have a continuous flow of private information about the unobserved mean-reverting growth rate of cash flows; one trader's information decays as other traders acquire similar information and trade on it; it also decays as a result of public information flow. To model market impact, each trader is assumed to optimize his trading taking into account its dynamic effect on prices, which influence the beliefs and trading of other traders. To keep matters simple, we assume that informed oligopolistic traders with the same degree of risk aversion disagree in a symmetric manner.

Overconfidence motivates trade. Traders do not share a common prior; instead, they agree to disagree about the informativeness of one another's signals. Each trader is "relatively overconfident" in that he believes his own signal to be more precise than other traders believe it to be. Unlike Grossman and Stiglitz (1980) or Kyle (1985), there are no noise traders and market makers. In the special case when traders believe other traders' signals are completely uninformative, the model implements the idea of Black (1986) of "trading on noise as if it were information." In the more general case when traders believe other traders' signals have some information, each trader believes that other traders "over-trade" on the basis of their private information, as in Kyle and Lin (2001) and Scheinkman and Xiong (2003).

In order to obtain linear trading strategies and linear information processing rules, we assume exponential utility and linear Gaussian information processes concerning future rates of dividend growth. We look for a steady-state symmetric linear equilibrium in which each trader applies Bayes law correctly given his beliefs and the dynamic equilibrium trading strategies of other traders. A symmetric linear equilibrium may exist; if so, it can be characterized as a solution of six quadratic polynomial equations in six unknowns, which we solve numerically. There also exists an obvious no-trade equilibrium with an undefined price: If each trader believes that all other traders will trade a zero quantity, it is optimal for them not to trade as well.

Our model builds on the economics literature describing single period and dynamic auctions.

We motivate the continuous model with an extensive discussion of a one-period model, which implements "bid-shading" in a manner similar to Kyle (1989) and Rostek and Weretka (2012). Traders exploit their market power by trading approximately one-half of the amount they believe would fully reveal their information. To obtain an equilibrium with positive trading volume, there needs to be "enough" disagreement. For other traders will be willing to take the opposite side of trades, each trader must believe that other traders' signals are slightly less than half as precise as the other traders believe their own signals to be.

Our continuous time model is closest to Vayanos (1999) and Du and Zhu (2013), who present dynamic models with symmetric strategic oligopolistic traders who smooth out their trading to reduce reduce market impact resulting from adverse selection. Their models are set in discrete time. Our continuous time approach brings out clearly the intuition that traders trade smoothly.

Their equilibria are significantly different from ours due to differences concerning the orthogonality of private information. Vayanos (1999) assumes traders receive orthogonal endowment shocks and trade for risk-sharing motives; the orthogonal endowment shocks are a form of private information. Du and Zhu (2013) assume traders' valuations have orthogonal "private values" components; these orthogonal private values components are also a form of private information. In our model, traders receive noisy signals about the same underlying fundamental; thus, their signals have an underlying positive correlation. The economically important implications of this positive correlation are precisely what our model is designed to capture. Because of the intrinsic strategic interactions resulting from positive correlation in signals, our equilibrium is more difficult to characterize and has substantially different properties from theirs. In Vayanos (1999) and Du and Zhu (2013), the horizon over which traders smooth trading depends on their risk aversion, not the correlation of their private signals. In our model, the horizon over which traders smooth trading depends strongly on the rate at which their signals decay due to information acquisition by others; risk aversion scales the size of inventories, not the urgency with which traders trade. Our model captures an economically significant trade-off faced by large asset managers in financial markets.

In both the one-period model and the continuous model, prices adjust immediately, revealing the average of all traders' valuations. Thus, each trader can infer from the price the average of other traders' valuations, which is all that he cares to know about their private information. We describe an equilibrium which is always in a steady state. Describing how the equilibrium reaches a steady state, from a starting point in which prices are not already fully revealing, raises issues related to Ostrovsky (2012), taking us beyond the scope of this paper.

Mathematical intuition and numerical calculations imply an existence condition suggesting that, with continuous trading, each trader must believe that his information is more that four times as precise as other traders believe it to be. The contrast between the factor of four in the continuous model and the factor of two in the one-period model is consistent with results in Vayanos (1999) and Du and Zhu

(2013) that *less* frequent trading facilitates *more* trade.

The model explains the apparent short-term nature of trading, even though trading may be motivated by private information about long-term growth rates in cash flows. Holding periods depend on the horizons over which the information content of traders' private signals become known to others, not on the horizon over which underlying cash flows unfold. Traders build positions when they acquire new information, and they unwind positions as other traders acquire the same information.

The model incorporates precisely the the "beauty contest" logic of Keynes (1936). Although traders are rational investors with long-term horizons, they take into account expectations of short-term price dynamics induced by other traders whom they perceive to "overtrade" on imprecise information. Consistent with Allen, Morris and Shin (2006) and contrary to the intuition of Keynes (1936), we show that beauty contest intuition dampens price volatility.

Since inventory levels are differentiable functions of time, our model has a meaningful concept of trading volume; in models with noise traders, trading volume is infinite since noise traders' inventories follow a Brownian motion. It is consistent with the empirical observation of high trading volume in the markets.

We model formally the conventional Wall Street wisdom that speed of trading affects prices. The empirical studies such as Chan and Lakonishok (1995), Keim and Madhavan (1997), and Dufour and Engle (2000) uniformly support this assertion. Holthausen, Leftwich and Mayers (1990) measure temporary and permanent price effects associated with block trades. They find that most of the adjustment occurs during the very first trade in a sequence, somewhat consistent with instantaneous price adjustment in our model. Almgren et al. (2005) calibrate price impact functions depending both on quantities traded and on the speed of trading, using a functional form very similar to the form we derive endogenously. Kyle and Obizhaeva (2013) further suggest that trading cost functions may be ultimately described just by a few parameters, if market microstructure invariance principles are imposed.

There are numerous papers that incorporate the effect of fast trading into classical finance problems. Given exogenous price impact functions explicitly or implicitly depending on the speed of trading, Brunnermeier and Pedersen (2005) studied price effects of a large trader unwinding his position in the presence of strategic traders, Carlin, Lobo and Viswanathan (2007) focused on the interaction between traders facing liquidity shocks, Longstaff (2001) analyzed the portfolio choice problem, Grinold and Kahn (1995), Almgren and Chriss (2000) as well as Obizhaeva and Wang (2013) derived optimal execution strategies for liquidation of an existing position. Our model makes the speed of trading endogenous and suggests a good way to model equilibrium price impact functions for those applications.

This paper is structured as follows. Section I presents a one-period model. Section II outlines a dynamic continuous-time model. Section III examines properties of smooth trading equilibrium. Section IV concludes. Proofs are in Appendix.

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I. One-period Model

To develop intuition for how equilibrium prices and quantities depend on the interaction between overconfidence and market power, we start with a one-period model. There are N traders who trade a risky asset with liquidation value $\tilde{v} \sim N(0, 1/\tau_v)$ against a safe numeraire asset with liquidation value of one. Traders maximize expected utility of terminal wealth based on utility functions with constant absolute risk aversion (CARA) and risk aversion parameter A. Each trader n is endowed with inventory of S_n shares of a risky asset, $n = 1, \ldots, N$. Since a risky asset is in zero net supply, the sum of inventories of all traders is zero.

Bayesian Updating.

All traders observe a public signal $\tilde{i}_0 = \tilde{v} + \tilde{e}_0$ with $\tilde{e}_0 \sim N(0, 1/\tau_0)$. There are also N private signals $\tilde{i}_n = \tilde{v} + \tilde{e}_n$ with $\tilde{e}_n \sim N(0, 1/\tau_n)$, $n = 1, \ldots, N$. The stock payoff $\tilde{v}$, the public signal error $\tilde{e}_0$, and N private signal errors $\tilde{e}_1, \ldots, \tilde{e}_N$ are independently distributed. Trader n observes signal $\tilde{i}_n$ privately but the equilibrium price, as discussed below, fully reveals the average of other traders' signals defined by $\tilde{i}_{-n} := \frac{1}{N-1} \sum_{m \neq n} \tilde{i}_m$.

Traders agree about the precision of the public signal τ_0 but agree to disagree about the precisions of private signals τ_n. Traders are "relatively overconfident" in that each trader believes his own signal is more precise than the signals of the other traders; specifically, each trader n believes that $\tau_n = \tau_H$ and $\tau_m = \tau_L$ when $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Let E_n and Var_n denote trader n's expectation and variance operators conditional on observing all signals $i_0, i_1, \ldots, i_N$. Using standard formulas for conditional means and variances of jointly normally distributed random variables, we define

$$\tau := Var_n^{-1}\{\tilde{v}\} = \tau_v + \tau_0 + \tau_H + (N-1)\tau_L, \tag{2}$$

and obtain

$$E_n\{\tilde{v}\} = \frac{\tau_0}{\tau} \cdot \tilde{i}_0 + \frac{\tau_H}{\tau} \cdot \tilde{i}_n + \frac{(N-1)\tau_L}{\tau} \cdot \tilde{i}_{-n}. \tag{3}$$

Utility Maximization with Market Power.

Traders are imperfect competitors who explicitly take into account the effect of their trading on prices. Suppose trader n believes the price is a function of the quantity x_n he trades, $p = P(x_n)$. He thinks that his terminal wealth $\tilde{W}_n = \tilde{v} \cdot (S_n + x_n) - P(x_n) \cdot x_n$ will be distributed as a normal random variable with the mean and variance given by

$$E_n\{\tilde{W}_n\} = E_n\{\tilde{v}\} \cdot (S_n + x_n) - P(x_n) \cdot x_n, \tag{4}$$

$$Var_n\{\tilde{W}_n\} = (S_n + x_n)^2 \cdot Var_n\{\tilde{v}\}. \tag{5}$$

Each trader n maximizes the exponential utility of his wealth,

$$E_n\{-e^{-A\cdot\tilde{W}_n}\} = -\exp\Big(-A\cdot E_n\{\tilde{W}_n\} + \frac{1}{2}A^2\cdot Var_n\{\tilde{W}_n\}\Big). \tag{6}$$

The problem is equivalent to maximizing monotonically transformed expected utility $-\frac{1}{A}\ln\big(-E_n\{-e^{-A\cdot\tilde{W}_n}\}\big)$. Plugging equations (2), (3), (4) and (5) into equation (6), the trader's optimization problems is to choose the quantity to trade x_n to solve the maximization problem

(7)

$$\max_{x_n}\left(\Big[\frac{\tau_0}{\tau}\cdot\tilde{i}_0 + \frac{\tau_H}{\tau}\cdot\tilde{i}_n + \frac{\tau_L}{\tau}\cdot(N-1)\cdot\tilde{i}_{-n}\Big]\cdot(S_n+x_n) - P(x_n)\cdot x_n - \frac{1}{2\tau}A\cdot(S_n+x_n)^2\right).$$

For a perfect competitor, $P(x_n)$ would be just a constant p which does not depend on x_n. In exercising market power, the oligopolistic trader takes into account how his choice of quantity x_n affects the price $P(x_n)$.

Conjectured Linear Strategies.

As in Kyle (1989), we assume a single-price auction in which traders submit demand schedules $X_n(i_0, i_n, S_n, p)$ to an auctioneer, who then calculates a market clearing price p. Suppose trader n conjectures that the other $N-1$ traders submit symmetric linear demand schedules

$$X_m(i_0, i_m, S_m, p) = \alpha\cdot i_0 + \beta\cdot i_m - \gamma\cdot p - \delta\cdot S_m, \quad n = 1,\ldots,N, \quad m\neq n. \tag{8}$$

From the market clearing condition $\sum_{m=1}^{N} X_m(i_0, i_m, S_m, p) = 0$ and the linear specification of demand for the other traders, it follows that $x_n + \sum_{m\neq n}(\alpha\cdot i_0 + \beta\cdot i_m - \gamma\cdot p - \delta\cdot S_m) = 0$. Since $\sum_{m=1}^{N} S_m = 0$, solving for p as a function of i_0, i_{-n}, S_n, and x_n yields price impact function of the following form

$$P(i_0, i_{-n}, S_n, x_n) = \frac{\alpha}{\gamma}\cdot i_0 + \frac{\beta}{\gamma}\cdot i_{-n} + \frac{1}{(N-1)\gamma}\cdot x_n + \frac{\delta}{(N-1)\gamma}\cdot S_n. \tag{9}$$

Under the assumption that trader n knows the value of i_{-n}, we plug equation (9) into equation (7) and use the first order condition to find his optimal demand,

(10)

$$x_n = \frac{\Big(\frac{\tau_0}{\tau}\cdot i_0 + \frac{\tau_H}{\tau}\cdot i_n + \frac{(N-1)\tau_L}{\tau}\cdot i_{-n}\Big) - \Big(\frac{\alpha}{\gamma}\cdot i_0 + \frac{\beta}{\gamma}\cdot i_{-n}\Big) - \Big(\frac{\delta}{(N-1)\gamma} + \frac{A}{\tau}\Big)\cdot S_n}{\frac{2}{(N-1)\gamma} + \frac{A}{\tau}}.$$

In the numerator of this equation, the first term is trader n's expectation of the liquidation value, the second term is the market clearing price when trader n trades a quantity of zero and has no inventory, the last term is the adjustment for existing inventory. In the denominator, the first and second terms reflect how trader n restricts the quantity traded due to market power and risk aversion, respectively.

As in Kyle (1989), even though trader n does not observe i_{-n} explicitly, he is still able to implement this optimal strategy by inferring i_{-n} from the market clearing price; as in Kyle (1989) and Du and Zhu (2013), the strategies are ex post optimal.

Define the constant $C := 1/((N-1)\gamma) + A/\tau + \tau_L/(\tau\beta)$. Solving for i_{-n} instead of p in the market clearing condition with linear conjectured strategies for the other traders, substituting this solution into equation above, and then solving for x_n, we derive a demand schedule $X_n(i_0, i_n, S_n, p)$ for trader n as a function of price p,

(11)

$$X_n(i_0, i_n, S_n, p) = \frac{1}{C} \cdot \left[\left(\frac{\tau_0}{\tau} - \frac{(N-1)\tau_L}{\tau} \frac{\alpha}{\beta} \right) \cdot i_0 + \frac{\tau_H}{\tau} \cdot i_n + \left(\frac{(N-1)\tau_L}{\tau} \frac{\gamma}{\beta} - 1 \right) \cdot p - \left(\frac{\tau_L}{\tau} \frac{\delta}{\beta} + \frac{A}{\tau} \right) \cdot S_n \right].$$

Equilibrium.

In a symmetric linear equilibrium, the strategy chosen by trader n must be the same as the linear strategy (8) conjectured for the other traders. Equating corresponding coefficients of variables i_0, i_n, P and S_n yields the system of four equations in terms of four unknowns α, β, γ, and δ:

$$\alpha = \frac{1}{C} \left(\frac{\tau_0}{\tau} - \frac{(N-1)\tau_L}{\tau} \frac{\alpha}{\beta} \right), \quad \beta = \frac{1}{C} \frac{\tau_H}{\tau}, \tag{12}$$

$$\gamma = -\frac{1}{C} \left(\frac{(N-1)\tau_L}{\tau} \frac{\gamma}{\beta} - 1 \right), \quad \delta = \frac{1}{C} \left(\frac{\tau_L}{\tau} \frac{\delta}{\beta} + \frac{A}{\tau} \right). \tag{13}$$

The unique solution is

$$\beta = \frac{(N-2)\tau_H - 2(N-1)\tau_L}{A(N-1)}, \tag{14}$$

$$\alpha = \frac{\tau_0}{\tau_H + (N-1)\tau_L} \cdot \beta, \quad \gamma = \frac{\tau}{\tau_H + (N-1)\tau_L} \cdot \beta, \quad \delta = \frac{A}{\tau_H - \tau_L} \cdot \beta.$$

THEOREM 1: *Define the constant* $\Delta_H := \tau_H - 2(\tau_H + (N-1)\tau_L)/N$. *In addition to a no-trade equilibrium, which always exists, there exists a unique symmetric equilibrium with linear trading strategies and non-zero trade if and only if* $\Delta_H > 0$. *Such an equilibrium has the following properties:*

1. *The equilibrium demand functions are given by equations (8) and (14).*
2. *The equilibrium quantity traded by trader* n *is*

$$x_n^* = \frac{\Delta_H}{A} \cdot \left(\tilde{i}_n - \tilde{i}_{-n} \right) - \delta \cdot S_n, \tag{15}$$

3. The equilibrium price is

(16)
$$P^* = \frac{\tau_0}{\tau} \cdot \tilde{i}_0 + \frac{\tau_H + (N-1)\tau_L}{\tau} \cdot \frac{1}{N} \sum_{m=1}^{N} \tilde{i}_m.$$

The second order condition for optimization (7) is equivalent to the denominator of equation (10) being positive, i.e., $\frac{2}{(N-1)\gamma} + \frac{A}{\tau} > 0$. Given the solution for γ in equation (14), this second order condition holds if and only if $\Delta_H > 0$. From equation (14), the condition $\Delta_H > 0$ also ensures that a trader's demand function is increasing in the trader's own private signal ($\beta > 0$), increasing in the public signal ($\alpha > 0$), decreasing in price ($\gamma > 0$), and decreasing in the trader's inventory ($\delta > 0$). From equation (15), the trader trades in the direction of his private signal and against the average of the signals of others.

We think of Δ_H as a measure of "disagreement," which is positive if and only if there is an equilibrium with trade. It reflects the extent by which each trader believes that the precision of his own private signal τ_H more than twice exceeds the precision of the average private signal in the market (including his own). The necessary condition $\Delta_H > 0$ is equivalent to the condition

(17)
$$\frac{\tau_H}{\tau_L} > 2 + \frac{2}{N-2}.$$

A symmetric linear equilibrium does not exist unless $N \geq 3$ and τ_H is sufficiently more than twice as large as τ_L. To obtain an equilibrium with positive trading volume, there needs to be "enough" disagreement. As in the model of Rostek and Weretka (2012) with "bid-shading," traders exploit their market power by trading approximately one-half of the amount they believe would fully reveal their information. If $N \to \infty$, an equilibrium exists only if each trader believes that his signal is more than twice as precise as other traders believe it to be, so that other traders will be willing to take the opposite side of his trades. When there is not enough disagreement, each trader wants to shade his bid more than the others, and the result is no trade.[2]

[2]When there is not enough disagreement to sustain an equilibrium with pure strategies, one might imagine that it is possible to have an equilibrium with mixed strategies. For mixed strategies to be an equilibrium, the trader must be indifferent across the various randomized choices of quantities he trades. For example, if we add normally distributed noise to quantities traded, symmetrically across all traders, a mixed strategy equilibrium requires the second order condition to be exactly zero. This means that the quadratic objective function reduces to a linear function, i.e., the denominator of equation (10) is zero. Since the trader has to be indifferent across various randomizations, this further implies that the linear function must be a constant, independently of the quantity traded. This assumption cannot hold, because a trader with a positive value of i_n would always want to buy unlimited quantities and a trader with a negative i_n would always want to sell unlimited quantities. Thus, an equilibrium with symmetric normally distributed noise cannot exist. When noise is not normally distributed or the equilibrium is not symmetric, the objective function is not quadratic any more, but it will still be difficult to find a mixed strategy equilibrium given that the sensitivity of utility to a the trader's own private information must be well-defined.

Equilibrium Properties.

The fully revealing equilibrium price (16) is the average of all traders' valuations of the risky asset, i.e., the price is the precision-weighted average of the public signal i_0 (with precision τ_0) and the N private signals i_n (with precision $[\tau_H + (N-1)\tau_L]/N$ each). Each trader believes that signals are weighted in equilibrium price incorrectly. Each trader believes that his own signal should receive a higher weight of τ_H and the other $N-1$ signals should each receive a lower weight of τ_L. There is no risk adjustment, because the risky asset is in zero-net supply and there are no noise traders.

Traders trade for both information and hedging motives. The equilibrium quantity traded x_n^* in equation (15) is a linear function of the deviation of a trader's signal from the average of other traders' signals $(\tilde{i}_n - \frac{1}{N}\sum_{m \neq n} \tilde{i}_m)$ and his inventory S_n.

Each trader "shades" the quantity traded relative to the quantity that a perfect competitor would trade to exercises his market power. To quantify this shading, define a trader's "target inventory" S_n^{TI} as the inventory such that he does not want to trade ($x_n^* = 0$), given from equation (15) by

$$S_n^{TI} = \frac{1}{A} \cdot \left(1 - \frac{1}{N}\right) \cdot (\tau_H - \tau_L) \cdot (\tilde{i}_n - \tilde{i}_{-n}). \tag{18}$$

Then, his optimal demand can be written

$$x_n^* = \delta \cdot (S_n^{TI} - S_n), \tag{19}$$

where the parameter δ is defined in equation (14).

These equations have a simple intuition. A target inventory S_n^{TI} is proportional to a trader's risk tolerance $1/A$ and the difference between his valuation and the valuation of other traders, which itself is proportional to overconfidence $\tau_H - \tau_L$ and to the difference between a trader's signal and the average signal of others $(\tilde{i}_n - \tilde{i}_{-n})$. Even if N goes to infinity, traders continue to agree to disagree and their target inventories do not converge to zero.

The parameter δ determines the speed with which traders adjust positions towards target levels. This parameter is always less than one, decreasing monotonically in precision τ_L from $\delta = 1 - 1/(N-1) < 1$ when $\tau_L = 0$ to $\delta = 0$ when $\tau_L = (1 - 1/(N-1))\tau_H/2$, which corresponds to $\Delta_H = 0$. If traders were perfect competitors, it could be shown that the competitive equilibrium price would be still defined by equation (16) but the optimal demand would be equal to $x_n^* = S_n^{TI} - S_n$, i.e., traders would move all the way from initial inventory S_n to target inventory S_n^{TI} and $\delta = 1$ in equation (19). Thus, monopoly power reduces the amount of trading relative to perfect competition.

Equation (9) implies that the price impact function has the form

$$P(x_n, S_n) = \lambda_0 + \lambda_S \cdot S_n + \lambda_x \cdot x_n. \tag{20}$$

Using equations (13) and (14), its coefficients are

$$\lambda_S := \frac{\delta}{(N-1)\gamma} = \frac{\tau_H + (N-1)\tau_L}{\tau} \cdot \frac{A}{(N-1)(\tau_H - \tau_L)}. \tag{21}$$

and

$$\lambda_x := \frac{\lambda_S}{\delta} = \frac{1}{(N-1)\gamma} = \frac{A}{N\Delta_H} \cdot \frac{\tau_H + (N-1)\tau_L}{\tau}. \tag{22}$$

In equilibrium, traders provide liquidity to one another because they agree to disagree about the quality of their respective signals. The liquidity measures λ_S and λ_x depend on the degree of disagreement and risk aversion. In the continuous-time model, which we consider next, the first component $\lambda_S \cdot S_n$ will be related to permanent linear impact as in Kyle (1985) and the second component $\lambda_x \cdot x_n$ will be related to temporary price impact determined by the speed of trading (x_n^* will be replaced by the derivative of the trader's inventory dS_n/dt). As we shall see below, the continuous time model sharpens the insights derived from this one-period model.

II. Continuous-time Model

There are N risk-averse oligopolistic traders who trade at price $P(t)$ a risky asset in zero net supply against a risk-free asset which earns constant risk-free rate r.

The risky asset pays out dividends at continuous rate $D(t)$. Dividends follow a stochastic process with mean-reverting stochastic growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$,

$$dD(t) = -\alpha_D \cdot D(t) \cdot dt + G^*(t) \cdot dt + \sigma_D \cdot dB_D(t). \tag{23}$$

The growth rate $G^*(t)$ follows an AR-1 process with mean reversion α_G and volatility σ_G:

$$dG^*(t) = -\alpha_G \cdot G^*(t) \cdot dt + \sigma_G \cdot dB_G(t). \tag{24}$$

The dividend is publicly observable, but the growth rate $G^*(t)$ is not observed by any trader.

Each trader n observes a continuous stream of private information $I_n(t)$ defined by the stochastic process

$$dI_n(t) = \tau_n^{1/2} \cdot \frac{G^*(t)}{\sigma_G \cdot \Omega^{1/2}} \cdot dt + dB_n(t), \quad n = 1, \ldots, N. \tag{25}$$

Since its drift $\tau_n^{1/2} \cdot G^*(t)/(\sigma_G \Omega^{1/2})$ is proportional to $G^*(t)$, each increment $dI_n(t)$ in the process $I_n(t)$ is a noisy observation of the unobserved growth rate $G^*(t)$. In equation (25), the parameters σ_G and Ω are scaling parameters which simplify the intuitive interpretations of the model. The "precision" parameter τ_n measures the informativeness of the signal $dI_n(t)$ as a signal-to-noise ratio describing how fast the information flow generates a signal of a given level of statistical significance. The precise interpretation of Ω is discussed below.[3]

Analogously to the one period model, we assume that each trader believes that his own private information $I_n(t)$ has "high" precision $\tau_n = \tau_H$ and the other traders' private information have "low" precision $\tau_m = \tau_L$ for $m \neq n$, with $\tau_H > \tau_L \geq 0$.

Each trader's information set at time t consists of the histories of the dividend process $D(s)$, the trader's own private information $I_n(s)$, and the market price $P(s)$, $s \in [-\infty, t]$. We assume that all traders process information rationally; they apply Bayes Law correctly given their possibly incorrect beliefs.

Using the scaling parameter Ω, we can express the information content of the publicly observable dividend $D(t)$ in a form consistent with the notation for private signals $I_n(t)$ in equation (25). Define $dI_0(t) := [\alpha_D \cdot D(t) \cdot dt + dD(t)] / \sigma_D$ and $\tau_0 := \Omega \cdot \sigma_G^2 / \sigma_D^2$ with $dB_0 := dB_D$. Then the public information $I_0(t)$ in the divided stream (23) can be written

$$dI_0(t) = \tau_0^{1/2} \cdot \frac{G^*(t)}{\sigma_G \cdot \Omega^{1/2}} \cdot dt + dB_0(t). \tag{26}$$

Observing the process $I_0(t)$ is informationally equivalent to observing the dividend process. The quantity τ_0 measures the precision of the dividend process in units analogous to the units of precision for private signals. This notation simplifies the Kalman filtering formulas we are about to derive.

We assume that the values of the parameters α_D, σ_D, α_G, σ_G, τ_H, τ_L, and Ω are common knowledge. It is common knowledge that $B_D(t), B_G(t), B_1(t), ...B_N(t)$ are independently distributed standardized Brownian motions. Traders stubbornly believe that their beliefs τ_H or τ_L about the precision parameters $\tau_0, \ldots, \tau_N$ are point estimates with no possibility of error. This stubborn belief structure of all traders is also common knowledge. Traders thus "agree to disagree" about the precisions of their private signals. Regardless of the true precisions of the signals, traders are "relatively overconfident" about the precision of their signals.

Let $S_n(t)$ denote the inventory of trader n at time t. Since the risky asset is in zero net supply, we have $\sum_{n=1}^{N} S_n(t, P(t)) = 0$. We conjecture that traders smooth out their trading, i.e., the trajectories of their inventories $S_n(t)$ are differentiable functions of time without diffusion terms. Intuitively, infinitely fast portfolio updating cannot be an equilibrium. If other traders traded infinitely fast, each trader would then believe that he could lower his execution costs by trading more slowly than the other traders—essentially by walking up or down the residual demand schedules

[3] In general, the units in which the signal $I_n(t)$ is measured do not affect the information content of the signal itself, as long as traders understand the scaling; thus, $I_n(t)$ has the same information content as $K \cdot I_n(t)$ for any constant K, assuming the value of the constant K is common knowledge. Since the innovation variance of the signal $dI_n(t)$ can be estimated arbitrarily precisely by observing an arbitrarily large number of past signals, traders do not disagree with the hypothesis that the innovation variance of the signal is one. Scaling the innovation variance of $I_n(t)$ in equation (25) to make it equal one is therefore a normalization without loss of generality.

they present to him—but all traders cannot trade more slowly than average. This is similar to Kyle (1985), where it turns out to be optimal for informed trader in Kyle (1985) to smooth out his trading so that his inventory is a continuous function of time. The noise traders in Kyle (1985) do not trade optimally; they generate high transactions costs by trading infinitely impatiently, so that their inventory is a diffusion, not a differentiable function of time. If they were modeled as hedgers motivated by endowment shocks, they would smooth their trading as well, like in Vayanos (1999), and this would "break" the equilibrium in Kyle (1985).

We therefore specify trading strategies and market clearing condition in terms of rates of trading, not shares traded. Each trader's trading strategy is assumed to be a mapping from his information set at time t into a "flow demand schedule" $X_n(t,.)$ defining his "trading intensity" as a function of its price $P(t)$. An auctioneer continuously calculates the market clearing price $P(t)$ such that $\sum_{n=1}^{N} X_n(t, P(t)) = 0$. The trader's inventory follows $dS_n(t)/dt = X_n(t, P(t))$. Each trader takes into account the effect of his trading on market prices.

Each trader chooses a consumption intensity $c_n(t)$ and trading strategy $X_n(t,.)$ to maximize an expected constant-absolute-risk-aversion (CARA) utility function. Let $U_n(c(s)) := -e^{-A\cdot c(s)}$ be an exponential utility function with a constant absolute risk aversion parameter A. Letting ρ denote a time preference parameter, trader n solves the maximization problem

$$\max_{\{c_n(t), X_n(t,.)\}} \quad E_t^n \left\{ \int_{s=t}^{\infty} e^{-\rho(s-t)} \cdot U(c_n(s)) \cdot ds \right\}, \tag{27}$$

subject to the inventory constraint $dS_n(t) = X_n(t, P(t))dt$ and the budget constraint $dW = (rW(t) + S_n(t)D(t) - c_n(t) - P(t)X_n(t, P(t)))dt$. The superscript n indicates that the expectation is taken with respect to the beliefs of trader n. The subscript t indicates that the expectation is taken with respect to trader n's information set at time t. As shown below, in equilibrium, each trader can infer the average of other traders' private signals from the history of prices, so all traders act as if they are fully informed with the same information $D(t), I_1(t), \ldots, I_N(t)$.

We will show numerically that if disagreement is large enough—i.e., if τ_H is sufficiently larger than τ_L—there will be trade based on private information. The perceived precisions τ_L and τ_H affect the equilibrium prices and quantities traded. Without overconfidence—e.g., in a model with rational expectations— there would be no trade after traders unwind their suboptimal initial endowments.

Bayesian Updating with Signals of Arbitrary Precision.

In general, discussing how beliefs of traders about the information content of signals affects the information content of prices is tricky because the discussion requires notation which keeps track of the unobserved true values of the parameters, the beliefs of an economist who studies the market outcomes, and the possibly incorrect beliefs of the traders in the market. A hypothetical economist who studies this equilibrium may assume precisions arbitrarily different from the traders in the market, and the traders in the market may, in principle, assume values arbitrarily different from other traders. Of course, by studying the equilibrium of the economy, an economist will not change the equilibrium but the economist's beliefs about the values of the precisions will affect how the economist interprets the information content of prices.

We therefore first study information processing for arbitrary "generic" beliefs $\bar{\tau}_0, \bar{\tau}_1, ...\bar{\tau}_N$ about the precisions.

Define $G(t) = E_t\{G^*(t)\}$, where the subscript t denotes conditioning on the history of the signals $I_0(s), \ldots, I_N(s)$ for $s \in [-\infty, t]$. Without loss of generality, we define $\bar{\Omega}$ as the error variance $\bar{\Omega} := Var\{(G^*(t) - G(t))/\sigma_G\}$. We assume a steady state in which $\bar{\Omega}$ is a constant which does not depend on time. Like a squared Sharpe ratio, $\bar{\Omega}$ measures the error variance in units of time. For example, if time is measured in years, $\bar{\Omega} = 4$ means that the estimate of $G^*(t)$ is "behind" the true value of $G^*(t)$ by an amount equivalent to four years of volatility unfolding at rate σ_G. There are simple and intuitive formulas for information processing:

LEMMA 1: *Given generic beliefs $\bar{\tau}_1, \ldots, \bar{\tau}_N$, let $\bar{\tau}$ denote the sum of precisions*

$$\bar{\tau} := \bar{\tau}_0 + \sum_{n=1}^{N} \bar{\tau}_n. \tag{28}$$

Then $\bar{\Omega}$ and $dG(t)$ satisfy

$$\bar{\Omega}^{-1} := Var^{-1}\left\{\frac{G^*(t) - G(t)}{\sigma_G}\right\} = 2 \cdot \alpha_G + \bar{\tau}, \tag{29}$$

$$dG(t) = -\left(\alpha_G + \bar{\tau}\right) \cdot G(t) \cdot dt + \sigma_G \cdot \bar{\Omega}^{1/2} \cdot \sum_{n=0}^{N} \bar{\tau}_n^{1/2} \cdot dI_n. \tag{30}$$

The error variance $\bar{\Omega}$ corresponds to a steady state that balances an increase in error variance due to stochastic change $dB_G(t)$ in the true growth rate with a reduction in error variance due to a mean-reversion of the true growth rate at rate α_G and arrival of new information with total precision $\bar{\tau}$.

Note that $\bar{\Omega}$ is a not a "free parameter," but is instead determined as an endogenous function of the other parameters. Equation (29) implies that Ω turns out to be the solution to the quadratic equation $\bar{\Omega}^{-1} = 2 \cdot \alpha_G + \bar{\Omega} \cdot \sigma_G^2/\sigma_D^2 + \sum_{n=1}^{N} \bar{\tau}_n$. In equations (25) and (26), we scaled the units with which precision is measured by the endogenous parameter Ω because this leads to simpler Kalman filtering expressions which bring out more clearly intuition about signal processing.

From equation (30), the estimate $G(t)$ can be conveniently written as the weighted sum of $N+1$ sufficient statistics H_n corresponding to information flow dI_n. Define the sufficient statistics $H_n(t)$ by

$$H_n(t) := \int_{u=-\infty}^{t} e^{-(\alpha_G + \bar{\tau})\cdot(t-u)} \cdot dI_n(u), \quad n = 0, 1, ...N, \tag{31}$$

which implies

$$dH_n(t) = -(\alpha_G + \bar{\tau}) \cdot H_n(t) \cdot dt + dI_n(t), \quad n = 0, 1, ...N. \tag{32}$$

Then $G(t)$ becomes a linear combination of sufficient statistics $H_n(t)$ with weights proportional to the square roots of the precisions $\bar{\tau}_n^{1/2}$:

$$G(t) = \sigma_G \cdot \bar{\Omega}^{1/2} \cdot \sum_{n=0}^{N} \bar{\tau}_n^{1/2} \cdot H_n(t). \tag{33}$$

The importance of each bit of information dI_n about the growth rate $G(t)$ decays exponentially at a rate $\alpha_G + \bar{\tau}$, which is the same for all of the signals. The half-life of a signal, $\ln 2/(\alpha_G + \bar{\tau})$ decreases as the "aggregate precision" $\bar{\tau}$ increases. Even though the true unobserved growth rate may have a long half life (small α_G), information about this growth rate may decay rapidly if aggregate precision $\bar{\tau}$ is large.

Note that equations (25), (26), and (30) imply that the estimate $G(t)$ mean-reverts to zero at a rate α_G while moving towards the true value G^* at rate τ:

$$dG(t) = -\alpha_G \cdot G(t) \cdot dt + \bar{\tau} \cdot (G^* - G) \cdot dt + \sigma_G \cdot \bar{\Omega}^{1/2} \cdot \sum_{n=0}^{N} \bar{\tau}_n^{1/2} \cdot dB_n(t). \tag{34}$$

Bayesian Updating by Traders in the Model.

Asset managers who trade based on statistical models typically take raw information and process it into "signals". Returns forecasts are then generated as functions of the signals. The signals are often scaled so that they have a meaningful interpretation in terms of intuition or statistics. Here, we can think of the information processes $I_n(t)$ as "raw information" and the sufficient statistics $H_n(t)$ as "signals." In equilibrium, traders believe that their signals forecast returns. Given the depth of the market, they trade on the signals with a aggressiveness that depends on the information content and the decay rate of the signals.

We next consider how traders "in the model" use their signals to update their beliefs about the unobserved growth rate $G^*(t)$.

Let $G_n(t) := E_t^n\{G^*(t)\}$ denote trader n's estimate of the unobserved growth rate $G^*(t)$ conditional on all information. The superscript n indicates that conditional distributions of growth rates are calculated by trader n based on his belief that his own signal signal has high precision τ_H and other traders' signals have low precision τ_L. The subscript t denotes, as before, conditioning on the history of all information $I_0(s) = D(t), I_1(s), \ldots, I_N(s)$, $s \in [-\infty, t]$.

It is common knowledge that each trader believes his own signal has high precision τ_H and other traders' signals have low precision τ_L. Thus, if we define

$$\tau := \tau_0 + \tau_H + (N-1)\tau_L, \qquad \Omega^{-1} := 2\alpha_G + \tau, \tag{35}$$

all traders agree that the error variance is given by $\bar{\Omega} = \Omega$ from equation (29), total precision is given by $\bar{\tau} = \tau$ from equation (28), with $\tau_0 = \Omega\sigma_G^2/\sigma_D^2$. Traders agree that the correct way to process available information is to construct signals $H_n(t), n = 0, ..N$ by plugging τ and Ω into equation (31) and (32). Because they disagree about the precisions, traders disagree about the weights used to aggregate the signals $H_n(t), n = 0, \ldots, N$ into an estimate of a growth rate in equation (33); each assigns a larger weight to his own signal than to others' signals.

Let $H_{-n}(t)$ denote the average of the other traders' signals $m \neq n$:

$$H_{-n}(t) := \frac{1}{N-1} \sum_{m=1,..,N, m\neq n} H_m(t), \tag{36}$$

Equation (33) implies that trader n's estimate of the true growth rate $G_n(t)$ can be expressed as a linear combination of three signals $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$:

$$G_n(t) := \sigma_G \cdot \Omega^{1/2} \cdot \left(\tau_0^{1/2} \cdot H_0(t) + \tau_H^{1/2} \cdot H_n(t) + (N-1)\tau_L^{1/2} \cdot H_{-n}(t)\right). \tag{37}$$

Trader n's optimal trading strategy depends on trader n's estimates of the unobserved growth rate $G^*(t)$ and his beliefs about the dynamic statistical relationship between this rate and the signals $H_0(t)$, $H_n(t)$ and $H_{-n}(t)$.

Define the $N+1$ processes dB_0^n, dB_n^n, and dB_m^n, $m = 1, \ldots N$, $m \neq n$, by

$$dB_0^n(t) = \tau_0^{1/2}(\sigma_G \Omega^{1/2})^{-1} \cdot (G^*(t) - G_n(t)) \cdot dt + dB_D(t), \tag{38}$$

$$dB_n^n(t) = \tau_H^{1/2}(\sigma_G \Omega^{1/2})^{-1} \cdot (G^*(t) - G_n(t)) \cdot dt + dB_n(t), \tag{39}$$

and

$$dB_m^n(t) = \tau_L^{1/2}(\sigma_G \Omega^{1/2})^{-1} \cdot (G^*(t) - G_n(t)) \cdot dt + dB_m(t). \tag{40}$$

Since trader n's forecast of the error $G^*(t) - G_n(t)$ is zero given his information set, these $N+1$ processes are independently distributed Brownian motions from the perspective of trader n. In terms of these Brownian motions, trader n thinks that signals change as follows:

$$dH_0(t) = -(\alpha_G + \tau) \cdot H_0(t) \cdot dt + \tau_0^{1/2} \cdot (\sigma_G \Omega^{1/2})^{-1} \cdot G_n(t) \cdot dt + dB_0^n(t), \tag{41}$$

$$dH_n(t) = -(\alpha_G + \tau) \cdot H_n(t) \cdot dt + \tau_H^{1/2} \cdot (\sigma_G \Omega^{1/2})^{-1} \cdot G_n(t) \cdot dt + dB_n^n(t), \tag{42}$$

(43)

$$dH_{-n}(t) = -(\alpha_G + \tau) \cdot H_{-n}(t) \cdot dt + \tau_L^{1/2} \cdot (\sigma_G \Omega^{1/2})^{-1} \cdot G_n(t) \cdot dt + \frac{1}{N-1} \sum_{m=1, m\neq n}^{N} dB_m^n(t).$$

Note that each signal drifts towards zero at rate $\alpha_G + \tau$ and drifts towards the optimal forecast $G_n(t)$ at a rate proportional related to the square root of the signal's precisions $\tau_0^{1/2}$, $\tau_H^{1/2}$, or $\tau_L^{1/2}$, respectively.

Utility Maximization with Market Power.

We conjecture a steady state value function $V(M_n, S_n, D, H_0, H_n, H_{-n})$, where M_n denote trader n's cash holdings (measured in dollars) and S_n denote trader n's holdings of the traded asset (measured in shares).

In a competitive model, a trader's value function depends on his wealth but does not depend on the decomposition of his wealth into his various security holdings. With imperfect competition, the decomposition of a trader's wealth into various security holdings does affect his value function because the trader cannot costlessly convert one security holding into cash or another security holding by trading at market prices. It is therefore necessary to keep track of both variables M_n and S_n separately.

Also, we expect the asset price to be a linear combination of two components: (1) a dividend level component linear in dividends $D(t)$ (measured in dollars per share per unit of time) and (2) a dividend-growth component linear in the variables $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$. We will use the concept of "no-regret" pricing, based on the intuition that by conditioning on the market price, the trader can achieve the same outcome that could be obtained if he directly observed the average of other traders' signals $H_{-n}(t)$. Therefore we include $H_{-n}(t)$ as a state variable in the value function and omit the price $P(t)$.

In deriving the equilibrium below, the problem is simplified if the three state variables $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$ are replaced with two "composite" signals, which we denote $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$. Define the weighting constant $\hat{A}$ by

$$\hat{A} := \tau_0^{1/2} \cdot (\tau_H^{1/2} + (N-1)\tau_L^{1/2})^{-1}. \tag{44}$$

Now define the two composite signals $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ by

$$\hat{H}_n(t) := H_n(t) + \hat{A} \cdot H_0(t) \tag{45}$$

$$\hat{H}_{-n}(t) := H_{-n}(t) + \hat{A} \cdot H_0(t) \tag{46}$$

Trader n's estimate of dividend growth rate can now be expressed as a function of the two composite signals $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ as

$$G_n(t) = \sigma_G \cdot \Omega^{1/2} \left(\tau_H^{1/2} \cdot \hat{H}_n(t) + (N-1)\tau_L^{1/2} \cdot \hat{H}_{-n}(t) \right). \tag{47}$$

In terms of the composite variables $\hat{H}_n$ and $\hat{H}_{-n}$, we conjecture (and verify below) a steady state value function of the form $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$. Letting $(c_n(t), X_n(t, .))$ denote the optimal consumption and investment policy, we have

$$V(M, S_n, D, \hat{H}_n, \hat{H}_{-n}) := \max_{\{c_n(t), X_n(t,.)\}} \quad E_t^n \left\{ \int_{s=t}^{\infty} -e^{-\rho(s-t) - A \cdot c_n(s)} \cdot ds \right\}, \tag{48}$$

The five state variables satisfy five stochastic differential equations

$$dM_n(t) = (r \cdot M_n(t) + S_n(t) \cdot D(t) - c_n(t) - P(x_t) \cdot X_n(t, P(t))) \cdot dt, \tag{49}$$

$$dS_n(t) = X_n(t, P(t)) \cdot dt, \tag{50}$$

$$dD(t) = -\alpha_D \cdot D(t) \cdot dt + G_n(t) \cdot dt + \sigma_D \cdot dB_0^n(t), \tag{51}$$

$$\begin{aligned} d\hat{H}_n(t) = & - (\alpha_G + \tau) \cdot \hat{H}_n(t) \cdot dt \\ & + (\tau_H^{1/2} + \hat{A}\tau_0^{1/2}) \cdot \Big(\tau_H^{1/2}(t) \cdot \hat{H}_n + (N-1)\tau_L^{1/2} \cdot \hat{H}_{-n}(t)\Big) \cdot dt \\ & + \hat{A} \cdot dB_0^n(t) + dB_n^n(t), \end{aligned} \tag{52}$$

$$\begin{aligned} d\hat{H}_{-n}(t) = & - (\alpha_G + \tau) \cdot \hat{H}_{-n}(t) \cdot dt \\ & + (\tau_L^{1/2} + \hat{A}\tau_0^{1/2}) \cdot \Big(\tau_H^{1/2} \cdot \hat{H}_n(t) \cdot + (N-1)\tau_L^{1/2} \cdot \hat{H}_{-n}(t)\Big) \cdot dt \\ & + \hat{A} \cdot dB_0^n(t) + \frac{1}{N-1} \sum_{m=1, m \neq n}^{N} dB_m^n(t), \end{aligned} \tag{53}$$

The seemingly complicated dynamics of $\hat{H}_n$ and $\hat{H}_{-n}$ in equations (52) and (53) can be derived from equations (41), (42), and (43).

The value function $V(\ldots)$ satisfies the transversality condition

$$\lim_{t \to +\infty} E^n\{e^{-\rho t} V(M_n(t), S_n(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))\} = 0. \tag{54}$$

Linear Conjectured Strategies.

Based on his information set, each trader submits a flow demand schedule for the rate at which he will buy the asset at time t as a function of the market clearing price. Trader n conjectures that the other $N-1$ traders, $m = 1, \ldots N, m \neq n$, submit symmetric linear demand schedules of the form

$$X_m(t) = \gamma_D \cdot D(t) + \gamma_H \cdot \hat{H}_m(t) - \gamma_S \cdot S_m(t) - \gamma_P \cdot P(t). \tag{55}$$

The demand schedules are defined by the four constants γ_D, γ_H, γ_S, and γ_P.

Let $x_n(t) = X_n(t, P(t))$ denote the quantity traded by trader n. From the market clearing condition and the linear conjecture for demand schedules of other traders, it follows that

$$x_n(t) + \sum_{m \neq n} \Big(\gamma_D \cdot D(t) + \gamma_H \cdot \hat{H}_m(t) - \gamma_S \cdot S_m(t) - \gamma_P \cdot P(t)\Big) = 0. \tag{56}$$

Since zero net supply implies $\sum_{m=1}^{N} S_m(t) = 0$, solving for $P(t)$ as a function of $x_n(t)$ yields the following price impact function trader n conjectures that he faces:

$$(57)\quad P(x_n(t)) = \frac{\gamma_D}{\gamma_P} \cdot D(t) + \frac{\gamma_H}{\gamma_P} \cdot \hat{H}_{-n}(t) + \frac{\gamma_S}{\gamma_P} \frac{1}{N-1} \cdot S_n(t) + \frac{1}{(N-1)\gamma_P} \cdot x_n(t).$$

The key difference from Kyle (1985) is that the residual demand curve is specified in terms of trader n's rate of trading $x_n(t)$ rather than in terms of the number of shares traded. This specification makes temporary price impact associated with the speed with which traders build their positions economically relevant in a manner not present in the model of Kyle (1985).

In the continuous model of Kyle (1985), the informed trader, who buys at rate $x(t)$, conjectures that price follows the process $dP(t) = \lambda \cdot (\sigma_U \cdot dB_U(t) + x(t) \cdot dt)$, where $dB_U(t)$ represents random noise trading. If the informed trader buys Q shares over a period of time Δt, he "walks up the demand schedule" and pays an expected average price

$$(58)\qquad P(t) + \frac{1}{2}\lambda \cdot Q.$$

His permanent price impact is $\lambda \cdot Q$. Regardless of how fast he trades, he has no temporary price impact.

By contrast, if trader n buys Q shares over the time interval Δt in our model, equation (57) implies that the unconditional expected price (e.g., assuming $H_{-n} = 0$) is

$$(59)\qquad P(t) + \frac{1}{2}\frac{\gamma_S}{\gamma_P}\frac{1}{N-1} \cdot Q + \frac{1}{(N-1)\gamma_P} \cdot Q/\Delta t.$$

The permanent price impact coefficient $\gamma_S/\gamma_P \cdot 1/(N-1) \cdot Q$ corresponds to λ in Kyle (1985). The additional term $1/(N-1) \cdot 1/\gamma_P \cdot Q/\Delta t$ represents temporary price impact. The temporary price impact cost is proportional to the speed $1/\Delta t$ with which trader n buys Q shares. It becomes arbitrarily large if he buys that quantity over an arbitrarily short time interval.

Plugging the price impact function (57) into the optimization problem (48), trader n solves for his optimal consumption and demand schedule. Imperfect competition requires trader n to take into account both his permanent and temporary price impact in choosing how fast to change his inventory. Trader n exercises monopoly power in choosing how fast to demand liquidity to profit from innovations in his private information. He also exercises monopoly power in choosing how fast to provide liquidity to the $N-1$ other traders who, according to trader n's beliefs, trade with overconfidence and make supplying liquidity profitable.

In equilibrium, the temporary price impact cost parameter γ_P represents compensation to the other traders for providing trader n with liquidity quickly. Intuitively, the symmetry of equilibrium trading strategies requires traders to believe they are being adequately compensated for both supplying and demanding liquidity in a manner consistent with market clearing.

Conjectured Value Function.

We conjecture and verify that the value function $V(M, S, D, \hat{H}_n, \hat{H}_{-n})$ has the specific quadratic exponential form

$$(60)\quad V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n}) = -\exp\Big(\psi_0 + \psi_M \cdot M_n + \frac{1}{2}\psi_{SS} \cdot S_n^2 + \psi_{SD} \cdot S_n D + \\ + \psi_{Sn} \cdot S_n \hat{H}_n + \psi_{Sx} \cdot S_n \hat{H}_{-n} + \frac{1}{2}\psi_{nn} \cdot \hat{H}_n^2 + \frac{1}{2}\psi_{xx} \cdot \hat{H}_{-n}^2 + \psi_{nx} \cdot \hat{H}_n \hat{H}_{-n}\Big).$$

The nine constants ψ_0, ψ_M, ψ_{SS}, ψ_{SD}, ψ_{Sn},ψ_{Sx},ψ_{nn},ψ_{xx}, and ψ_{nx} have values consistent with a steady state equilibrium.

Notice that "wealth" does not appear in the value function because wealth is not well-defined. Trader n is always influencing the mark-to-market value of his risky inventory by choosing his rate of trading $x_n(t) = dS_n(t)/dt$. Instead, the two components of wealth—cash M_n and inventories S_n—enter the utility function directly. The term ψ_M measures the utility value of cash.

The terms ψ_{SS}, ψ_{SD}, ψ_{Sn},ψ_{Sx} measure the utility value of risky asset holdings. The ψ_{nn}, ψ_{xx}, and ψ_{nx} terms capture the value of future trading opportunities based on current public and private information. The value of trading on innovations to future information is built into the constant term ψ_0.

Characterization of Steady-State Symmetric Equilibrium with Linear Trading Strategies and Quadratic Value Functions.

To solve the trader's optimization problem, we use the "no-regret" approach found in Kyle (1989): Instead of solving for a demand function which depends on price, we suppose instead that the trader observes his equilibrium residual supply schedule, which reveals the value of $H_{-n}(t)$, then picks the optimal point on this residual supply schedule. We then show that this optimal point can be implemented with a linear demand schedule.

To solve for a steady state equilibrium, it is necessary to determine simultaneously values for the four γ-parameters defining the optimal demand schedule in equation (55) and the nine ψ-parameters defining the value function in equation (60). The solution to these equations is discussed in the Appendix. We obtain the following theorem:

THEOREM 2: ***Characterization of Equilibrium.*** *There always exists a no-trade equilibrium. In addition, there may exist a steady state equilibrium with symmetric linear flow trading strategies of the form conjectured in equation (55) and a value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ for trader n satisfying the quadratic conjecture in equation (60). Such an equilibrium has the following properties:*

The parameters ψ_{Sx}, γ_H, γ_S, and γ_D satisfy

$$(61)\quad \psi_{Sx} = \frac{N-2}{2}\psi_{Sn}, \quad \gamma_H = \frac{N\gamma_P}{2\psi_M}\psi_{Sn}, \quad \gamma_S = -\frac{(N-1)\gamma_P}{\psi_M}\psi_{SS}, \quad \gamma_D = \frac{\gamma_P}{\psi_M}\psi_{SD}.$$

The parameters ψ_M and ψ_{SD} satisfy

$$\psi_M = -rA, \quad \psi_{SD} = -\frac{rA}{r+\alpha_D}, \tag{62}$$

and ψ_0 satisfies equation

$$\psi_0 = 1 - \log\{r\} + \frac{1}{r}\left(-\rho + \frac{1}{2}(1+\hat{A}^2)\psi_{nn} + \frac{1}{2}\left(\frac{1}{N-1} + \hat{A}^2\right)\psi_{xx} + \hat{A}^2\psi_{nx}\right). \tag{63}$$

The six constants γ_P, ψ_{SS}, ψ_{Sn}, ψ_{nn}, ψ_{xx}, and ψ_{nx} satisfy the six polynomial equations (109)-(114) in the Appendix. The second order condition requires downward sloping demand schedules, implying $\gamma_P > 0$ and $\psi_{SS} > 0$, and $\psi_{Sn} < 0$.

Define the average of traders' expected growth rates $\bar{G}(t)$ by

$$\bar{G}(t) := \frac{1}{N}\sum_{n=1}^{N} G_n(t), \tag{64}$$

and define the constants C_L and C_G by

$$C_L := -\frac{\psi_{Sn}}{2\psi_{SS}}, \quad C_G := \frac{\psi_{Sn}}{2\psi_M}\frac{N(r+\alpha_D)(r+\alpha_G)}{\sigma_G\Omega^{1/2}(\tau_H^{1/2} + (N-1)\tau_L^{1/2})}. \tag{65}$$

Trader n's optimal consumption satisfies equation

$$c_n^*(t) = -\frac{1}{A}\cdot\log(\psi_M \cdot V(t)/A). \tag{66}$$

Trader n's optimal flow demand schedule $x_n^*(t)$ makes inventories $S_n(t)$ a differentiable function of time such that

$$x_n^*(t) = \frac{dS_n(t)}{dt} = \gamma_S \cdot \Big(C_L \cdot (H_n(t) - H_{-n}(t)) - S_n(t)\Big). \tag{67}$$

The equilibrium price can be written

$$P^*(t) = \frac{D(t)}{r+\alpha_D} + \frac{C_G \cdot \bar{G}(t)}{(r+\alpha_D)(r+\alpha_G)}. \tag{68}$$

Note there is always a trivial no-trade equilibrium, as in the one-period model. If each trader submits a no-trade demand schedule $X_n(t,.) \equiv 0$, then such a no-trade demand schedule is optimal for all traders. This is not a symmetric linear equilibrium in which an auctioneer can establish a meaningful market price.

Equations (67) and (68) imply that the equilibrium with trade has a surprisingly simple structure in which quantities adjust to new information slowly while prices adjust instantaneously. Equation (68) implies that each trader has a target inventory proportional to the difference between his own private signal $H_n(t)$ and the average of other traders' private signals $H_{-n}(t)$ inferred from prices. Each trader continuously moves his inventory towards the target inventory so that the difference decays at rate γ_S. Equation (68) implies that the price is a linear function of the weighted average of all traders expected growth rates. Price is also the precision-weighted average of the public signal $H_0(t)$ (with precision $\tau_0^{1/2}$) and the N private signals $H_n(t)$ (with precision $[\tau_H^{1/2} + (N-1)\tau_L^{1/2}]/N$ each). The price responds instantaneously to innovations in each trader's private information, so that the average of all signals is immediately revealed. This occurs despite the fact that, to reduce trading costs resulting from adverse selection, each trader intentionally slows down his trading to reduce other traders' estimates of the magnitude of his private signal.

Implied Price Impact Model.

Since we have an equilibrium model with imperfect competition, we can explicitly calculate the effect on prices if a trader deviates from his optimal inventory policy $S_n^*(t)$ and instead holds inventories denoted $S_n(t)$, assumed to be a differentiable function of time with $x_n(t) := dS_n(t)/dt$. As a result of the deviation, the old equilibrium price path $P^*(t)$ will be changed to a different price path, denoted $P(t)$, given by

$$P(t) = P^*(t) + \lambda_S \cdot (S_n(t) - S_n^*(t)) + \lambda_x \cdot (x_n(t) - x_n^*(t))\,. \tag{69}$$

From equation (57), the constants λ_S and λ_x are given by

$$\lambda_S := \frac{\gamma_S}{(N-1)\cdot\gamma_p}, \quad \lambda_x := \frac{1}{(N-1)\cdot\gamma_p}. \tag{70}$$

The term $\lambda_S \cdot (S_n(t) - S_n^*(t))$ represents permanent price impact, linear in the numbers of shares. The term $\lambda_x \cdot (x_n(t) - x_n^*(t))$ represents transitory price impact linear in the rate of trading. Larger trades and faster trading result in larger permanent and temporary price changes.

An Existence Condition.

We implement the characterization of equilibrium in Theorem 2 by attempting to solve the equations numerically. As expected, numerical algorithms do not always find an equilibrium with trade satisfying Theorem 2.

Although we have not been able to prove analytically the conditions under which equilibrium exists, extensive numerical experimentation supports the following intuitive argument: Like the one period model, we expect equilibrium with trade to exist only if there is enough disagreement. With continuous trading, each trader tries to exercise monopoly power by smoothly walking along the residual demand schedules of other traders rather than trade at one market clearing price. If ΔP denotes the price impact of trading smoothly, then the average transactions price incorporates a realized price impact cost of approximately $\Delta P/2$. Compared with a single round of trading, the price impact cost is one-half the cost of trading the entire quantity in one auction with impact ΔP. Each trader therefore believes that this smooth trading strategy extracts more value from other traders than the single price of a one-shot auction. For traders to be willing to take the other side of the smooth trades of their competitors, traders must believe that their competitors' signals are only about half as precise as in the one period model. With continuous trading, existence of an equilibrium with trade may therefore require "more disagreement" than in the one-period model—by a factor of approximately two.

To convert this intuitive argument into mathematics, fix all of the exogenous parameters except for the number of "other" traders $N-1$ and the "low" precision of their signals τ_L. Now allow $N-1$ and τ_L to vary such that the total precision $(N-1)\tau_L$ of other traders is constant. If $N-1$ is very large and τ_L is very small, there is a huge degree of disagreement, each trader is small relative to the market, and an equilibrium should exist which resembles perfect competition or monopolistic competition . As $N-1$ shrinks and τ_L increases, eventually a point is reached such that there is not enough disagreement to support an equilibrium. Just before this point is reached, the parameter γ_P—which measures the liquidity of the market—should fall to a value close to zero; the equilibrium should involve very little trade; and the value function should resemble a no-trade equilibrium. The value of $N-1$ such that $\gamma_P = 0$ defines defines a "critical" value N^* (not necessarily an integer) such that equilibrium exists if and only if $N > N^*$.

This intuitive argument leads to a mathematically precise existence condition derived from the six equations in six unknowns (109)-(114) in the Appendix. Plug $\gamma_P = 0$ into these equations, representing no market liquidity. Now, holding the other exogenous parameters constant—allow $N-1$ and τ_L to vary so that $(N-1)\tau_L$ is constant and the six equations have a solution. With $\gamma_P = 0$, it is clear that $\psi_{nn} = \psi_{nx} = \psi_{xx} = 0$ solves the last three of the six equations, consistent with the intuition that information has no value if there is no market liquidity. It is straightforward to show that a solution to the first three equations requires that the critical value N^* satisfy $\tau_H/\tau_L = (2+2/(N^*-2))^2$. We therefore conjecture that an equilibrium with trade, consistent with Theorem 2, exists if and only if

$$\frac{\tau_H}{\tau_L} > \left(2 + \frac{2}{N-2}\right)^2. \tag{71}$$

Our extensive examination of numerical solutions to the six equations in six unknowns supports this conjecture. We have found that precisely one solution with downward sloping demand schedules ($\gamma_P > 0$) is discovered when existence condition (71) is satisfied, and no solution with downward sloping demand schedules is discovered when inequality (71) is reversed.

Since the existence condition for the one-period model can be written $\tau_H/\tau_L > 2 + 2/(N-2)$, the existence condition for the continuous model in equation (71) is the square of the existence condition for the one-period model. For relatively large N, existence in the one period model requires that each trader believe the high precision τ_H of his own signal to be somewhat more than twice as large as the low precisions τ_L of the other $N-1$ traders. In the continuous model, existence requires that each trader believe the high precision τ_H of his signal to be somewhat more that four times larger than the low precisions τ_L of the other $N-1$ traders. These numerical results are consistent with the spirit of Vayanos (1999) and Du and Zhu (2013), who find that increasing the interval between rounds of trading increases trading by providing traders with a "commitment" not to trade between trading rounds.

Numerical Comparative Statics Results.

Next, we analyze numerically how a degree of competition and overconfidence affect the solution. We keep the total precision τ fixed and vary the degree of overconfidence measured by the ratio of precisions τ_H/τ_L and the degree of competition measured by the number of traders N.

Figure 1 shows the effect of changes in degree of overconfidence τ_H/τ_L on parameters γ_S, γ_P, C_G, C_L, $1/\lambda_S$ and $1/\lambda_x$. We assume that $r = 0.01, A = 1, \alpha_D = 0.1$, $\alpha_G = 0.02, \sigma_D = 0.5, \sigma_G = 0.1, \tau_0 = \Omega\sigma_G^2/\sigma_D^2 = 0.0054$ and $N = 100$. We change τ_H and τ_L, keeping the total precision fixed at $\tau = 7.4$. Higher values of the ratio τ_H/τ_L correspond to higher degree of overconfidence. Note that the equilibrium exists only when τ_H is sufficiently large relative to τ_L, as described in equation (71), otherwise the numeric algorithm for solving the system (109)-(114) does not converge to the solution.

The coefficient C_G decreases monotonically as τ_H/τ_L increases. The more traders disagree with each other, the more they discount actions of others and therefore dampen the equilibrium price sensitivity to the average signal. The rate of inventory adjustment γ_S and the sensitivity of trading rate to prices γ_P increases, as the degree of disagreement increases. The coefficient C_L related to target inventories is a non-monotonic function of τ_H/τ_L. The price impact coefficients λ_S and λ_x decreases, as the degree of disagreement increases, since traders are willing to provide more liquidity to presumably less informed counterparties. As τ_H/τ_L increases, the model converges to the market of Black (1986), in which everybody thinks that others are noise traders with $\tau_L = 0$. In the limit, λ_S and λ_x are close to zero, γ_S converges to infinity, and C_G converges to some fixed level.

Figure 2 shows the effect of changes in degree of competition N on parameters γ_S, γ_P, C_G, C_L, $1/\lambda_S$ and $1/\lambda_x$. In order to isolate the effect of competition from the effect of overconfidence, we assume that $\tau_L = 0$ and fix $\tau = 1.4$. This ensures that changes in N do not change the total precision: There is the same amount of information, but information is dispersed among more market participants. Higher values of N correspond to higher degree of competition. When N increases, however, the effective risk aversion of the market A/N changes as well, and the market converges to a risk-neutral case. We assume that $r = 0.01, A = 1, \alpha_D = 0.1$, $\alpha_G = 0.02, \sigma_D = 0.5, \sigma_G = 0.1$ and $\tau_0 = \Omega\sigma_G^2/\sigma_D^2 = 0.0279$.

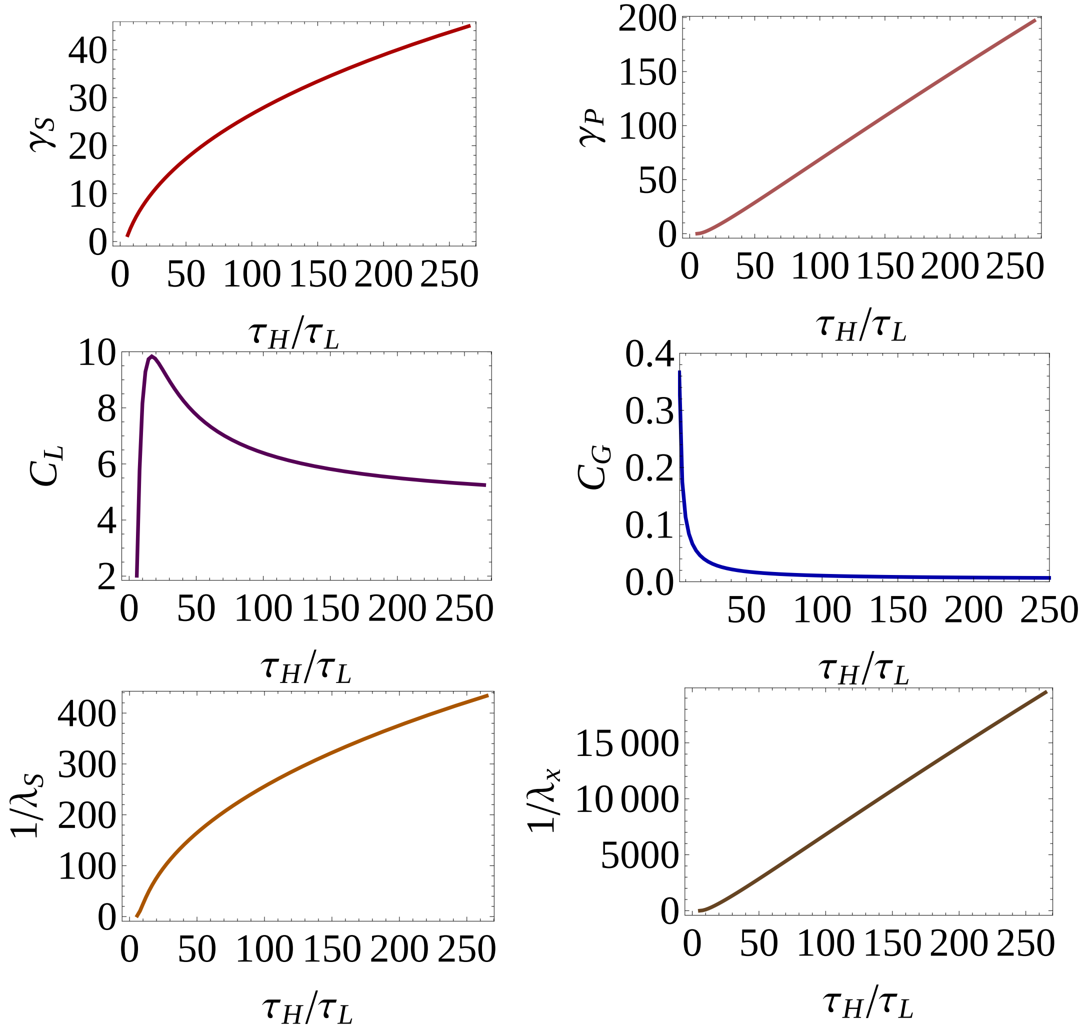

FIGURE 1. COEFFICIENTS γ_S, γ_P, C_G, C_L, $1/\lambda_S$ AND $1/\lambda_x$ AGAINST τ_H/τ_L WHILE FIXING $\tau = 7.4$.

Figure 2 shows that the speed of adjusting positions γ_S towards targets increases with the number of traders N, as each trader thinks that the risk bearing capacity of the market in aggregate increases and it becomes less and less costly for traders to trade aggressively towards their target inventories. The coefficient C_L defining target inventories increases with N and converges to the constant level after $N = 150$. The coefficient C_G is monotonically decreasing with N. The sensitivity of investors' order to market price γ_P increases with N. Both price impact coefficients λ_S and λ_x decrease, as the number of traders N increases, keeping the total precision fixed.

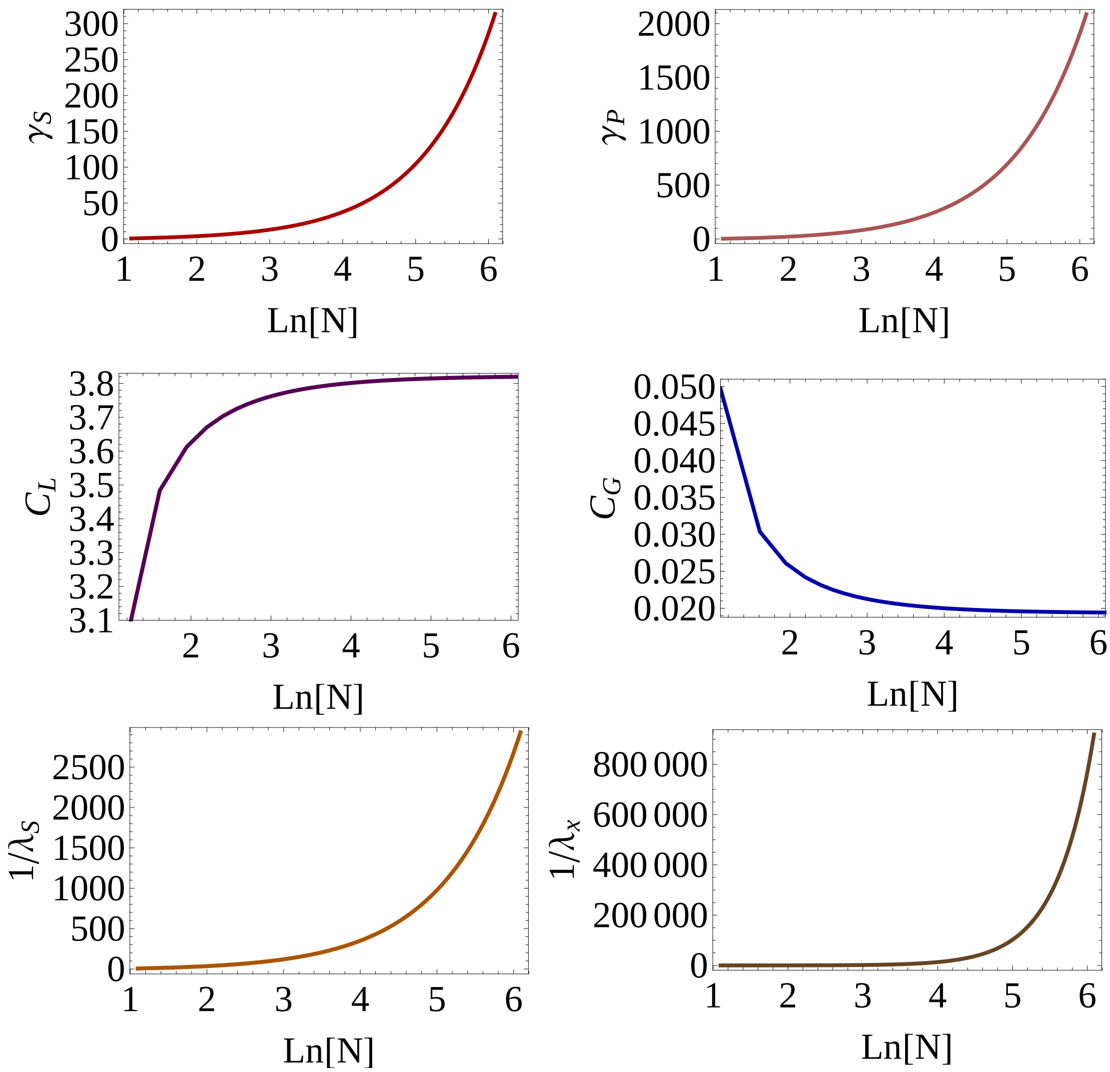

FIGURE 2. γ_S, γ_P, C_G, C_L, $1/\lambda_S$ AND $1/\lambda_x$ AGAINST N WHILE FIXING $\tau = 1.4$ AND $\tau_L = 0$

.

III. Implications

This section discusses properties of prices, quantities, and trading strategies in more detail.

"Dampened" Prices Reflect a Keynesian Beauty Contest.

Define the "fundamental value" of the risky asset as the expected present value of all future dividends based on all information, discounted at the risk-free rate r. From the perspective of trader n, it can be shown that the fundamental value of the risky asset is given by a version of Gordon's growth formula based on trader n's expected growth rate $G_n(t)$:

$$F_n(t) = \frac{D(t)}{r+\alpha_D} + \frac{G_n(t)}{(r+\alpha_D)(r+\alpha_G)}. \tag{72}$$

Since the risky asset is in zero net supply, intuition might suggest that the equilibrium price is the average estimate of fundamental value $\sum_n F_n(t)/N$ obtained by replacing $G_n(t)$ with $\bar{G}(t)$ in equation (72). This intuition is precisely consistent with the one period model. Surprisingly, in the model with continuous trading, this intuition turns out to be wrong! A comparison of equations (68) and (72) reveals that the equilibrium price would be equal to the average of the N traders' estimates of fundamental value only if $C_G = 1$. In our numerical calculations, we always find $0 < C_G < 1$; thus, we conjecture that C_G must always be less than one. This conjecture implies that, with continuous trading, the equilibrium price is "dampened" version of Gordon's growth formula, with dampening factor $0 < C_G < 1$. Even if all N traders unanimously agree on the same expected growth rate $G_n(t) = \bar{G}(t)$, the equilibrium price will reflect a dampened implied growth rate $C_G \cdot \bar{G}(t)$.

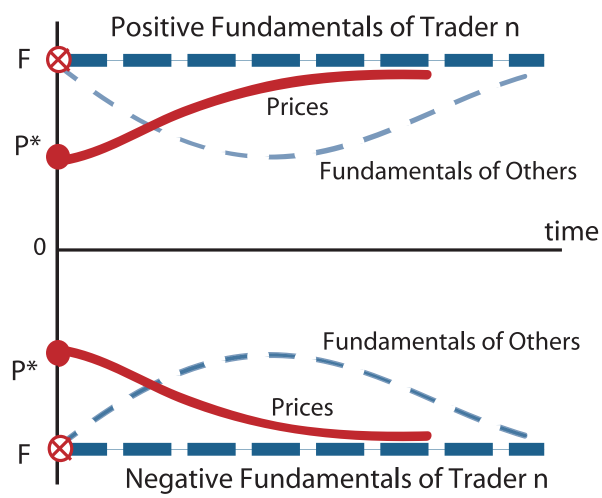

Figure 3. Dynamics of Prices and Fundamentals From Perspective of a Trader.

Figure shows the projected dynamics of a trader's estimate of the fundamental value in dark blue, the projected dynamics of other traders' estimates of the fundamental value in light blue, and the resulting equilibrium prices in red; all three variables are from perspective of a trader. All traders agree on fundamental value F and its dynamics, but each trader thinks that his forecast will differ from forecasts of others in the future and the equilibrium price P^* differs from fundamental value F.

Since the dampening result $0 < C_G < 1$ contrasts with our one-period model of imperfect competition, the intuitive explanation must be based on having many rounds of trading, not imperfect competition. Indeed, Kyle and Lin (2001) find a similar dampening result in a competitive model of continuous trading.

To understand the intuition, suppose that all traders happen to have the same optimistic growth rate estimates $G_n(t) = \bar{G}(t) > 0$. If there were no future rounds of trading, all traders would be happy to hold market clearing inventories of zero if the price reflected $C_G = 1$. Now suppose that traders believe there will be many future rounds of trading but—contrary to equilibrium—believe that the market clearing price will be based on $C_G = 1$, both currently and the the future, rather than $C_G < 1$. Trader n believes that his own growth estimate $G_n(t)$ will decay to zero at rate α_G, consistent with the discount factor $1/(r + \alpha_G)$ in equation (72). But trader n also believes that the growth estimates of the other $N - 1$ traders, $G_m(t)$, $m \neq n$, will decay to zero at a rate faster than α_G, because other traders will become more bearish due to misinterpreting the precision of their signals. Thus, trader n believes that, in the short run, other traders will become more bearish, driving down prices, even though current prices reflect an appropriate consensus buy-and-hold valuation. Since trader n believes that prices will fall in the short run, he will start selling now, with the intention of buying later when prices offer a more attractive value. Since all traders have the same beliefs, all traders will want to sell at prices reflecting $C_G = 1$, and this breaks an equilibrium with the incorrect conjecture $C_G = 1$. A symmetrically similar argument also breaks the equilibrium conjecture $C_G = 1$ if all traders have pessimistic beliefs $G_n(t) = \bar{G}(t) < 0$. This intuition suggests the conjecture $0 < C_G < 1$.

Figure 3 highlights informally patterns in the evolution of trader's expectations. The dark dashed (blue) horizontal lines represent the cumulative expected returns trader n believes he would realize if prices reflected his own expected growth rate $G_n(t)$ and not the average of others' expected growth rates. Since the line is horizontal, trader n would be comfortable holding a target inventory of zero, consistent with clearing the market for the zero-net-supply asset. The light dashed (blue) line represents the cumulative expected returns trader n believes he would realized if prices reflected the average of all traders beliefs about expected growth rates and all N traders started with the same expected growth rate $G_m(t) = \bar{G}(t)$. The line first moves towards zero and then moves back to its initial level, consistent with the interpretation that trader n believes that the other traders' estimate of expected growth rates will first move towards zero (since they were based on signals with less precision that other traders thought they had), then move back towards the initial value (since trader n believes his own estimate will be proven correct in the long run). Since all traders expect prices to deviate apart from the long term mean in the short run, traders will not want to hold inventories of zero. They will want to sell if the consensus growth estimate is positive (top of figure) and buy if the consensus growth estimate is negative (bottom of figure). This leads to an equilibrium price depicted by the dard solid (red) lines. In equilibrium, when traders have positive expected growth rates, they expect returns to be slightly negative in the short run but positive in the long run, eventually reflecting the traders' common expected growth rates $G_m(t) = \bar{G}(t)$.

Our model bears a resemblance to the beauty contest described by Keynes (1936):[4]

> "For most of these persons are, in fact, largely concerned, not with making superior long-term forecasts of the probable yield on an investment over its whole life, but with foreseeing changes in the conventional basis of valuation a short time ahead of the general public. They are concerned not with what an investment is really worth to a man who buys it 'for keeps,' but with what the market will value it at, under the influence of mass psychology, three months or a year hence."

As in Keynes (1936), traders in our model trade based on short-term price dynamics rather than hold-to-maturity values. As Keynes puts it, "it is not sensible to pay 25 for an investment of which you believe the prospective yield to justify value of 30, if you also believe that the market will value it at 20 three months hence."

Keynes also believed that since financial markets are dominated by short-term speculation rather than long-term enterprise, they are not too different from a casino and exhibit excessive volatility. In contrast to Keynes, short-term trading dynamics dampens price volatility in our model relative to the volatility of fundamental value. This result is similar to Allen, Morris and Shin (2006), who also find that prices in a beauty contest exhibit inertia and react sluggishly to changes in the fundamental value. While Allen, Morris and Shin (2006) consider a model with differential information and their result is based on the idea that the average of martingales is not a martingale, the results in our model are based on disagreement about whether expected growth rate estimates imply martingale returns or not.

To summarize, each trader believes that equilibrium prices usually differ from fundamentals, prices do not follow a martingale, and price changes are predictable. Kyle, Obizhaeva and Wang (2013) explore implications for the equilibrium returns in more detail.

Trading Strategies Follow A Partial Adjustment Process.

With continuous trading, the equilibrium trading strategies have a simple form similar to the one-period model. Let $S_n^{TI}(t)$ denote the "target inventory" of trader

[4] "...Professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of ones judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees."

n, defined as the inventory level $S_n^{TI}(t)$ such that the informed trader chooses not to trade $x_n^*(t) = 0$. From equation (67), the target inventory $S_n^{TI}(t)$ is given by

$$S_n^{TI}(t) = C_L \cdot (H_n(t) - H_{-n}(t)) \,. \tag{73}$$

Trader n targets a long position if his own signal $H_n(t)$ is greater than the average signal of other traders $H_{-n}(t)$ and targets a short position if his own signal is less than the average signal of others. Trader n follows a partial adjustment strategy, with his inventory $S_n(t)$ converging towards its optimal level $S_n^{TI}(t)$ at rate γ_S:

$$x_n^*(t) = \frac{dS_n(t)}{dt} = \gamma_S \cdot \big(S_n^{TI}(t) - S_n(t)\big) \,. \tag{74}$$

While the target inventory levels $S_n^{TI}(t)$ and therefore the speed of trading $x_n^*(t)$ change like a diffusion of order $dt^{1/2}$ due to arrival of new information, the actual inventories change differentiably at rate of order dt.

When a trader observes a new signal, he updates his estimate of the growth rate, recalculates his target inventory, and immediately adjusts the rate of trading towards the new target. Since block trades are infinitely expensive, a trader does not trade immediately to the new target but rather adjusts his inventories slowly, taking into account his market power. As soon as a trader changes the speed of trading, the price of a risky asset instantaneously moves to a new equilibrium level, even though a trader has not traded a single share yet.

Equations (73) and (74) imply that inventories $S_n(t+s)$ have the integral representation

$$S_n^*(t+s) = e^{-\gamma_s \cdot T} \cdot \Big(S_n^*(t) + \int_{u=t}^{t+s} e^{-\gamma_s \cdot (t-u)} \cdot \gamma_S \cdot C_L \cdot (H_n(u) - H_{-n}(u)) \cdot du\Big). \tag{75}$$

The equation has a simple intuition. Traders accumulate inventories gradually based on their current level of disagreement $H_n(t) - H_{-n}(t)$; inventories accumulated based on past disagreement are gradually liquidated at rate γ_S. If signals $H_n(t)$ and $H_{-n}(t)$ do not change, the price does not change, but trader n will continue to trade based on the level of his "past" disagreement with the market $H_n(t) - H_{-n}(t)$.

Transaction Costs Depend on Quantities and Speed of Trading.

Equation (69) can also be used to to calculate the out-of-equilibrium price effect of a "new" trader $n = N+1$ who silently enters the market and acquires inventories $\bar{S}_{N+1}(t)$, unbeknownst to the other N traders. Since the new trader does not actually trade in equilibrium, we plug $S_{N+1}(t) = x_{N+1}(t) = 0$ into equation (69). By affecting prices, the new trader incurs permanent and temporary price impact costs, denoted $\tilde{C}$. We measure these costs using the concept of implementation shortfall, as described by Perold (1988). The expected price impact costs are given by

$$E\{\tilde{C}\} = E\left\{\int_{u=t}^{\infty} (P(u) - P^*(u)) \cdot \bar{x}(u) \cdot du\right\}. \tag{76}$$

The actual expected implementation shortfall depends on how the new trader trades. Here are two simple examples.

Example 1: Suppose the new trader $N+1$ enters the market at date $t=0$ and acquires a random block of shares $\tilde{B}$, uncorrelated with signals $H_n(t)$, $n = 1, \ldots, N$, by trading at the constant rate $\bar{x}(t) = \tilde{B}/T$ over the interval $[0, T]$. Using $\lambda_S = \gamma_S \cdot \lambda_x$ from equation (70), new trader $N+1$'s expected "implementation shortfall" is given by

$$E\{C\} = \Big(\lambda_S + \frac{\lambda_x}{T/2}\Big) \cdot \frac{\tilde{B}^2}{2} = \lambda_S \cdot \Big(1 + \frac{1}{\gamma_S} \cdot \frac{1}{T/2}\Big) \cdot \frac{\tilde{B}^2}{2}. \tag{77}$$

Example 2: Suppose instead that the new trader enters the market at date $t=0$ and acquires the random inventory $\tilde{B}$ by trading at rate $x_{N+1}(t) = \gamma \cdot (\tilde{B} - \bar{S}_{N+1}(t))$. Then his inventory evolves as $\bar{S}(t) = \tilde{B} \cdot (1 - e^{-\gamma \cdot t})$, with $\bar{S}(t) \to \tilde{B}$ as $t \to \infty$. The implementation shortfall is given by

$$E\{C\} = \Big(\lambda_S + \gamma \cdot \lambda_x\Big) \cdot \frac{\tilde{B}^2}{2} = \lambda_S \cdot \Big(1 + \frac{\gamma}{\gamma_S}\Big) \cdot \frac{\tilde{B}^2}{2}. \tag{78}$$

If the new trader chooses a speed of execution γ equal to the equilibrium speed of execution γ_S, then the permanent cost $\lambda_S \cdot \bar{S}^2/2$ is equal to the temporary cost $\lambda_S \cdot \bar{S}^2/2 \cdot \gamma/\gamma_S$; thus, both are equal to a half of the total cost $E\{C\} = \lambda_S \cdot \bar{S}^2$. The intuition for this result is the same as the intuition for traders "in the model." Since each trader is trying to walk up the demand schedule of the other traders by slowing down his trading, all traders cannot trade simultaneously as price discriminating monopolists. In a symmetric equilibrium, each trader expects to break even providing liquidity to other traders. This requires each trader to pay out his potential monopoly profit from walking the demand curve, $\lambda_S \cdot \bar{S}^2/2$ (exactly half the costs incurred due to permanent impact), to others in the form of temporary price impact. Note that each trader $n = 1, \ldots, N$ still expects to make a profit because traders disagree with one another about the fundamental value of the asset.

In both examples, faster execution leads to larger temporary price impact but has no effect on permanent price impact. Infinitely fast block trades with $T \to 0$ or $\gamma \to \infty$ become infinitely expensive due to temporary price impact costs. Infinitely slow trades with $T \to \infty$ or $\gamma \to 0$, which new trader executed slowly walking up the demand schedule as a price discriminating monopolist, incur only the permanent impact cost $\lambda_S \cdot \tilde{B}^2/2$ and no temporary impact costs. If the new trader unwinds his position at a later date, the trader recovers the permanent costs but he incurs the temporary impact costs again.

The functional form of the price impact function (69) is not entirely new. The same price impact model can be found in Grinold and Kahn (1995) and Almgren and Chriss (2000). These researchers specify the price impact functions exogenously, not based on an equilibrium model, but rather based on empirical practitioner wisdom that fast trading increases temporary price impact costs. Our contribution is to show that this price impact model, considered reasonable in practitioner-oriented research based on empirical applicability, actually arises as an equilibrium implication of a model in which traders optimize their trading strategies rationally taking into account their price impact.

The price impact model in equation (69) contrasts with other models in the finance literature. Obizhaeva and Wang (2013) suggest an alternative model in which—rather than decaying instantaneously—the temporary price impact decays gradually over time at an exponential rate.

The temporary impact implicitly exists in most models, in which large orders can be executed as sequences of many small trades. For example, the temporary impact implicitly exists in Kyle (1985), where smooth trading of informed traders does not incur any temporary costs but block trades of noise trader do induce temporary costs. Informed traders submit many small autocorrelated trades X of order dt at price increments of $\lambda \cdot X$ of order dt, and their bid-ask spread cost of order dt^2 is economically inconsequential. Noise traders submit large block trades X of order dB at price increments of $\lambda \cdot X$ of order dB, and continuously pay a bid-ask spread of order $dt^{1/2}$ to market makers. Block trades of noise traders effectively make prices deviate from equilibrium levels for short periods of time dt. In some sense, the model of Kyle (1985) is of a binary nature: Block trades are penalized with finite temporary bid-ask costs, but as long as trading is smoothed out it does not incur any temporary costs, even when the speed of smooth trading is very high. Our model does not inherit that binary nature: The temporary impact is proportional to the speed of trading for smooth trading and infinite for any block trade.

Fast Execution Can Lead To Flash Crashes.

Equation (69) can also be used to describe what happens if trader n "in the model," $1 \leq n \leq N$, silently deviates from his optimal trading strategy. Suppose that at date T, trader n, instead of trading towards his target inventory at equilibrium rate γ_S, deviates from his optimal strategy by trading towards his target inventory at some arbitrarily faster or slower rate $\gamma \neq \gamma_S$ and implements the strategy

$$(79) \qquad \bar{x}_n(T+t) = \gamma \cdot (S_n^{TI}(T+t) - \bar{S}_n(T+t)), \quad t \geq 0.$$

Equation (79) becomes the equilibrium strategy when $\gamma = \gamma_s$. The inventory level $\bar{S}_n(t)$ then coincides with the equilibrium inventory level $S_n(t)$ before date T, but deviates afterwards, so that for any $t \geq 0$,

$$(80) \quad \bar{S}_n(T+t) = e^{-\gamma \cdot t} \cdot \Big(S_n(T) + \int_{u=T}^{T+t} e^{-\gamma \cdot (T-u)} \cdot \gamma \cdot C_L \cdot (H_n(u) - H_{-n}(u)) \cdot du \Big).$$

To consider an analytically tractable example, suppose that just before date T, the values of $H_n(T^-) > 0$ and $H_0(T^-) = H_{-n}(T^-) = 0$ and trader n happens to hold his positive target inventory, i.e.,

$$S_n(T) = S_n^{TI}(T^-) = C_L \cdot H_n(T^-) > 0, \tag{81}$$

$$P(T^-) = \frac{\gamma_S}{(N-1)\gamma_P} \cdot S_n(T) > 0. \tag{82}$$

Suddenly, the signals change so that $H_n(T) = 0$ whereas $H_0(t)$ and $H_{-n}(t)$ remain to be zero. As a result, trader n begins to liquidate his inventory, moving from inventory $S_n(T)$ towards a new target inventory of zero. Trader n's expected inventories at dates $T + t$, $t > 0$, conditional on information at date T, are given by

$$E_T^n\{\bar{S}_n(T+t)\} = e^{-\gamma \cdot t} \cdot S_n(T). \tag{83}$$

Using $\mathrm{E}_T^n\{P(T+t)\} = 0$, equation (69) implies that expected prices are given by

$$\mathrm{E}_T^n\{\bar{P}(T+t)\} = -\frac{\gamma - \gamma_S}{(N-1)\gamma_P} \cdot e^{-\gamma t} \cdot S_n(T). \tag{84}$$

If trader n, like other traders, follows the equilibrium strategies with $\gamma = \gamma_S$, then the price immediately falls to zero and is expected to stay there.

Figure 4 shows expected paths of future prices based on equation (84) in panel A and future inventories based on equation (80) in panel B for different horizons, as trader n sells $S_n(T) = 1,000$ shares over time.

We consider the two cases. Trader n implements his execution at a rate five times faster than the equilibrium rate ($\gamma = \gamma_S \cdot 5$, dotted lines) and at a rate five times slower than the equilibrium rate ($\gamma = \gamma_S/5$, solid lines). Also, since dynamics depends on the total precision in the economy, we consider two models: (1) the model with low total precision $\tau = 9.95$ with $\tau_0 = 0.004$, $\tau_L = 0.05$ and $\tau_H = 5$, implying equilibrium price $P_1(T^-) =$ and equilibrium $\gamma_{S1} = 35.8$ (light color) and (2) the model of high total precision $\tau = 14.09$ with $\tau_0 = 0.0028$, $\tau_L = 0.0708$ and $\tau_H = 7.08$, implying equilibrium price $P_2(T^-) =$ and equilibrium $\gamma_{S2} = 50.6$ (dark color). The other parameters $r = 0.01, A = 1, \alpha_D = 0.1, \alpha_G = 0.02, \sigma_D = 0.5, \sigma_G = 0.1$, and $N = 100$; $D(T) = 0$.

When a trader sells at a rate five times slower than the equilibrium rate, $\gamma = \gamma_S/5$, the price is expected to drop immediately, but only about 1/5 as much as in the equilibrium. Slowing down execution reduces transitory price impact, but the price eventually converges to the equilibrium level. Saving on transactions costs by selling at higher-than-equilibrium prices at the beginning is more than offset by losing later on being unable to trade on the information quickly enough.

When a trader sells at a rate five times faster than the equilibrium rate, $\gamma = \gamma_S \cdot 5$, the price is expected to drop sharply, by about five times as much as in equilibrium. Speeding up execution exacerbates transitory price impact and elevates transactions costs. As the price converges to the equilibrium, the price path exhibits a distinct V-shaped pattern. The figure shows that the price gap disappears as time passes,

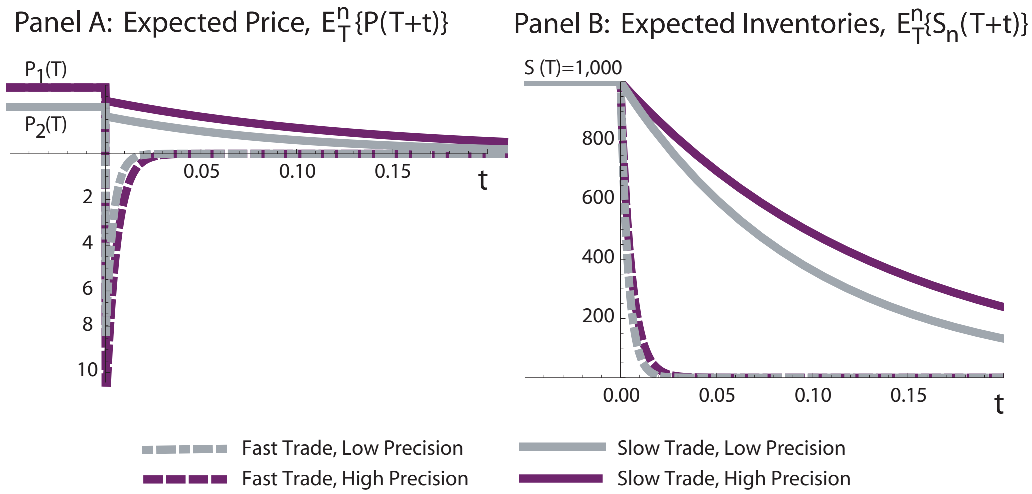

FIGURE 4. THE EXPECTED PRICE AND INVENTORIES DYNAMICS.

The figure shows $E^n_T\{P(T+t)\}$ and $E^n_T\{S_n(T+t)\}$ for horizons $T+t$, $t \geq 0$. The parameters are $r = 0.01, A = 1, \alpha_D = 0.1, \alpha_G = 0.02, \sigma_D = 0.5, \sigma_G = 0.1, N = 100, S_n(T) = S^{TI}_n(T) = 1,000$, and $S^{TI}_n(T+) = 0$. Paths during fast trading, $\gamma = \gamma_S \cdot 5$, are in dotted lines. Paths during slow trading, $\gamma = \gamma_S/5$, are in solid lines. Low total precision case (light color): $\tau = 9.95$ with $\tau_0 = 0.004$, $\tau_L = 0.05$ and $\tau_H = 5$. High total precision case (dark color): $\tau = 14.0882$ with $\tau_0 = 0.0028$, $\tau_L = 0.0708$ and $\tau_H = 7.08$.

and this adjustment is quicker when the total precision and therefore price resilience is higher.

This calibrated price response is very similar to the price pattern observed during the flash crash of May 6, 2010, when the E-mini S&P 500 futures price plunged by 5% over a five-minute period and then quickly recovered all of the earlier losses after the CME's pre-programmed circuit breakers triggered a five-second pause in futures trading. Staffs of the CFTC and SEC (2010a,b) reported that the flash crash was triggered by an automated execution algorithm that sold S&P 500 E-mini futures worth approximately $4 billion. Kyle and Obizhaeva (2013) note that market microstructure invariance would imply a price impact of only 0.61% and attributed the difference from actual price changes to unusually fast execution of the order. Indeed, the entire order was executed over a twenty-minute period, while orders of similar magnitude would normally expected to be executed over horizons of at least several hours. Our model implies that the selling should occur *after* prices crash, while the market recovers. This is reasonably consistent with the pattern observed during the Flash Crash, since most of the $4 billion in selling took place after the market had crashed and while prices were recovering.

The Value Function and Marking to Market.

The value function of each trader is such that the implied valuation of his risky position reflects its illiquidity as well as the potential disagreement between a trader and the rest of the market about its fundamental value.

In trader n's value function (60), the value of inventories can be expressed in monetary units by scaling by ψ_M. Letting $P_n(t)$ denote the dollar value of one unit of inventories in trader n's value function, we have

$$P_n(t) = \frac{\psi_{SD}}{\psi_M} \cdot D(t) + \frac{\psi_{Sn}}{\psi_M} \cdot \hat{H}_n(t) + \frac{\psi_{Sx}}{\psi_M} \cdot \hat{H}_{-n}(t) + \frac{\psi_{SS}}{2\psi_M} \cdot S_n(t). \tag{85}$$

The ψ_{SS} term adjusts the value of inventories for their riskiness. The ψ_{SD} term, with $\psi_{SD}/\psi_M = 1/(r+\alpha_D)$ from equation (62), measures the cash-flow value of dividends. Intuitively, the ψ_{Sn} and ψ_{Sx} terms average together in some manner—influenced by the level of market liquidity—both trader n's expectation of the value of cash flows and the market price $P(t)$.

If the market has almost no liquidity, the trader's value of inventories $P_n(t)$ will be close to a no-trade value reflecting only trader n's estimate of the expected present value of cash flows, discounted for risk . Using equations (37) and (72), the expected present value of cash flows is given by

$$F_n(t) = \frac{1}{r + \alpha_D} \cdot D(t) + \frac{\sigma_G \Omega^{1/2} \cdot \tau_H^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \cdot \hat{H}_n(t) + \frac{\sigma_G \Omega^{1/2} \cdot (N-1)\tau_L^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \cdot \hat{H}_{-n}(t). \tag{86}$$

If the market is almost perfectly competitive, implying that trader n can convert his inventories into cash at current market prices with no market impact costs, then the value of inventories $P_n(t)$ will correspond closely to the mark-to-market value $P(t)$, which is given using equation (57) by

$$P(t) = \frac{\psi_{SD}}{\psi_M} \cdot D(t) + \frac{\psi_{Sn}}{\psi_M} \cdot \frac{1}{2} \cdot \hat{H}_n(t) + \frac{\psi_{Sn}}{\psi_M} \cdot \frac{N-1}{2} \cdot \hat{H}_{-n}(t). \tag{87}$$

Using $\psi_{Sx} = (N-2)\psi_{Sn}/2$ from equation (61), it can be shown from equations (85) and (87) that

$$P_n(t) = P(t) + \frac{\psi_{Sn}}{\psi_M} \cdot \frac{1}{2} \cdot (H_n(t) - H_{-n}(t)) + \frac{\psi_{SS}}{2\psi_M} \cdot S_n(t). \tag{88}$$

The difference between trader n's private valuation $P_n(t)$ and the market price $P(t)$ has an interesting economic interpretation. Trader n's target inventory is $S_n^{TI}(t) = C_L \cdot (H(_n(t) - H_{-n}(t))$. Using definitions of parameters $\psi_{Sn}, \psi_{SS}, \gamma_S$, and λ_S from equations (61), (65), (70), it can be shown that

$$P_n(t) = P(t) - \frac{1}{2}\lambda_S \cdot S_n(t) + \lambda_S \cdot S_n^{TI}(t). \tag{89}$$

In privately valuing his risky inventories at $P_n(t)$, trader n makes two adjustments to market prices $P(t)$. First, he marks the market price downwards by $\frac{1}{2}\lambda_S \cdot S_n(t)$, the *average* permanent impact cost of that he would incur if he liquidated his current inventories $S_n(t)$ at the slowest rate possible. Second, he marks the price upward by $\lambda_S \cdot S_n^{TI}(t)$, the *marginal* permanent impact cost of acquiring his target inventory $S_n^{TI}(t)$ in the market. For example, if trader n values the asset at \$40 in his value function, but the market price is \$30, then trader n may think that he will make on average \$15 per each share he hold based on his favorable information but lose \$5 per share when liquidating that position.

From trader n's value function (60), the dollar value of future trading opportunities based on current information is given by

$$\frac{1}{2} \cdot \frac{\psi_{nn}}{\psi_M} \cdot \hat{H}_n^2 + \frac{1}{2} \cdot \frac{\psi_{xx}}{\psi_M} \cdot \hat{H}_{-n}^2 + \frac{\psi_{nx}}{\psi_M} \cdot \hat{H}_n \cdot \hat{H}_{-n} \tag{90}$$

Intuition suggests that the value of trading opportunities should always be non-negative: a trader will not trade on information unless the trader expects the trades to be profitable. Consistent with this intuition, our numerical results always find that (90) is a positive definite quadratic form. The symmetry of this quadratic form about zero is consistent with the symmetry of the zero-net-supply model about zero; long positions are just as profitable as short positions.

Market Liquidity.

Practitioners often observe that market liquidity is ephemeral. The mathematics of our model suggests related hypotheses concerning the stability or continuity of liquidity.

Equations (104) result from imposing symmetry on symmetric linear strategies. While intuition suggests that these equations might determine the four gamma-parameters $\gamma_D, \gamma_H, \gamma_S$ and γ_P as functions of the nine psi-parameters, these equations do not actually determine γ_P as a function of the psi-parameters; instead, they determine the three gamma-parameters γ_D, γ_H, and γ_S and generate the psi-parameter restriction $\psi_{Sx} = (N/2 - 1)\psi_{Sn}$. The sensitivity of trading to price γ_P is left undetermined.

The intuition for this result is that the objective function is a quadratic function of $x_n(t)/((N-1)\gamma_P)$. The first order condition determines only this ratio, i.e., the optimal strategy $x_n(t)$ for each level of market liquidity $(N-1)\gamma_P$, and the implied value function is linear in the level of market liquidity $(N-1)\gamma_P$.

Mathematically, the level of market liquidity $(N-1)\gamma_P$ must be such that the values of the psi-parameters in the value function remain constant over time at just the right ratio consistent with equilibrium. If the equilibrium were not in a steady state, the six equations (109)-(114) in the Appendix would not be polynomials but would instead be differential equations with the zeros on the right-hand-side changed to derivatives of the corresponding psi-parameters with respect to time. If market liquidity $(N-1)\gamma_P$ is different from its equilibrium level, the values of psi-parameters such as ψ_{Sn} and ψ_{Sx} will over time wander away from their equilibrium levels. Indeed, the modified equations (110) and (111) will imply that ψ_{Sn} and ψ_{Sx} will change by the same margin proportional to $\gamma_P(N-1)\psi_{Sn}\psi_{SS}/(rA)$ but in the opposite directions, and the equality $\psi_{Sx} = (N/2-1)\psi_{Sn}$ would eventually be violated. The economic intuition for this is the following: If the equality $\psi_{Sx} = (N/2-1)\psi_{Sn}$ becomes an inequality, then (depending on the direction of the inequality) *either* traders will want to supply liquidity more aggressively than they demand it *or* traders will want to demand liquidity more aggressively than they supply it—*for any level of market liquidity* $(N-1)\gamma_P$.

Thus, the level of market liquidity $(N-1)\gamma_P$ is pinned down at a value consistent with the steady-state dynamics in equations (109)-(114) and the equilibrium psi-parameter restriction. At the same time, traders choose an intensity of trading which is consistent with the equilibrium market depth.

A somewhat analogous delicacy in determination of equilibrium arises in the continuous model of Kyle (1985), where the optimization problem of the informed trader is linear in the intensity with which the informed trader trades. This linearity does not place a restriction on the trading strategy of the informed trader by itself, but instead requires market depth to be constant. If market depth were not constant, then either the informed trader would try to destabilize prices and generate unbounded profits or would want to incorporate information into prices aggressively. The informed trader's optimal trading strategy must be consistent with a constant level of market depth, but this consistency condition comes from the conditions determining market depth, not the conditions determining the optimal trading strategy. In equilibrium, the informed trader ultimately does not care how aggressively he incorporates his private information into prices. Nevertheless, to sustain the equilibrium, the informed trader must choose to trade with an intensity that incorporates information into prices at a constant rate, consistent with constant market depth.

When we think about how this delicate relationship between the level of market liquidity and desire to supply and demand liquidity plays out in actual markets, we are left with the intuition that the level of liquidity is probably somewhat indeterminate over very short period of time. The model says literally that private information is manufactured at a constant rate over time. As a practical matter, more private information might be manufactured on trading days than weekends, more during the day than overnight. The spirit of the model is that, at any time during the week, market liquidity is in some sense proportional to the rate at which private information is being manufactured. But what happens if this proportionality is violated? For example, what happens if private information is manufactured at a constant rate twenty-four hours per day, seven days per week, while market liquidity is much greater during business hours than in the evenings? In this case, the equality $\psi_{Sx} = (N/2-1)\psi_{Sn}$ will be violated, but perhaps in an economically insignificant manner. Traders will be just about as happy with constant market liquidity twenty-four hours per day and three times higher liquidity during eight business hours with closed markets the rest of the day. Whether this intuition is correct is an interesting issue for further research.

IV. Conclusion

We develop a model with overconfidence and market power where traders agree to disagree based on the flow of public and private information and trade with each other, taking into account their market power. This model provides a framework for thinking about intertemporal properties of market liquidity, transaction costs, and market prices.

The idea that securities markets offer a flow equilibrium rather than a stock equilibrium may seem far-fetched at first glance. Yet recent trends in the way liquidity is supplied and demanded in electronic markets are in many ways consistent with the way our model predicts liquidity to be supplied and demanded.

Relative to order processing by human clerks, electronic processing of orders has dramatically reduced the fixed costs of executing an order. The result has been a dramatic reduction in the average size of the trades "printed" in price reporting systems and a correspondingly large increase in the number of orders and messages routed to various trading venues. In previous decades, a trader may have purchased 100,000 shares of stock in a single block trade for 100,000 shares. Nowadays, a trader might execute 1,000 orders for 100 shares each, over a matter of several hours. Although the trader's inventory would not theoretically be a differentiable function of time, there is a small economic difference between purchasing continuously at the constant rate of 100 shares per minute and an "order shredding" strategy of purchasing numerous 100 share increments at random (poisson) time intervals spaced about one minute apart.

Our model predicts that there should be vanishingly small market depth available at a given point in time; instead, market depth is predicted to be made available only over time. In today's markets, the actual level of market depth available at the "top of the book"—i.e., at the best bid and the best offer—is influenced by tick size (the smallest units in which prices are allowed to fluctuate) and by rules for time and price priority. Since time priority mandates execution of the older resting limit orders before newer ones with the same limit price, time priority encourages traders to place bids and offers into the limit order book in order to have their orders executed ahead of others who want to trade at the same price at around the same time. Relative to our model, time priority creates an externality among traders which results in more depth in the limit order book than would otherwise be present. This microstructure externality may be more important when minimum tick size is large. Between 1997 and 2001, the minimum tick size in the U.S. equity markets was reduced by a factor of 12.5 from one-eighth of a dollar to one cent. Our model implies an infinitesimal tick size. Thus, today's markets probably have more instantaneous market depth available than our theory would imply, but they may have less instantaneous depth available than they would have if tick size were larger.

Our model of smooth order flow implements in a precise mathematical manner ideas about market liquidity describe informally by Black (1995). Black envisioned a future frictionless market for exchanges as "an equilibrium in which traders use indexed limit orders at different levels of urgency but do not use market orders or conventional limit orders." In that equilibrium, there will be no conventional liquidity available for market orders and conventional limit orders. Placement of indexed orders onto the market will move the price by an amount increasing in level of urgency. Our model implements Black's intuition in a precise mathematical manner.

Algorithms for executing orders to trade stocks and futures contracts have for years been incorporating the idea of urgency. For example, algorithms based on VWAP ("Volume Weighted Average Price") have become popular. In a VWAP trade, a trader chooses a target number of shares to trade, a time frame (say one day) and a participation rate (say 5% of volume). The higher the participation rate, the greater the trader's impatience.

In the future, exchanges might change order matching rules to implement limit orders conforming to the intuition of our model. For example, exchanges might approximate our flow model by having frequent batch auctions, say once per second, consistent with Budish, Cramton and Shim (2013). Limit orders could easily be implemented with a time parameter. For example, at trader who might in today's market place a limit order to buy 10,000 shares at a price of $40 per share might instead enter an order to purchase one share per second for at a price of $40 or better. The order would be filled over 10,000 seconds if the price moved below $40 per share and stayed there. It would take longer, or perhaps never be fully executed, if the price moved above $40 per share and stayed there. Clearly, such orders would reduce the high levels of message traffic which attempt to similar strategies using thousands of conventional limit orders.

REFERENCES

Allen, Franklin, Stephen Morris, and Hyun Song Shin. 2006. "Beauty Contests and Iterated Expectations in Asset Markets." *Review of Financial Studies*, 19(3): 719–752.

Almgren, Robert, and Neil Chriss. 2000. "Optimal Execution of Portfolio Transactions." *Journal of Risk*, 3: 5–39.

Almgren, Robert, Chee Thum, Emmanuel Hauptmann, and Hong Li. 2005. "Direct Estimation of Equity Market Impact." *Risk*, 18(7): 58–62.

Black, Fischer. 1986. "Noise." *The Journal of Finance*, 41: 529–543.

Black, Fischer. 1995. "Equilibrium Exchanges." *Financial Analysts Journal*, 51: 23–29.

Brunnermeier, Markus K., and Lasse H. Pedersen. 2005. "Predatory Trading." *Journal of Finance*, 60(4): 1825–1863.

Budish, Eric, Peter Cramton, and John Shim. 2013. "The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response." Working Paper, University of Chicago.

Carlin, Bruce, Miguel Lobo, and S. Viswanathan. 2007. "Episodic Liquidity Crises: Cooperative and Predatory Trading." *Journal of Finance*, 62(5): 2235–2274.

Chan, Louis, and Josef Lakonishok. 1995. "The Behavior of Stock Prices Around Institutional Trades." *Journal of Finance*, 50(4): 1147–1174.

Dufour, Alfonso, and Robert Engle. 2000. "Time and the Price Impact of a Trade." *Journal of Finance*, 55(6): 2469–2498.

Du, Songzi, and Haoxiang Zhu. 2013. "Ex Post Equilibria in Double Auctions of Divisible Assets." Working Paper, MIT Sloan School of Management.

Grinold, Richard C., and Ronald N. Kahn. 1995. *Active Portfolio Management: Quantitative Theory and Applications.* Probus Pub Co.

Grossman, Sanford J., and Joseph E. Stiglitz. 1980. "On the Impossibility of Informationally Efficient Markets." *American Economic Review*, 70(3): 393–408.

Holthausen, Robert W., Richard W. Leftwich, and David Mayers. 1990. "Large-Block Transactions, the Speed of Response, and Temporary and Permanent Stock-Price Effects." *Journal of Finance*, 26(1): 71–95.

Keim, Donald B., and Ananth Madhavan. 1997. "Transaction Costs and Investment Style: An Inter-Exchange Analysis of Institutional Equity Trades." *Journal of Financial Economics*, 46(3): 265–292.

Keynes, John M. 1936. *The General Theory of Employment, Interest and Money.* Palgrave Macmillan, United Kingdom.

Kyle, Albert S. 1985. "Continuous Auctions and Insider Trading." *Econometrica*, 53(6): 1315–1335.

Kyle, Albert S. 1989. "Informed Speculation with Imperfect Competition." *Review of Economic Studies*, 56: 317–355.

Kyle, Albert S., and Anna A. Obizhaeva. 2013. "Market Microstructure Invariance: Theory and Empirical Tests." Working Paper, University of Maryland.

Kyle, Albert S., and Tao Lin. 2001. "Continuous Speculation with Overconfident Competitors." Working Paper.

Kyle, Albert S., Anna A. Obizhaeva, and Yajun Wang. 2013. "A Market Microsructure Theory of the Term Structure of Asset Returns." Working Paper, University of Maryland.

Longstaff, Francis. 2001. "Optimal Portfolio Choice and the Valuation of Illiquid Securities." *Review of Financial Studies*, 14: 407–431.

Milgrom, Paul, and Nancy Stokey. 1982. "Information, Trade and Common Knowledge." *Journal of Economic Theory*, 26(1): 17–27.

Obizhaeva, Anna A., and Jiang Wang. 2013. "Optimal Trading Strategy and Supply/Demand Dynamics." *Journal of Financial Markets*, 16(1): 1–32.

Ostrovsky, Michael. 2012. "Information Aggregation in Dynamic Markets with Strategic Traders." *Econometrica*, 80(6): 2595–2647.

Perold, André. 1988. "The Implementation Shortfall: Paper vs. Reality." *Journal of Portfolio Management*, 14(3): 4–9.

Rostek, Marzena, and Marek Weretka. 2012. "Price Inference in Small Markets." *Econometrica*, 80(2): 687–711.

Scheinkman, José A., and Wei Xiong. 2003. "Overconfidence and Speculative Bubbles." *Journal of Political Economy*, 111: 1183–1219.

Staffs of the CFTC and SEC. 2010a. *Preliminary Findings Regarding the Market Events of May 6, 2010.* Report of the Staffs of the CFTC and SEC to the Joint Advisory Committee on Emerging Regulatory Issues. May 18, 2010. "Preliminary Report".

Staffs of the CFTC and SEC. 2010b. *Findings Regarding the Market Events of May 6, 2010.* Report of the Staffs of the CFTC and SEC to the Joint Advisory Committee on Emerging Regulatory Issues. September 30, 2010. "Final Report".

Vayanos, Dimitri. 1999. "Strategic Trading and Welfare in a Dynamic Market." *Review of Economic Studies*, 66(2): 219–254.

Appendix

Proof of Theorem 1

For the second order condition to be positive, we need to have $\frac{2}{(N-1)\gamma} + \frac{A}{\tau} > 0$, *i.e.*,

$$\text{(91)} \qquad \frac{A}{\tau}\frac{N\tau_H}{(N-2)\tau_H - 2(N-1)\tau_L} > 0.$$

Therefore, the second order condition holds if and only if $(N-2)\tau_H - 2(N-1)\tau_L > 0$. Substituting (14) into (11), we get trader n's optimal demand x_n^*. Substituting it into the market clearing condition $\sum_{m=1}^{N} X_m(i_0, i_m, p) = 0$, we get the equilibrium price P^*. Q.E.D.

Proof of Lemma 1

Applying Kalman-Bucy filter to the filtering problem summarized by equation (24) for signals and equations (25) and (26) for observations, we find that the filtering estimate is defined by Itô differential equation:

$$\text{(92)} \quad dG(t) = -\alpha_G \cdot G(t) \cdot dt + \sum_{n=0}^{N} \sigma_G^2 \cdot \bar{\Omega} \cdot \frac{\bar{\tau}_n^{1/2}}{\sigma_G \cdot \bar{\Omega}^{1/2}} \Big(dI_n(t) - G(t) \cdot \frac{\bar{\tau}_n^{1/2}}{\sigma_G \cdot \bar{\Omega}^{1/2}} \cdot dt \Big).$$

and the mean-square filtering error of the estimate $G(t)$, denoted as $\sigma_G^2 \cdot \Omega$, is defined by Riccati differential equation:

$$\text{(93)} \qquad \sigma_G^2 \dot{\bar{\Omega}} = -2\alpha_G \cdot \sigma_G^2 \cdot \bar{\Omega} + \sigma_G^2 - \sigma_G^4 \bar{\Omega}^2 \sum_{n=0}^{N} \Big(\frac{\bar{\tau}_n^{1/2}}{\sigma_G \bar{\Omega}^{1/2}} \Big)^2.$$

Rearranging the terms in the first equation yields equation (30). Using the steady-state assumption that $\dot{\bar{\Omega}} = 0$ and solving the second equation for $\bar{\Omega}$ yield (29). Q.E.D.

Proof of Theorem 2

One might expect that the solution of the maximization problem will yield solutions for the nine ψ-parameters as functions of the four γ-parameters. One might also expect that imposing symmetry by equating the four optimal γ-parameters, implied by trader n's optimal trading strategy, to the four conjectured γ-parameters will yield solutions for the four γ-parameters as functions of the nine ψ-parameters. In principle, one could then expect a solution to the thirteen equations in thirteen unknowns to describe a steady-state equilibrium, if one exists.

Although this is the intuition for the solution methodology, the solution does not work in this straightforward manner. The four equations for the γ-parameters do not determine γ_P as a function of the nine psi parameters. Instead, the solution to the four γ-equations implies a restriction on the ψ-parameters which must hold in a steady state equilibrium. Since this restriction has an interesting economic interpretation, we discuss it in some detail below.

Suppressing a subscript n for notational simplicity, the Hamilton-Jacobi-Bellman (HJB) equation corresponding to the conjectured value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ in equation (48) is

$$\begin{aligned}(94)\quad &\max_{c,x} \left\{U(c) - \rho V + \frac{\partial V}{\partial M_n}(rM_n + S_n D - c - P(x)\cdot x) + \frac{\partial V}{\partial S_n}x\right\} + \\ &+\frac{\partial V}{\partial D}(-\alpha_D D + \sigma_G \Omega^{1/2}\sqrt{\tau_H}\hat{H}_n + \sigma_G\Omega^{1/2}(N-1)\sqrt{\tau_L}\hat{H}_{-n}) + \\ &+\frac{\partial V}{\partial \hat{H}_n}\Big(-(\alpha_G+\tau)\hat{H}_n(t) + (\sqrt{\tau_H} + \hat{A}\sqrt{\tau_0})(\sqrt{\tau_H}\hat{H}_n + (N-1)\sqrt{\tau_L}\hat{H}_{-n})\Big) + \\ &+\frac{\partial V}{\partial \hat{H}_{-n}}\Big(-(\alpha_G+\tau)\hat{H}_{-n}(t) + (\sqrt{\tau_L} + \hat{A}\sqrt{\tau_0})(\sqrt{\tau_H}\hat{H}_n + (N-1)\sqrt{\tau_L}\hat{H}_{-n})\Big) + \frac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma_D^2 + \\ &+\frac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_n^2}(1+\hat{A}^2) + \frac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_{-n}^2}(\frac{1}{N-1}+\hat{A}^2) + \left(\frac{\partial^2 V}{\partial D\partial \hat{H}_n} + \frac{\partial^2 V}{\partial D\partial \hat{H}_{-n}}\right)\hat{A}\sigma_D + \frac{\partial^2 V}{\partial \hat{H}_n\partial \hat{H}_{-n}}\hat{A}^2 = 0.\end{aligned}$$

For the specific quadratic specification of the value function in equation (60), the Hamilton-Jacobi-Bellman (HJB) equation becomes

(95)

$$\begin{aligned}&\min_{c,x} \left\{-\frac{e^{-Ac}}{V} - \rho + \psi_M(rM_n + S_n\cdot D - c - P(x)\cdot x) + (\psi_{SS}S_n + \psi_{SD}D + \psi_{Sn}\hat{H}_n + \psi_{Sx}\hat{H}_{-n})x\right\} + \\ &+\psi_{SD}S_n(-\alpha_D D + \sigma_G\Omega^{1/2}\sqrt{\tau_H}\hat{H}_n + \sigma_G\Omega^{1/2}(N-1)\sqrt{\tau_L}\hat{H}_{-n}) + \\ &+(\psi_{Sn}S_n + \psi_{nn}\hat{H}_n + \psi_{nx}\hat{H}_{-n})\Big(-(\alpha_G+\tau)\hat{H}_n(t) + (\sqrt{\tau_H}+\hat{A}\sqrt{\tau_0})(\sqrt{\tau_H}\hat{H}_n + (N-1)\sqrt{\tau_L}\hat{H}_{-n})\Big) + \\ &+(\psi_{Sx}S_n + \psi_{xx}\hat{H}_{-n} + \psi_{nx}\hat{H}_n)\Big(-(\alpha_G+\tau)\hat{H}_{-n}(t) + (\sqrt{\tau_L}+\hat{A}\sqrt{\tau_0})(\sqrt{\tau_H}\hat{H}_n + (N-1)\sqrt{\tau_L}\hat{H}_{-n})\Big) + \\ &+\frac{1}{2}\psi_{SD}^2 S_n^2\sigma_D^2 + \frac{1}{2}\Big((\psi_{Sn}S_n + \psi_{nn}\hat{H}_n + \psi_{nx}\hat{H}_{-n})^2 + \psi_{nn}\Big)(1+\hat{A}^2) + \\ &+\frac{1}{2}\Big((\psi_{Sx}S_n + \psi_{xx}\hat{H}_{-n} + \psi_{nx}\hat{H}_n)^2 + \psi_{xx}\Big)(\frac{1}{N-1} + \hat{A}^2) + \\ &+\Big((\psi_{Sn}+\psi_{Sx})S_n + (\psi_{nn}+\psi_{nx})\hat{H}_n + (\psi_{xx}+\psi_{nx})\hat{H}_{-n}\Big)\psi_{SD}S_n\hat{A}\sigma_D \\ &+((\psi_{Sn}S_n + \psi_{nn}\hat{H}_n + \psi_{nx}\hat{H}_{-n})(\psi_{Sx}S_n + \psi_{xx}\hat{H}_{-n} + \psi_{nx}\hat{H}_n) + \psi_{nx})\hat{A}^2 = 0.\end{aligned}$$

The solutions for optimal consumption is

$$(96)\qquad c_n^*(t) = -\frac{1}{A}\cdot\log\Big(\frac{\psi_M\cdot V(t)}{A}\Big).$$

In the HJB equation (95), the price $P(x)$ is linear in x based on equation (57). Plugging $P(x)$ from equation (57) into the HJB equation (95) yields a quadratic function of x which captures the effect of trader n's trading rate x_n on prices. Because the conjectured value function is a quadratic of the state variables, the optimal trading strategy is a linear function of the state variables given by

$$(97)\quad x_n^*(t) = \frac{(N-1)\gamma_P}{2\psi_M} \cdot \Bigg[\left(\psi_{SD} - \frac{\psi_M \gamma_D}{\gamma_P}\right) \cdot D(t) + \left(\psi_{SS} - \frac{\psi_M \gamma_S}{(N-1)\gamma_P}\right) \cdot S_n(t) + \psi_{Sn} \cdot \hat{H}_n(t) + \left(\psi_{Sx} - \frac{\psi_M \gamma_H}{\gamma_P}\right) \cdot \hat{H}_{-n}(t)\Bigg].$$

The derivation of this optimal trading strategy assumes that trader n observes the values of $D(t)$, $S_n(t)$, $\hat{H}_n(t)$, and $\hat{H}_{-n}(t)$. Although trader n does not actually observe $\hat{H}_{-n}(t)$, he can implement the optimal quantity x_n^* by submitting an appropriate linear demand schedule. We can think of this demand schedule as a linear function of $P(t)$ whose intercept is a linear function of $D(t)$, $S_n(t)$, and $\hat{H}_n(t)$. Trader n can infer from the market-clearing condition (56) that $\hat{H}_{-n}$ is given by

$$(98)\quad \hat{H}_{-n}(t) = \frac{\gamma_P}{\gamma_H}\left(P(t) - D(t) \cdot \frac{\gamma_D}{\gamma_P}\right) - \frac{1}{(N-1)\gamma_H} \cdot x_n^*(t) - \frac{\gamma_S}{(N-1)\gamma_H} \cdot S_n(t).$$

Plugging equation (98) into equation (97) and solving for $x_n^*(t)$ implements the optimal trading strategy $x_n^*(t)$ as a linear demand schedule which depends on the price $P(t)$ and state variables $\hat{H}_n$, $S_n(t)$, $D(t)$, which the trader directly observes. This schedule is given by

$$(99)\quad x_n^*(t) = \frac{(N-1)\gamma_P}{\psi_M} \cdot \left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1} \cdot \Bigg[\left(\psi_{SD} - \psi_{Sx}\frac{\gamma_D}{\gamma_H}\right) \cdot D(t) + \left(\psi_{SS} - \psi_{Sx}\frac{\gamma_S}{(N-1)\gamma_H}\right) \cdot S_n(t) + \psi_{Sn} \cdot \hat{H}_n(t) + \left(\psi_{Sx}\frac{\gamma_P}{\gamma_H} - \psi_M\right) \cdot P(t)\Bigg].$$

Symmetry requires that this demand schedule be the same as the demand schedule conjectured for the $N-1$ other traders. Equating the coefficients of $D(t)$, $\hat{H}_n(t)$, $S_n(t)$, and $P(t)$ in equation (99) to the conjectured coefficients γ_D, γ_H, $-\gamma_S$, and $-\gamma_P$ places four restrictions that the values of the γ-parameters and ψ-parameters must satisfy in a symmetric equilibrium with linear trading strategies.

$$(100)\quad \frac{(N-1)\gamma_P}{\psi_M} \cdot \left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1} \cdot \left(\psi_{SD} - \psi_{Sx}\frac{\gamma_D}{\gamma_H}\right) = \gamma_D,$$

$$(101)\quad \frac{(N-1)\gamma_P}{\psi_M} \cdot \left(1 + \frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1} \cdot \psi_{Sn} = \gamma_H,$$

$$(102)\qquad \frac{(N-1)\gamma_P}{\psi_M}\cdot\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\cdot\left(\psi_{SS}-\psi_{Sx}\frac{\gamma_S}{(N-1)\gamma_H}\right)=-\gamma_S,$$

$$(103)\qquad \frac{(N-1)\gamma_P}{\psi_M}\cdot\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\cdot\left(\psi_{Sx}\frac{\gamma_P}{\gamma_H}-\psi_M\right)=-\gamma_P,$$

Solving this system, we obtain the four equations in terms of the four unknowns, ψ_{Sx}, γ_H, γ_S, and γ_D. The solution is

$$(104)\quad \psi_{Sx}=\frac{N-2}{2}\psi_{Sn},\quad \gamma_H=\frac{N\gamma_P}{2\psi_M}\psi_{Sn},\quad \gamma_S=-\frac{(N-1)\gamma_P}{\psi_M}\psi_{SS},\quad \gamma_D=\frac{\gamma_P}{\psi_M}\psi_{SD}.$$

Plugging the last equation into equation (97) implies that traders will not trade on public information. It is intuitively obvious that traders cannot trade on the basis of the public information $D(t)$ because all traders would want to trade in the same direction. Substituting equation (104) into equation (97) yields the solution for optimal strategy.

$$(105)\qquad x_n^*(t)=\gamma_S\cdot\Big(C_L\cdot(H_n(t)-H_{-n}(t))-S_n(t)\Big).$$

Plugging (96) and (97) back into the Bellman equation and setting the constant term, coefficients of M_n, S_nD, S_n^2, $S_n\hat{H}_n$, $S_n\hat{H}_{-n}$, $\hat{H}_n^2$, $\hat{H}_{-n}^2$, and $\hat{H}_n\hat{H}_{-n}$ to be zero, we obtain nine equations. There are in total nine equations in nine unknowns γ_P, $\psi_0, \psi_M, \psi_{SD}, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and ψ_{nx}.

By setting the constant term, coefficient of M, and coefficient of SD to be zero, we get

$$(106)\qquad \psi_M=-rA,$$

$$(107)\qquad \psi_{SD}=-\frac{rA}{r+\alpha_D},$$

$$(108)\quad \psi_0=1-\log\{r\}+\frac{1}{r}\left(-\rho+\frac{1}{2}(1+\hat{A}^2)\psi_{nn}+\frac{1}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{xx}+\hat{A}^2\psi_{nx}\right).$$

In addition, by setting the coefficients of $S_n^2, S_n\hat{H}_n, S_n\hat{H}_{-n}, \hat{H}_n^2, \hat{H}_{-n}^2$ and $\hat{H}_n\hat{H}_{-n}$ to be zero, we get the following six polynomial equations about six unknowns $\gamma_P, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and ψ_{nx}:

(109)

$$\underline{S_n^2}:\ -\frac{1}{2}r\psi_{SS}-\frac{\gamma_P(N-1)}{rA}\psi_{SS}^2+\frac{r^2A^2\sigma_D^2}{2(r+\alpha_D)^2}+\frac{1}{2}(1+\hat{A}^2)\psi_{Sn}^2+\frac{1}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\frac{(N-2)^2}{4}\psi_{Sn}^2$$

$$-\frac{rA}{r+\alpha_D}\hat{A}\sigma_D\frac{N}{2}\psi_{Sn}+\hat{A}^2\frac{N-2}{2}\psi_{Sn}^2=0,$$

(110)

$$\underline{S_n\hat{H}_n}: \quad -r\psi_{Sn}-\frac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn}-\frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}\sqrt{\tau_H}+a_1\psi_{Sn}+\frac{N-2}{2}a_4\psi_{Sn}+(1+\hat{A}^2)\psi_{nn}\psi_{Sn}+$$

$$+\frac{N-2}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{nx}\psi_{Sn}-\frac{rA}{r+\alpha_D}\hat{A}\sigma_D\left(\psi_{nn}+\psi_{nx}\right)+\hat{A}^2\psi_{nx}\psi_{Sn}+\frac{N-2}{2}\hat{A}^2\psi_{nn}\psi_{Sn}=0,$$

(111)

$$\underline{S_n\hat{H}_{-n}}: \quad -r\frac{N-2}{2}\psi_{Sn}+\frac{\gamma_P(N-1)}{rA}\psi_{SS}\psi_{Sn}-\frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}(N-1)\sqrt{\tau_L}+\left(a_3+\frac{N-2}{2}a_2\right)\psi_{Sn}+(1+\hat{A}^2)\psi_{Sn}\psi_{nx}+$$

$$+\frac{N-2}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{xx}\psi_{Sn}-\frac{rA}{r+\alpha_D}\hat{A}\sigma_D\left(\psi_{xx}+\psi_{nx}\right)+\hat{A}^2\psi_{xx}\psi_{Sn}+\frac{N-2}{2}\hat{A}^2\psi_{nx}\psi_{Sn}=0,$$

(112)

$$\underline{\hat{H}_n^2}: \quad -\frac{r}{2}\psi_{nn}-\frac{\gamma_P(N-1)}{4rA}\psi_{Sn}^2+a_1\psi_{nn}+a_4\psi_{nx}+\frac{1}{2}(1+\hat{A}^2)\psi_{nn}^2+\frac{1}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{nx}^2+\hat{A}^2\psi_{nn}\psi_{nx}=0,$$

(113)

$$\underline{\hat{H}_{-n}^2}: \quad -\frac{r}{2}\psi_{xx}-\frac{\gamma_P(N-1)}{4rA}\psi_{Sn}^2+a_2\psi_{xx}+a_3\psi_{nx}+\frac{1+\hat{A}^2}{2}\psi_{nx}^2+\frac{1}{2}\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{xx}^2+\hat{A}^2\psi_{xx}\psi_{nx}=0,$$

(114)

$$\underline{\hat{H}_n\hat{H}_{-n}}: \quad -r\psi_{nx}+\frac{\gamma_P(N-1)}{2rA}\psi_{Sn}^2+a_3\psi_{nn}+a_4\psi_{xx}+(a_1+a_2)\psi_{nx}+(1+\hat{A}^2)\psi_{nn}\psi_{nx}+$$

$$+\left(\frac{1}{N-1}+\hat{A}^2\right)\psi_{xx}\psi_{nx}+\hat{A}^2\left(\psi_{nn}\psi_{xx}+\psi_{nx}^2\right)=0,$$

where $a_1=-\alpha_G-\tau+\sqrt{\tau_H}(\sqrt{\tau_H}+\hat{A}\sqrt{\tau_0})$, $a_2=-\alpha_G-\tau+(N-1)\sqrt{\tau_L}(\sqrt{\tau_L}+\hat{A}\sqrt{\tau_0})$, $a_3=(\sqrt{\tau_H}+\hat{A}\sqrt{\tau_0})(N-1)\sqrt{\tau_L}$, $a_4=(\sqrt{\tau_L}+\hat{A}\sqrt{\tau_0})\sqrt{\tau_H}$.

If this system of equations has a solution, then the solution defines a "flow equilibrium" with symmetric linear trading strategies. We find that solution numerically. Note there is always a trivial no-trade equilibrium, as in one-period model. If each trader submits a demand schedule $X_n(t,.)\equiv 0$, then such a no-trade demand schedule is optimal for all traders. This is not a symmetric linear equilibrium in which an auctioneer can establish a meaningful market price.

The transversality condition is equivalent to $r>0$: From the HJB equation and equations (109)-(114), we have

(115)

$$E_t^n\{dV(M_n(t),S_n(t),D(t),\hat{H}_n(t),\hat{H}_{-n}(t))\}=-(r-\rho)V(M_n(t),S_n(t),D(t),\hat{H}_n(t),\hat{H}_{-n}(t))dt.$$

This yields

(116)

$$E_t^n\{e^{-\rho(T-t)}V(M_n(T),S_n(T),D(T),\hat{H}_n(T),\hat{H}_{-n}(T))\}=e^{-r(T-t)}V(M_n(t),S_n(t),D(t),\hat{H}_n(t),\hat{H}_{-n}(t)),$$

which implies that the transversality condition (54) is satisfied if $r > 0$. The second order condition is equivalent to $\gamma_P > 0$: For the minimum in the optimization problem (95) to exist, the second order condition requires the 2×2 matrix

$$\begin{pmatrix} -\frac{A^2}{V} & 0 \\ 0 & \frac{2 \cdot r \cdot A}{(N-1)\gamma_P} \end{pmatrix} \tag{117}$$

be positive definite. Since value function V is negative, this condition holds when demand schedules are downward sloping ($\gamma_P > 0$). Q.E.D.