Dynamic Debt Runs with Strategic Complementarity

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Abstract

I prove that the dynamic debt run model, initially developed by He and Xiong (2012, henceforth, HX), has a unique equilibrium *without* restricting attention to a specific type of equilibria. To derive this result, I need to impose one additional condition to HX (2012), which I call *costly liquidation.* Under this assumption, the economy in HX (2012) exhibits strategic complementarity: each creditor chooses a more conservative rollover strategy if others do so. If the economy satisfies this property, I can use iterative elimination of dominated regions to obtain a unique equilibrium. In equilibrium, it is shown that the creditors indeed use symmetric threshold rollover strategies, meaning that they run if and only if a firm's time-varying fundamental falls below a certain threshold *and* they use exactly the same threshold. Moreover, the alternative proof of this paper allows me to introduce ex-ante heterogeneous creditors to HX (2012) with no effort. Specifically, I consider three motivating examples: the economies consisting of 1) long-term creditors and short-term creditors, 2) junior creditors and senior creditors, and 3) creditors with heterogeneous beliefs. For each of these examples, I investigate which group has a larger incentive to run and so generates the first wave of a crisis. I find that ceteris paribus long-term/junior/pessimistic creditors run earlier than short-term/senior/optimistic creditors, respectively. (JEL G01, C72, G20)

1 Introduction

I revisit and extend the dynamic debt run model initially developed by He and Xiong (2012). It basically studies a dynamic coordination problem between creditors of a single firm. The creditors in this economy face strategic uncertainty about other creditors' rollover decisions because a firm's fundamental is time-varying and their maturities are staggered. In spite of HX's (2012) apparent contributions to debt run literature, researchers have struggled to extend it partly because it seems challenging to characterize equilibria for such an economy using guess-and-verification argument, as done in HX (2012). I find a certain condition that induces this economy to satisfy *strategic complementarity*: each creditor chooses a more conservative strategy if other creditors do so. As long as this property holds, I can use iterative elimination of dominated regions to prove there exists a unique equilibrium. Here, I *do not* restrict attention to a special type of rollover strategies, which differentiates

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this paper from existing dynamic debt run literature; e.g., Cheng and Milbradt (2012), Schroth, E., G. A. Suarez, and L. A. Taylor (2014), and HX (2012). The only assumption I make on the creditors' strategy space is that they use Markov strategies: they make rollover decisions only based on the current fundamental, not on the entire history of it. Even with this relaxed strategy space, I can show that in equilibrium, the creditors use symmetric threshold strategies, meaning that they run if and only if the firm's time-varying fundamental falls below a certain threshold and they use exactly the same threshold.

The key assumption that makes our economy exhibit strategic complementarity is what I call *costly liquidation*. In fact, when the firm runs out of its resources to defend running creditors, it has to liquidate its asset in a secondary market and distribute it to the creditors. But, what would happen if liquidation cost is cheap? To be precise, suppose that the liquidation value of debt can be larger than its continuation value at some economic outcomes, that is, the creditors are expected to receive more when the firm liquidates the asset today than when it survives and continues to pay coupons to the creditors. The individual creditors will then be (ironically) *better off* if other creditors choose more conservative rollover strategies to shut down the firm as soon as possible. As a result, strategic complementarity may be violated in this economy. In this paper, I will rule out such a scenario by assuming liquidation is costly because 1) it allows me to develop more economically intuitive method to prove the uniqueness of equilibrium, 2) it allows me to introduce ex-ante heterogeneous creditors to the original model with no effort, and 3) it is realistic enough for a crisis period.[1]

The basic proof argument for the unique equilibrium is almost in the same line as those in Frankel, Morris, and Pauzner (2003, henceforth, FMP) and Frankel and Pauzner (2000, henceforth, FP); the former studies static global games with private noises and the latter studies dynamic global games where agents make asynchronous actions as in our economy. Specifically, in addition to strategic complementarity, I will show that our economy satisfies (under some additional mild conditions) the following properties: 1) dominated regions: for sufficiently high (low) fundamental values, it is optimal to renew (run) regardless of other creditors' strategies, 2) state monotonicity: the present value of debt of any individual creditor is strictly increasing in the fundamental when other creditors use *threshold* strategies, and 3) weak strategic complementarity: again, in the case where other creditors use threshold strategies, any individual creditor's value function increases more when the fundamental improves by some amount than when other creditors decrease their rollover thresholds by the same amount. Notice that state monotonicity and weak strategic complementarity are proved to hold only for the cases where other creditors use threshold strategies. It may sound weird at first because I have emphasized I would not restrict my attention to those specific strategies. However, these two properties in that restricted form are sufficient to prove there is a unique equilibrium; more will be discussed later in section 3.2. In addition, slightly different from FMP (2003) and FP (2000), I use weak strategic complementarity more systematically to prove the existence of a unique equilibrium. The importance of weak strategic complementarity was emphasized

[1] Since the costly liquidation condition is not imposed in HX (2012), their original model may not satisfy strategic complementarity. Nonetheless, they obtained a unique equilibrium by imposing some other condition to make the creditor's value function satisfy a certain single crossing property; see condition (7) and Lemma 3 in HX (2012). However, without strategic complementarity, it looks hard to apply iterative elimination of dominated regions to find an equilibrium and that is why HX (2012) could not avoid focusing on a specific type of equilibrium.

by Vives (2005, 2014): it literally says that for an equilibrium to be uniquely determined, strategic complementarity must be *lessened*. It turns out that when our economy satisfies weak strategic complementarity, the slope of a creditor's best response to other creditors' strategies becomes moderate, which can be used to verify there exists only one strategy profile that survives iterative elimination of dominated regions. In fact, FMP (2003) and FP (2000) developed a more general approach to deal with a continuum of actions and path-dependent action-switching thresholds, respectively, which is not necessary for our setting. But, the idea that strategic complementarity must be weak is also there. Although the proof methodology of this paper is similar to those in FMP (2003) and FM (2000), this paper contributes to global games literature by showing that their proof concept developed over a highly stylized economy, where they *directly* impose strategic complementarity on agents' (flow) utility, can be fruitfully applied to more general dynamic economies.

To justify the usefulness and flexibility of the alternative proof of this paper, I introduce ex-ante heterogeneous groups of creditors to the original model. It is shown that in equilibrium, the creditors use partially symmetric threshold strategies in the sense that they use the same rollover threshold *within* groups. I will then consider three motivating example economies: the economies consisting of 1) long-term creditors and short-term creditors, 2) junior creditors and senior creditors, and 3) creditors with heterogeneous beliefs. I find that ceteris paribus long-term/junior/pessimistic creditors run more hastily compared to short-term/senior/optimistic creditors, respectively. This is because from an individual creditor's perspective, a short-term debt is more valuable than a long-term debt because 1) the former allows her to readjust rollover decisions more frequently and 2) no matter which debt she purchases, she will face the same degree of strategic uncertainty from other creditors because she is atomic in the economy. One can understand the second and the third examples in a similar manner. This result is consistent with the following empirical facts: 1) according to Anderson and Gascon (2009), maturities of commercial papers drastically shortened during fall of 2008, which could be interpreted as firms' restructuring efforts to avoid imminent runs from long-term creditors and 2) according to Egan, Hortaçsu, and Matvos (2014), as Citibank faced more distress than JPMorgan during 2008, Citibank's market share of uninsured deposits decreased, whereas that of JPMorgan increased, suggesting uninsured depositors are more run-prone than insured depositors.

However, this result does not imply that a naive policy of issuing more debt that induces lower incentives to run will deter firm's default. Specifically, on one hand, regarding the third example above, if the fraction of optimistic creditors grows, both optimistic and pessimistic creditors in the economy will choose lower rollover thresholds and so the firm's life span is extended. This type of positive outcome is, however, no longer true for the other two examples. To see why, if the firm issues more short-term debts, then the creditors will face larger rollover risks from each other, even though more creditors now can enjoy more frequent decision readjustment rights. If the former effect dominates the latter, an excessive issue of short-term debts will lead to earlier default of the firm. Likewise, with respect to seniority, if the firm overissues senior debts, the payoffs of creditors will be diluted because the firm has to insure more creditors now. Thus, the creditors might increase their rollover thresholds, which will again worsen the crisis. Indeed, the numerical results shown in Figure 7 show that under a reasonable choice of parameters, all the creditors raise their rollover thresholds as the share of short-term debt or senior debt grows.

Angeletos, et. al. studies dynamic global games where agents learn about the qualtity of the fundamental using past and newly arrived information. But all agents make synchronous actions.

The paper is organized as follows: section 2 revisits the original model of HX (2012) and introduces ex-ante heterogeneous creditors to it. Section 3 characterizes an equilibrium. Section 4 considers the three example economies and analyzes which group of creditors has a larger incentive to run. Section 5 discusses certain parameter resrictions that induce our economy to exhibit the four main properties mentioned above. Section 6 concludes. All technical proofs are included in Appendix.

2 The Model

Section 2.1 revisits the original dynamic debt run model of HX (2012) but without restricting attention to symmetric threshold equilibrium. Section 2.2 extends the original model by introducing ex-ante heterogeneous groups of creditors.

2.1 The Model with Ex-Ante Identical Creditors

2.1.1 Environment

Consider a firm that has an investment opportunity in an illiquid asset that requires a cost of \$1. This asset produces a constant cash flow of rdt over each time interval $[t, t+dt]$. It matures at a random time according to a Poisson process with intensity ϕ. When it expires at date τ_ϕ, it produces a final payoff of y_{τ_ϕ}. The time-t value of the final payoff (observable to every agent in this economy), y_t, follows a geometric Brownian motion with a constant drift μ and volatility σ:

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t, \tag{1}$$

where Z_t is a standard Brownian motion. So, assuming all agents in this economy are risk neutral, the first-best value of the firm's asset equals

$$\begin{aligned} F(y_t) &= E_t\Big[\int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho(\tau_\phi - t)} y_{\tau_\phi}\Big] \\ &= \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t, \end{aligned}$$

where ρ is the common discount rate of the agents. Because of the simple one-to-one relationship between y_t and $F(y_t)$, I can refer both of them as the fundamental of the firm. To exclude an explosion of the first-best value of the asset, I assume

$$\rho + \phi > \mu.$$

To finance this investment, the firm issues debt to a continuum of creditors of measure 1, where each creditor can purchase only one contract. Specifically, the firm borrows \$1 from each creditor, and pays her interest rdt over each period $[t, t+dt]$ and a face value of

\$1 at maturity. As in Leland (1998), each debt is assumed to mature at time τ_λ randomly according to a Poisson process with intensity λ. This assumption implies that the average maturity of debt is equal to $\frac{1}{\lambda}$. Put another way, in aggregate, a fraction λ of outstanding debts, randomly chosen, reach their maturity over each time period $[t, t+dt]$. When a creditor meets maturity, she can choose whether to run or renew.[2] If she runs, she receives the promised face value; if she renews, she receives nothing today but continues to receive the interest payments until the next maturity date.

The firm can default before its asset expires if maturing creditors keep running on their debts over time. Specifically, if a certain creditor runs, the firm draws on pre-committed credit line to defend her, i.e., a certain credit-line provider pays \$1 net the present (continuation) value of debt to keep the running creditor with the firm.[3] But the credit line is imperfect and thus cannot defend running creditors all the time. For simplicity, as in HX (2012), I assume that the credit line fails in a memoryless fashion with intensity proportional to the aggregate measure of running creditors and θ, where $\frac{1}{\theta}$ indicates the reliability of the credit line. Let τ_θ denote the random time when the firm indeed defaults. At this event, the firm liquidates the asset in a secondary market at the price

$$\alpha F(y_{\tau_\theta}) =: L + l y_{\tau_\theta},$$

where $0 \leq \alpha \leq 1$, and distributes $\min\{L + l y_{\tau_\theta}, 1\}$ equally to all the creditors.

It remains to figure out how the (aggregate) measure of running creditors is determined in this model. Unlike HX (2012), I do not assume that the creditors use symmetric threshold strategies. So, in the next section, I will describe a strategy space associated to the creditors.

2.1.2 Strategy Space and Individual's Problem

The only assumption I make on the strategy space of the creditors is that they use Markov strategies: their strategies depend only on the current fundamental, not on the entire history of it. Specifically, consider a certain individual creditor $j \in [0,1]$. At time t, she decides whether to run or not only based on y_t, not $\{y_s : s \leq t\}$. I define such a (pure) rollover strategy as a $\{0,1\}$-valued measurable function $q^j(y)$ over $\mathbb{R}_{\geq 0}$: $q_i^j(y_t) = 1$ indicates she runs and $q_i^j(y_t) = 0$ indicates she renews at time t. Then, the aggregate fraction of running creditors at time t is given by

$$m(y_t) = \int_0^1 \lambda q^j(y_t) dj. \tag{2}$$

As all the creditors use Markov strategies, the aggregate run intensity m also satisfies the Markov property. Moreover, since q^j can be any $\{0,1\}$-valued function, $m(y)$ can be (potentially) any real-valued positive function which does not exceed λ. Some technical issue regarding this statement will be discussed later.

[2] I use the words run and withdraw or roll over and renew interchangeably.

[3] One can also assume that when a creditor runs, the credit-line provider pays her \$1 and takes over the debt from her. But this interpretation may sound weired when we consider creditors with heterogeneous beliefs.

Now, recall that the Poisson intensity of the credit-line failure is proportional to $m(y_t)$ and θ. Since each creditor is atomic in this economy, her decision cannot influence $m(\cdot)$ at all. Thus, she takes $m(\cdot)$ as given. More important, the information about m is sufficient for any creditor, say j, to make a rollover decision; she does not need to know the entire information about $q^k(\cdot)$ for all other $k \neq j$. Therefore, in this coordination game with a continuum of creditors, without loss of generality, I can call m an *aggregate action* of the opponent creditors, from creditor j's perspective. Let $V(y;m)$ denote the expected payoff of individual creditor j; since all the individual creditors face the same m, the value function must be all the same across them.[4]

According to the environment of our economy and the Markov strategy assumption, the value function of a certain individual creditor must be equal to

$$V(y_t;m) = \max_{q^j(\cdot)\in\{0,1\}} E_t[\int_t^\tau e^{-\rho(s-t)} r ds + e^{-\rho(\tau-t)}(f_\phi(y_\tau)1_{\tau=\tau_\phi} + f_\theta(y_\tau)1_{\tau=\tau_\theta} + \\ \{q^j(y_\tau) + (1-q^j(y_\tau))V(y_\tau;m)\}1_{\tau=\tau_\lambda})], \quad (3)$$

where

$$\tau = \min\{\tau_\phi, \tau_\theta, \tau_\lambda\}, \quad f_\phi(y) = \min\{y,1\} \quad \text{and} \quad f_\theta(y) = \min\{L+ly,1\}.$$

Notice that once she assumed that all other creditors use Markov strategies, due to the dynamic programming principle, it is also optimal for her to use the Markov strategy. That optimal strategy can be obtained by solving the following HJB equation:[5]

$$0 = \max_{q^j(y)\in\{0,1\}} \{r + \phi f_\phi(y) + \theta m(y)(f_\theta(y) - V(y)) + \lambda q^j(y)(1-V(y)) - (\rho+\phi)V(y) + \\ \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy}\}. \quad (4)$$

Note that I used the fact that each of the termination events, i.e., τ_ϕ, τ_θ, and τ_λ, occurs with a probability of dt order and independently across each other, so they can occur simultaneously only with a probability of dt^2 or dt^3 order, which is negligible.

Unfortunately, one uncomfortable technical issue arises. Although I have mentioned that $m(\cdot)$ can be any measurable function between 0 and λ, equation (4) may not admit a unique solution for all those m. To overcome this issue, I *simply* assume that creditors exclude those irregular m when they take arbitrary (hypothetical) aggregate actions. Under this simplification, I redefine the set of aggregate actions as

$$\mathcal{M} = \{m \mid m : \mathbb{R}_{\geq 0} \to [0,\lambda] \text{ which makes HJB equation (4) well posed.}\},$$

and each creditor is assumed to play a coordination game only against aggregate actions from this $\mathcal{M}$. It is beyond the scope of this paper to precisely characterize the collection $\mathcal{M}$.

[4] The creditors can potentially choose *asymmetric* strategies because when $V(y;m) = 1$, they are indifferent between running and rolling over. Especially, if $\{y : V(y;m) = 1\}$ has a strictly positive measure, asymmetric strategies between them is not a negligible issue. However, such a case will not occur as long as the economy satisfies state monotonicity.

[5] Of course, this equation is equivalent to

$$0 = r + \phi f_\phi(y) + \theta m(y)(f_\theta(y) - V(y)) + \lambda \max\{1-V(y),0\} - (\rho+\phi)V(y) + \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy}.$$

2.1.3 Equilibrium

An equilibrium for this economy can be defined as follows. It consists of rollover strategies $\{q^j(\cdot)\}_j$ of the individual creditors such that

$$m(y) = \int_0^1 \lambda q^j(y)dj \in \mathcal{M}$$

and for each j, $q^j(\cdot)$ solves equation (4), given the above m.

2.2 The Model with Ex-Ante Heterogeneous Creditors

This section develops a dynamic debt run model with finitely many groups of ex-ante heterogeneous creditors.

The basic environment is the same as in the benchmark model. But it will be more convenient to use a log scale in place of a normal scale to unfold my proof argument for the uniqueness of equilibrium. So from now on, I will directly start with the log scale by using the following change of variable: $u_t = \log y_t$. The evolution of the fundamental u_t then follows

$$du_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dZ_t.$$

In fact, this change of the scale is legitimate because it will turn out that a unique strategy profile surviving iterative elimination of dominated regions under one coordinate system will still survive it under another coordinate system as long as the transformation between the two scales is monotone. By relabeling, I can further change the evolution process to

$$du_t = \mu dt + \sigma dZ_t.$$

Put another way, the fundamental in our economy can be assumed to be either an arithmetic or a geometric Brownian motion with a constant drift and volatility.

There are N groups of creditors and the creditors in group i occupies a measure ξ_i of the total population, where $\sum \xi_i = 1$. Any single debt issued to each creditor in group i matures according to a Poisson process with intensity λ_i. The creditors still use Markov strategies. So, the rollover strategy of a certain individual creditor j in group i is determined by some $\{0,1\}$-valued function over $\mathbb{R}$, $q_i^j(u)$: she runs if $q_i^j(u_t) = 1$ and rolls over if $q_i^j(u_t) = 0$. The aggregate size of running creditors at time t is then given by

$$m(u_t) = \sum_{i=1}^{N} \int_0^{\xi_i} \lambda_i q_i^j(u_t)dj. \tag{5}$$

Again, I will call this m an aggregate action of the opponent creditors, which can be any positive function less than

$$\Lambda := \sum_{i=1}^{N} \lambda_i \xi_i.$$

Let $V_i(u;m)$ be the expected payoff of each creditor in group i, given m. Likewise to (4), it satisfies

$$0 = \max_{q_i^j(u)\in\{0,1\}} \{r_i(u) + \phi_i f_{\phi,i}(u) + \theta_i m(u)(f_{\theta,i}(u) - V_i(u)) + \lambda_i q_i^j(u)(1 - V_i(u)) - (\rho_i + \phi_i)V_i(u) + \mu_i V_{iu} + \frac{\sigma_i^2}{2} V_{iuu}\}, \quad (6)$$

where

$r_i(u):$ interest rate offered to each creditor in group i
$f_{\phi,i}(u):$ final payoff to to each creditor in group i, when the firm's asset expires
$f_{\theta,i}(u):$ final payoff to each creditor in group i, when the firm liquidates its asset
$\rho_i:$ subjective discount rate of each creditor in group i
$\phi_i, \theta_i, \mu_i, \sigma_i:$ subject belief of each creditor in group i about ϕ, θ, μ, and σ, respectively.

Here, $f_{\phi,i}$ and $f_{\theta,i}$ can also reflect creditors' heterogeneous beliefs because they, especially $f_{\theta,i}$, generally involve the first-best value of debt. Moreover, I assume that r_i, $f_{\phi,i}$, and $f_{\theta,i}$ are bounded, continuous, and piecewise smooth. I then define the set of aggregate actions as

$$\mathcal{M} = \{m \mid m : \mathbb{R} \to [0, \Lambda] \text{ which makes HJB equation (6) well posed.}\}.$$

Each creditor is assumed to take arbitrary aggregate actions only from $\mathcal{M}$. To avoid abstract exposition, I plot all the possible termination scenarios in Figure 1 by *assuming* the creditors use threshold strategies.

Finally, an equilibrium for this economy can be defined as follows. It consists of rollover strategies $\{q_i^j(\cdot)\}_{i,j}$ such that

$$m(u) = \sum_{i=1}^{N} \int_0^{\xi_i} \lambda_i q_i^j(u) dj \in \mathcal{M}$$

and $q_i^j(\cdot)$ solves equation (6) for each i and j, given the above m.

3 Characterization of Equilibrium

This section proves that this economy has a unique equilibrium under certain conditions. In equilibrium, the creditors will use partially symmetric threshold strategies, meaning that the creditors run if and only if the fundamental falls below a certain threshold and they use exactly the same threshold within groups. Before I start, I need to introduce some relevant notations to deal with (partially symmetric) threshold strategies below. Let $w = (w_1, ..., w_N) \in \mathbb{R}^N$ be an arbitrary vector and assume that each creditor in group i runs if and only if $u_t < w_i$, that is, for all j,

$$q_i^j(u) = \begin{cases} 1 & \text{if } u < w_i \\ 0 & \text{otherwise.} \end{cases}$$

Then, the aggregate run size at time t equals

$$m(u_t; w) = \sum_{i=1}^{N} \lambda_i \xi_i 1_{u_t < w_i}. \quad (7)$$

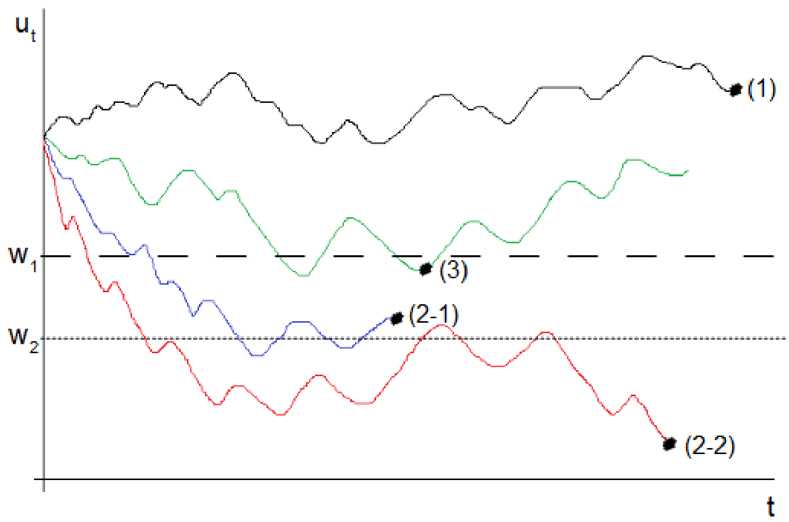

Figure 1: This graph plots all the possible termination scenarios when there are only two ex-ante heterogeneous groups of creditors. Each creditor in group i is assumed to run if and only if u_t falls below w_i. Case 1: The firm's asset itself expires. Case 2-1: The credit line fails when only the creditors in group 1 run. Case 2-2: The credit line fails when the creditors in both groups run. Case 3: A certain individual creditor runs at her maturity and the credit line successfully defends her, so the firm survives.

In such a special case, I will often use $V_i(u; w)$ in place of $V_i(u; m)$. (To avoid notational confusion, keep in mind that m is a real-valued function and w is a real-valued vector.)

3.1 Assumptions

This section presents sufficient conditions for our economy to have a unique equilibrium.

C1. *Costly liquidation*: For each i, I assume

$$V_i(u; \Lambda) \geq f_{\theta,i}(u), \quad \forall u. \tag{8}$$

It says that the *continuation* value of debt must be larger than its *liquidation* value for all possible outcomes u, even when all other creditors decide to run unconditionally (i.e., $m \equiv \Lambda$). This condition will be used to show that our economy exhibits complementarity and state monotonicity. To see why, suppose that the liquidation value of debt is larger than its continuation value at some economic outcomes. Then the individual creditors might be better off if other creditors choose more conservative strategies to shut down the firm as soon as possible, which violates strategic complementarity. See Figure 2. One can understand state monotonicity in a similar manner: if the liquidation cost is too cheap, the creditors might want the fundamental to deteriorate further when the economy is bad, to force the firm to liquidate its asset as soon as possible. In this paper, as mentioned in the introduction, I will rule out these unusual cases by assuming liquidation is costly.

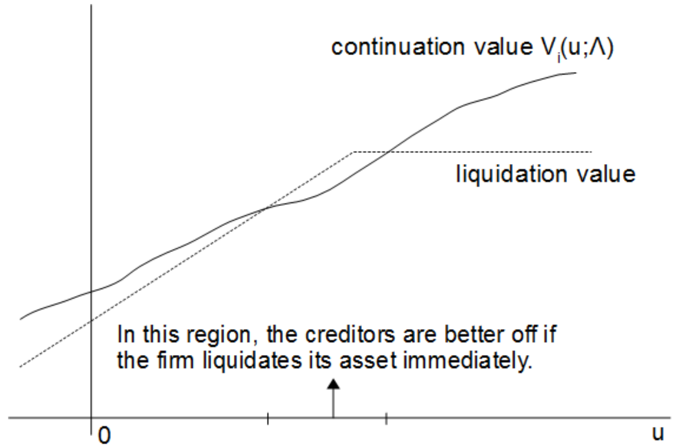

Figure 2: If the liquidation value of debt is higher than its continuation value at some economic outcomes, then the creditors might want other creditors to choose more conservative strategies to shut down the firm as soon as possible. In this scenario, strategic complementarity can be violated.

C2. *Boundary conditions*: Suppose that arbitrary aggregate actions m_1 and $m_2 \in \mathcal{M}$ such that $m_1(u) \leq m_2(u)$ for all u are given. Also, let an arbitrary vector of rollover thresholds $w \in \mathbb{R}^N$ be given. I assume that for each i,

$$\lim_{u \to -\infty} V_i(u; 0) < 1, \quad \lim_{u \to \infty} V_i(u; \Lambda) > 1, \tag{9}$$

$$\lim_{u \to -\infty} V_i(u; m_1) \geq \lim_{u \to -\infty} V_i(u; m_2), \quad \lim_{u \to \infty} V_i(u; m_1) \geq \lim_{u \to \infty} V_i(u; m_2), \tag{10}$$

$$\lim_{u \to -\infty} V_{iu}(u; w) \geq 0, \quad \lim_{u \to \infty} V_{iu}(u; w) \geq 0, \tag{11}$$

$$\lim_{u \to -\infty} V_{iu}(u; w) + \sum_j V_{iw_j}(u; w) \geq 0, \quad \lim_{u \to \infty} V_{iu}(u; w) + \sum_j V_{iw_j}(u; w) \geq 0. \tag{12}$$

These conditions can be interpreted as follows. First of all, the boundary values at $u = -\infty$ and $u = \infty$ describe the creditor's expected payoff when the economy is extremely bad and extremely good, respectively. Specifically, condition (9) assumes that if u is sufficiently small (large), then it is optimal to run (renew) even though nobody (everybody) runs. This condition is needed for our economy to have dominated regions. Condition (10) assumes that if the aggregate action m_1 is dominated by another aggregate action m_2 state by state, then her expected payoff for the former dominates that for the latter at least at the extreme economic outcomes. Condition (11) assumes that when other creditors use *threshold* strategies, her expected payoff must be increasing in u at least at the two extreme outcomes. Condition (12) assumes that at the extremes, her expected payoff increases more when the fundamental improves by some amount than when other creditors increase their rollover thresholds by the same amount. Basically, conditions (10) through (12) assume that strategic complementarity, state monotonicity, and weak strategic complementarity hold at least when the economy is sufficiently bad or sufficiently good.

C3. *Increasing payoffs*: For each i, $r_i(u)$, $f_{\phi,i}(u)$, and $f_{\theta,i}(u)$ are increasing in u and more importantly, at least one of $r_i(u)$ and $f_{\phi,i}(u)$ is *strictly* increasing in u over some *nontrivial* interval in $\mathbb{R}$. This condition will be used to rule out multiple equilibria.

For the sake of concreteness, in section 5, I will provide certain parameter restrictions to make some example economies satisfy the above assumptions.

3.2 Properties of the Value Function

This section proves that the value function associated to each creditor satisfies the above emphasized properties: strategic complementarity, state monotonicity, dominated regions, and weak strategic complementarity.

Let me begin with strategic complementarity: If aggregate action $m_1 \in \mathcal{M}$ is dominated by another one $m_2 \in \mathcal{M}$ (state by state), then the present value of debt corresponding to the former dominates that corresponding to the latter. This result is intuitive because if more creditors run, then they impose larger rollover threats to the firm. However, I would like to mention again that it actually may not hold if the liquidation cost is very cheap.

Theorem 3.1 (Strategic Complementarity)**.** For any aggregate actions m_1 and m_2 in $\mathcal{M}$ such that

$$m_1(u) \leq m_2(u), \quad \forall u,$$

it holds that for each i,

$$V_i(u; m_1) \geq V_i(u; m_2), \quad \forall u. \tag{13}$$

Proof. See Appendix A.1. □

Remark 1. In fact, strategic complementarity is in general defined in terms of the difference in (flow) utility between two arbitrary actions; for instance, see FMP (2003) or Vives (2005). In our economy, those two actions are running and rolling over: if a creditor runs, then she receives the face value of 1 and *leaves* the market; if she rolls over, she is expected to receive the *continuation* value of debta in the future. Thus, one can rewrite (13) as

$$V_i(u; m_1) - 1 \geq V_i(u; m_2) - 1, \quad \forall u$$

to make it more consistent with the standard definition in literature.

This theorem and condition C1 (costly liquidation) immediately imply the following corollary.

Corollary 3.2. For each i and any $m \in \mathcal{M}$, it holds that

$$V_i(u; m) \geq f_{\theta,i}(u), \quad \forall u.$$

So far, I have not restricted attention to threshold strategies. However, the previous and the following theorems will indeed imply that it suffices to consider only that type of strategies to prove the uniqueness of equilibrium. Before I explain why it is, I will first state the following theorem. Consider a certain individual creditor, say, in group i. Then, Theorem 3.3 says: if other creditors use partially symmetric threshold strategies represented by $w = (w_1, ..., w_N)$, her value function becomes strictly increasing in u. This result makes sense because the increment in u 1) raises the interest rate and the final payoffs paid when the firm's project terminates, *and* 2) makes the firm farther away from the running

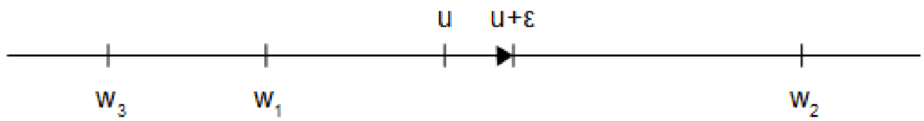

Figure 3: This plot describes an externality driven benefit. When the fundamental u increases, it makes the firm farther away from the running zone of group j creditors if $w_j < u_t$ or pushes the firm to escape from that running zone if $u_t < w_j$, for each j.

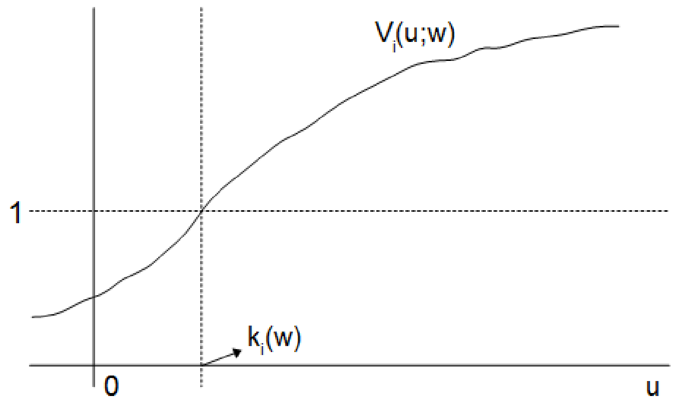

Figure 4: This figure plots the value function $V_i(u; w)$ and the best response function $k_i(w)$ of the creditors in group i, given w.

zone of group j creditors if $w_j < u_t$ or pushes the firm to escape from that running zone if $u_t < w_j$, for each j. See Figure 3. I call the former effect the fundamental driven benefit and the latter the externality driven benefit.

Theorem 3.3 (State Monotonicity)**.** For each i and $w \in \mathbb{R}^N$, $V_i(u; w)$ is strictly increasing in u. In addition, $V_i(u; 0)$ and $V_i(u; \Lambda)$ are also strictly increasing in u.

Proof. See Appendix A.2. □

Remark 2. As before, one can also state this theorem as $V_i(u; w) - 1$ is strictly increasing in u to make it more consistent with the conventional definition of state monotonicity; see FMP (2003). Also, I will occasionally use $V_i(u; -\infty)$ and $V_i(u; \infty)$ in place of $V_i(u; 0)$ and $V_i(u; \Lambda)$, respectively, for notational consistency.

Due to this theorem, together with conditions (9) and (10), I know that for any $w \in \mathbb{R}^N$, $V_i(u; w)$ is strictly increasing in u from below 1 to above 1 and so there exists a unique point, $k_i(w)$, such that

$$V_i(k_i(w); w) = 1.$$

See Figure 4, for instance. The same holds for $V_i(u; -\infty)$ and $V_i(u; \infty)$ as well. Denote by $k_i(-\infty)$ and $k_i(\infty)$, respectively, those two points that satisfy

$$V_i(k_i(-\infty); -\infty) = 1 \quad \text{and} \quad V_i(k_i(\infty); \infty) = 1.$$

Now, why Theorems 3.1 and 3.3 allow me to focus on partially symmetric threshold strategies? First, Theorem 3.1 implies that for each i and any $m \in \mathcal{M}$,

$$V_i(u; -\infty) \le V_i(u; m) \le V_i(u; \infty), \quad \forall u.$$

(Here, $V_i(u; m)$ may not be monotone in u because m can be any general aggregate action.) As a result, no matter what m is, each creditor in group i decides to run if $u < w^{l1} = k_i(-\infty)$ and roll over if $u > w^{u1} = k_i(\infty)$. See the left panel in Figure 5. Therefore, all the creditors in the economy rationally eliminates those lower and upper dominated regions, and focus on the aggregate actions m of the following form:

$$m(u) = \sum_{j=1}^{N} \int_0^{\xi_i} \lambda_i q_i^j(u) dj,$$

where

$$q_i^j(u) = 0, \ \forall u < w^{l1} \quad \text{and} \quad q_i^j(u) = 1, \ \forall u > w^{u1}. \tag{14}$$

Denote by $\mathcal{M}_1$ the collection of those m. Of course, $\mathcal{M}_1 \subseteq \mathcal{M}$.

I can do further. Theorem 3.1 again implies that the best and the worse scenarios, among those which can be derived from $\mathcal{M}_1$, come from w^{l1} and w^{u1}, respectively. In other words, for each i and any $m \in \mathcal{M}_1$,

$$V_i(u; w^{u1}) \le V_i(u; m) \le V_i(u; w^{l1}), \quad \forall u.$$

See the right panel in Figure 5. But, Theorems 3.1 and 3.3 imply that

$$w^{l1} \le w^{l2} := k_i(w^{l1}) \le w^{u2} := k_i(w^{u1}) \le w^{u1}, \quad \forall i. \tag{15}$$

Therefore, the creditors can now eliminate more regions and focus on m of the following form:

$$m(u) = \sum_{j=1}^{N} \int_0^{\xi_i} \lambda_i q_i^j(u) dj,$$

where

$$q_i^j(u) = 0, \ \forall u < w^{l2} \quad \text{and} \quad q_i^j(u) = 1, \ \forall u > w^{u2}.$$

Denote by $\mathcal{M}_2$ the collection of those m. Of course, again $\mathcal{M}_2 \subseteq \mathcal{M}_1$.

The above procedure, so-called iterative elimination of dominated regions, tells me that in order to prove there exists a unique equilibrium, it suffices to show

$$k^n(-\infty) \nearrow w^* \text{ and } k^n(\infty) \searrow w^* \text{ for some } w^*, \text{ as } n \to \infty. \tag{16}$$

Therefore, without loss of generality, I can assume that the creditors use partially symmetric threshold strategies and my ultimate goal becomes to show the desired result (16). In particular, I call $k_i(w)$ the group i creditors' best response to other creditors' threshold strategies represented by w. To begin with, I first restate the two important properties which I have already shown.

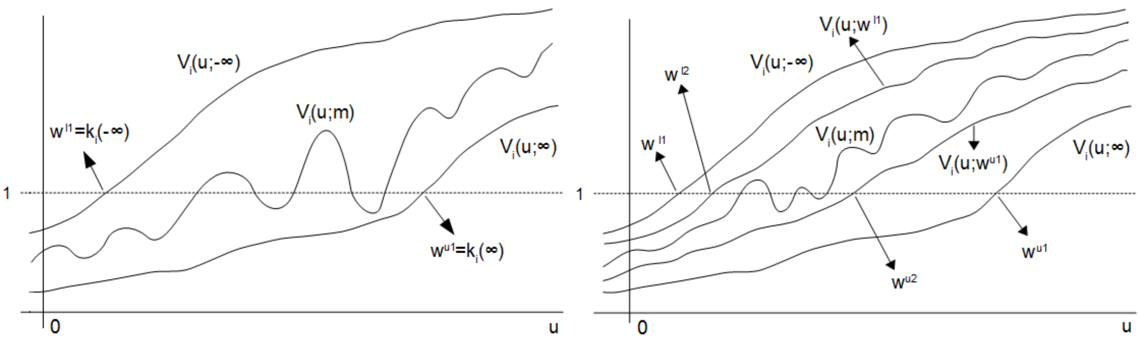

Figure 5: Dominated regions. In the first stage, for any aggregate action $m \in \mathcal{M}$, it is optimal for any creditor in group i to run and roll over if $u < w^{l1}$ and if $u > w^{u1}$, respectively. In the second stage, for any aggregate action $m \in \mathcal{M}_1$, it is optimal for any creditor in group i to run and roll over if $u < w^{l2}$ and if $u > w^{u2}$, respectively.

Corollary 3.4 (Dominated Regions)**.** The best response function is bounded. That is, for each i,

$$A \leq k_i(w) \leq B, \quad \forall w,$$

where

$$A = \min_j k_j(-\infty) \quad \text{and} \quad B = \max_j k_j(\infty).$$

Proof. It immediately comes from Theorems 3.1 and 3.3. □

Corollary 3.5 (Strategic Complementarity and State Monotonicity)**.** The best response increases in the rollover threshold chosen by any other creditors. That is, for each i and j, it holds[6]

$$k_{iw_j}(w) \geq 0, \quad \forall w \in \mathbb{R}^N.$$

Proof. It immediately comes from Theorems 3.1 and 3.3, as briefly seen in (15). □

Finally, I discuss weak strategic complementarity: A certain individual creditor's value function increases more when the fundamental improves by ϵ than when all the other creditors increase their rollover thresholds by the same amount. As a result, she only needs to increase her rollover threshold by strictly less than ϵ. I will discuss the intuition of this argument after I formally state it.

Theorem 3.6 (Weak Strategic Complementarity)**.** For each i, it holds

$$\sum_{j=1}^{N} k_{iw_j}(w) < 1, \quad \forall w \in \mathbb{R}^N. \tag{17}$$

Proof. See Appendix A.3. □

[6]Proving a relevant regularity property of the best response function without using a closed-form solution is beyond the scope of this paper. Let me assume it is differentiable.

In fact, inequality (17) is equivalent to

$$\frac{\partial}{\partial \epsilon} k_i(w_1 + \epsilon, ..., w_N + \epsilon)|_{\epsilon=0} < 1, \quad \forall w.$$

It formally justifies the above statement that the individual creditor only needs to increase her rollover threshold strictly less than ϵ.

To understand (17), recall

$$V_i(k_i(w); w) = 1, \quad \forall w.$$

Differentiating it with respect to w in the direction of $(1, ..., 1)$, I have

$$\begin{aligned} 0 &= \frac{\partial}{\partial \epsilon} V_i(k_i(w + \boldsymbol{\epsilon}); w + \boldsymbol{\epsilon})|_{\epsilon=0} \\ &= V_{iu}(k_i(w)) \sum_j k_{iw_j}(w) + \sum_j V_{iw_j}(k_i(w); w), \end{aligned} \tag{18}$$

where $\boldsymbol{\epsilon} = (\epsilon, ..., \epsilon)$. Because $V_{iu} > 0$, in order to prove (17) it suffices to show

$$V_{iu}(u) + \sum_j V_{iw_j}(u; w) > 0, \quad \forall u, w. \tag{19}$$

Literally, it means that the marginal benefit of u *outweighs* the sum of the marginal losses of w_j across all j. It makes sense because while the increment in u has both fundamental driven and externality driven benefits, the increment in w_j has only the externality driven loss; see the discussion prior to Theorem 3.3. More important, the externality driven benefit of u exactly cancels out the sum of the externality driven losses of w_j across all j. This is because even if w_1, ..., w_N, and u are all increased by the same amount ϵ, the probability distribution of the firm's default at any moment of time in the future will be unchanged because 1) the fundamental u_t follows an arithmetic Brownian motion with a constant drift and volatility and 2) the credit line fails in a memoryless fashion with a constant credit-line reliability. Because of this symmetric cancellation, only the fundamental driven marginal benefit of u survives, and so the result (19) follows.

3.3 Unique Equilibrium

This section finally proves the existence of a unique equilibrium. First, I will show that there is a unique threshold strategy profile $w^* = (w_1^*, ..., w_N^*)$ that satisfies

$$k_i(w^*) = w_i^*, \quad \forall i.$$

For this purpose, it suffices to show that the best response profile

$$k := (k_1, ..., k_N) : \prod_{j=1}^{N} [A, B] \to \prod_{j=1}^{N} [A, B] \tag{20}$$

is a contraction mapping, i.e.,

$$\|k(w^1) - k(w^2)\| < C\|w^1 - w^2\|, \quad \forall w^1, w^2 \in \prod_j [A, B]$$

for some $C < 1$. Suppose this result is indeed true. Then, (remember I have not restricted my attention to threshold strategies), the contraction mapping theorem actually tells us that for any $w \in \mathbb{R}^N$, it holds

$$\lim_{n\to\infty} k^n(w) = w^*,$$

in which the convergence is motonone due to Corollary 3.5. Therefore, it utimately proves our desired result (16) and so a unique equilibrium emerges.

Hence, it remains to show that the best response profile k is a contraction mapping from $\prod_j [A, B]$ into itself. In the case of $N = 1$, the best response k obviously becomes a contraction mapping because $[A, B]$ is compact (dominated regions) and $k'(w)$ is positive (strategic complementarity and state monotonicity) but strictly less than 1 (weak strategic complementarity). For the case of $N > 1$, it can be shown that k becomes a contraction mapping in the sup norm; see the proof below. The methodological connection between iterative elimination of dominated regions and the contraction mapping theorem was also discovered by Mathevet (2010).

Theorem 3.7 (Unique Equilibrium)**.** There exists a unique partially symmetric threshold equilibrium for our economy.

Proof. See Appendix A.4. □

3.4 Limit Uniqueness

FMP (2003) and FP (2000) show that there still exist a unique equilibrium even for very general prior beliefs or general aggregate shock processes if some controlling parameter vanishes. I can obtain an analogous result for our economy. Let $\mu(u) = \bar{\mu} + a\eta(u)$ and $\sigma(u) = \bar{\sigma} + b\zeta(u)$, where $\bar{\mu}$, a, b and $\bar{\sigma} > 0$ are constants and η and ζ satisfy the standard regularity conditions; see chapter IV.2 in Fleming and Soner (2006). Suppose u_t evolves according to

$$du_t = \mu(u_t)dt + \sigma(u_t)dZ_t.$$

When $a = b = 0$, I have already shown that the best response profile k becomes a contraction mapping with some $C < 1$. But even if I perturb a and b by a small amount, the best response profile for the new economy remains as a contraction mapping with slightly different A, B, and $C < 1$.[7] So a unique equilibrium still emerges. For example, if the fundamental follows a mean-reverting process, then a unique equilibrium exists as long as the mean reversion rate is sufficiently small.

4 Who Runs First?

This section investigates the following three examples: the economies consisting of 1) long-term creditors and short-term creditors, 2) junior creditors and senior creditors, and 3) creditors with heterogeneous beliefs. I find that ceteris paribus long-term/junior/pessimistic creditors run more hastily compared to short-term/senior/optimistic creditors, respectively.

[7] I assume that the individual's problem (6) is stable, that is, $V_i(u; m)$ is continuous in the data.

In this section, I return to the initial setting where the fundamental y_t follows the geometric Brownian motion (1).

4.1 With Respect to Maturity

Suppose that λ_i can be different across the creditors in different groups but all the other parameters remain the same.[8] Also, since I have already shown that the creditors use partially symmetric threshold strategies in equilibrium, without loss of generality I can assume that the creditors use threshold strategies represented by some $z = (z_1, ..., z_N)$: each creditor in group i is assumed to run if and only if $y_t < z_i$. Then the HJB equation faced by an individual creditor in group i can be written as

$$0 = r + \phi f_\phi(y) + \theta m(y, z)(f_\theta(y) - V_i(y)) + \lambda_i \max\{1 - V_i(y), 0\} - (\rho + \phi)V_i(y) + \mu y V_{iy} + \frac{\sigma^2}{2} y^2 V_{iyy}, \quad (21)$$

where

$$f_\phi(y) = \min\{y, 1\}, \quad f_\theta(y) = \min\{L + ly, 1\}, \quad \text{and} \quad m(y, z) = \sum_j \lambda_j \xi_j 1_{y<z_j}.$$

In this environment, as mentioned above, long-term creditors run more pre-emptively than short-term creditors. The reason is that a short-term debt is more beneficial to an individual creditor because 1) she can readjust rollover decisions more frequently and 2) no matter which debt she purchases, she cannot influence the aggregate rollover risks to the firm, (which means that the likelihood of the liquidation is unchanged), because she is atomic in the economy. Therefore, short-term creditors must be less sensitive to economic fluctuations, justifying their smaller incentives to run.

Theorem 4.1. Ceteris paribus, the long-term creditors run earlier than the short-term creditors. That is, if $\lambda_i < \lambda_j$ for some i and j, then it holds

$$z_j^* < z_i^*.$$

In fact, the following stronger property can be obtained:

$$V_i(y; z) < V_j(y; z), \quad \forall y, z.$$

Proof. See Appendix A.5. □

One needs to be careful to interpret this result. It only tells that the short-term creditors have less incentives to run than the long-term creditors when the share of short-term debts is *fixed*; it does not tell that issuing more short-term debts will deter the firm's default. To see why, suppose there are only two groups of creditors and let $\lambda_S = \lambda_1$ and $\lambda_L = \lambda_2$. ξ denotes the fraction of the short-term creditors. Then, observe that increasing the share of short-term debts by Δ reduces the size of the first run only from the long-term creditors by $\lambda_L \Delta$, whereas it *increases* the size of the second run from all the creditors by

[8] Of course, r_i can vary depending on maturity. But in this section I focus on the effect of maturity.

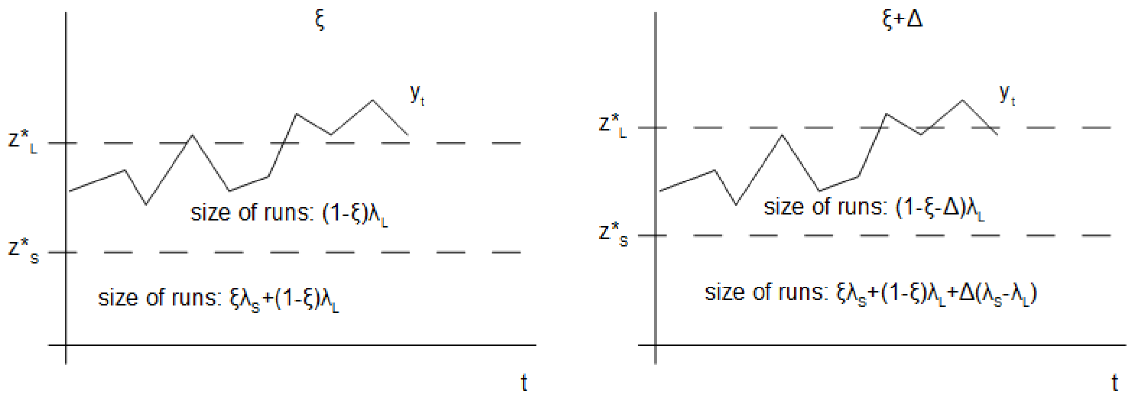

Figure 6: This figure explains how the intensities of the first and the second runs change as the firm increases the share of short-term debts by Δ. The positions of z_L^* and z_S^* are not an important issue at least in this figure.

$(\lambda_S - \lambda_L)\Delta$. See Figure 6. Therefore, it is ambiguous whether or not the creditors are better off from the increased share of short-term debts, even though more creditors can now readjust their rollover decisions more frequently. Studying this trade-off involved in maturity shortening rigorously will be an important future research topic. Here, I only report one numerical result under a reasonable choice of the model parameters. Let me postpone an in-depth discussion of it. The left panel in Figure 7 indeed tells that both short-term and long-term creditors lift their rollover thresholds when the firm issues more short-term debts. It means that the above mentioned positive effects are dominated by the increased intensity of the second run.

4.2 With Respect to Seniority

This section studies an economy with creditors of different seniority: one group of creditors has a right to collect money subordinate to that of creditors in the other group. The former (latter) is referred to as junior (senior) creditors. Let ξ be the fraction of the senior creditors. From a certain individual creditor's perspective, I assume that the senior creditors run if $y_t < z_S$ and the junior creditors run if $y_t < z_J$. Let $z = (z_S, z_J)$. On one hand, when the firm's asset expires, each senior creditor receives

$$f_{\phi,S}(y) = \frac{\min\{y, \xi\}}{\xi},$$

whereas each junior creditor receives

$$f_{\phi,J}(y) = \frac{\min\{y - \min\{y, \xi\}, 1 - \xi\}}{1 - \xi}.$$

On the other hand, when the firm defaults because of the creditors' runs, then each senior creditor receives

$$f_{\theta,S}(y) = \frac{\min\{L + ly, \xi\}}{\xi},$$

whereas each junior creditor receives

$$f_{\theta,J}(y) = \frac{\min\{L + ly - \min\{L + ly, \xi\}, 1 - \xi\}}{1 - \xi}.$$

In this setting, the HJB equation face by the creditors in group $i \in \{S, J\}$ can be written as

$$\rho V_i(y) = r + \phi(f_{\phi,i}(y) - V_i(y)) + \theta m(y, z)(f_{\theta,i}(y) - V_i(y)) + \lambda \max\{1 - V_i(y), 0\}+ \mu y V_{iy} + \frac{\sigma^2}{2} y^2 V_{iyy}, \quad (22)$$

where

$$m(y, z) = \lambda\big(\xi 1_{y<z_S} + (1 - \xi) 1_{y<z_J}\big).$$

In this economy, the junior creditors will run earlier than the senior creditors. This is because although the creditors still cannot influence the aggregate rollover risks to the firm, unlike the previous example, they receive different amounts of money when the project terminates. So, the junior creditors fear the firm's default more than the senior creditors. Therefore, the junior creditors are more run-prone.

Theorem 4.2. Ceteris paribus, the junior creditors run earlier than the senior creditors, that is,

$$z_S^* < z_J^*.$$

In fact, the following stronger property holds

$$V_J(y; z) < V_S(y; z), \quad \forall y, z.$$

Proof. See Appendix A.6. □

However, again, it is possible that issuing more senior debts will shorten the firm's life span. On one hand, different from the previous example, as the firm issues more senior debts, the size of the second run *remains* the same, whereas the size of the first run still reduces to $\lambda(1 - \xi - \Delta)$. Thus, it is good for the firm. Nevertheless, as the share of senior debts grows, the payoffs of both individual senior and junior creditors are diluted. In other words, the more creditors are insured, the less money the creditors collect when the firm defaults; see that all of $f_{\phi,S}$, $f_{\phi,J}$, $f_{\theta,S}$, and $f_{\theta,J}$ are decreasing in ξ. Thus, it is undetermined whether or not the creditors are better off from the increased share of senior debts. The numerical result, shown in the right panel in Figure 7, tells that both junior and senior creditors raise their rollover thresholds when the firm issues more senior debts, which means the dilution effect of excessively issued senior debts dominates the effect of the reduced size of runs.[9]

[9]The reason why the size of the senior debts only ranges from 0.55 to 0.95 is that if ξ is less than L, then all senior creditors receive \$1 when the asset is liquidated, regardless of the current state of the fundamental, which makes condition (8) not be satisfied. Also, even for ξ close to L, it is highly likely that condition (8) fails by the similar reason.

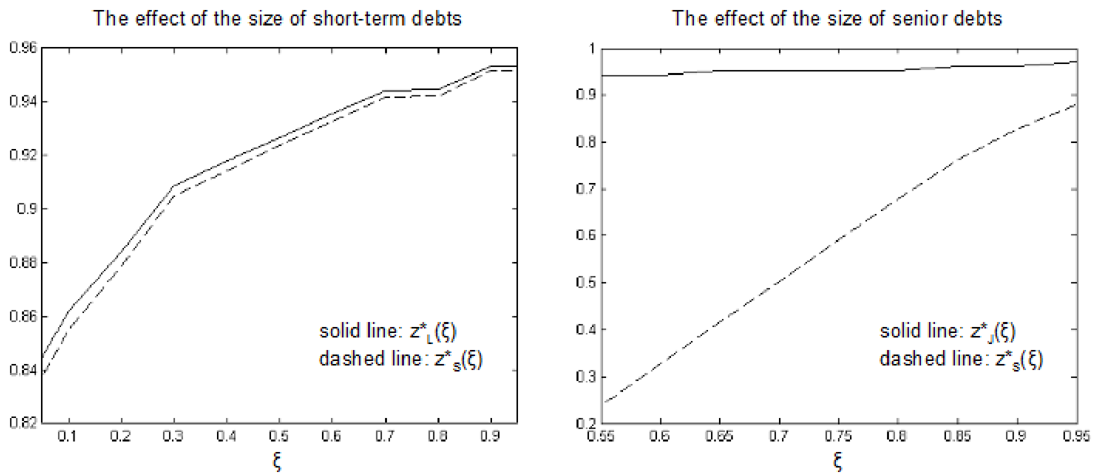

Figure 7: The left panel plots the equilibrium rollover thresholds, $z_L^*(\xi)$ and $z_S^*(\xi)$, chosen by long-term and short-term creditors, respectively, by varying the share of short-term debts, ξ. It tells that all the creditors raise their rollover thresholds as the firm issues more short-term debts. The choice of the parameters is $\rho = 1.5\%,\ r = 7\%,\ \phi = 0.077,\ \alpha = 55\%,\ \sigma = 10\%,\ \mu = 1.5\%,\ \lambda_L = 5,\ \lambda_S = 30,$ and $\theta = 1$, most of which are taken from HX (2012) and Schroth, Suarez, and Taylor (2014). On the other hand, the right panel plots the equilibrium rollover thresholds, $z_J^*(\xi)$ and $z_S^*(\xi)$, chosen by junior and senior creditors, respectively, by varying the share of senior debts, ξ. It tells that all the creditors raise their rollover thresholds as the firm issues more senior debts. The same parameters as above but $\lambda = 10$ are used.

4.3 With Respect to Heterogeneous Beliefs

This section studies the rollover decisions of creditors when they have heterogeneous beliefs. On one hand, it is easy to understand that pessimistic creditors run earlier than optimistic creditors because the present value of debt increases in the fundamental and so in the creditor's belief about the asset's growth rate. But, at the moment I am not able to confirm who is more run-prone between creditors with different beliefs about the fundamental volatility.

Theorem 4.3. Ceteris paribus, the optimistic creditors run earlier than the pessimistic creditors. That is, if $\mu_i < \mu_j$ for some i and j, then it holds

$$z_j^* < z_i^*.$$

In fact, the following stronger property can be obtained:

$$V_i(y; z) < V_j(y; z), \quad \forall y, z.$$

Proof. See Appendix A.7. □

In this economy, unlike the previous two examples, I can derive some theoretical result regarding the change in the fraction of a certain group of creditors. In short, when the fraction of the optimistic creditors increases, both optimistic and pessimistic creditors will lower rollover thresholds and thus the firm will survive longer. More in general, suppose there are two groups of creditors with the same maturity and let ξ be the fraction of the creditors in group 1. Further, I assume that the creditors' payoff structures (r_i, $f_{\phi,i}$, and $f_{\theta,i}$) are independent of ξ. Without loss of generality, at equilibrium, I can say that the creditors in group 1 run later than the group 2 creditors. Then it can be shown that as the fraction of the creditors in group 1 rises, all the creditors in the economy will lower their rollover thresholds. This is simply because the size of runs is mitigated but the negative effects discussed above (the increased aggregate rollover risks and the diluted final payoffs) do not occur any more.

Theorem 4.4. Suppose there are two groups of creditors. Let $\lambda_1 = \lambda_2 = \lambda$ and r_i, $f_{\phi,i}$, and $f_{\theta,i}$ be independent of ξ, where ξ is the fraction of the creditors in group 1. Also, let $z_1^*(\xi) \leq z_2^*(\xi)$. Then, it holds

$$\frac{\partial z_i^*(\xi)}{\partial \xi} \leq 0, \quad \forall i = 1, 2.$$

Proof. See Appendix A.8. □

Remark 3. This theorem does not require $V_1(y; z^*(\xi)) \geq V_2(y; z^*(\xi))$ for all y. The condition $z_1^*(\xi) \leq z_2^*(\xi)$ is sufficient to derive the above result.

5 Parameter Restrictions

In this section, I discuss under which parameters the example economies described in section 4 satisfy conditions C1 and C2. First, consider the economy with creditors of different maturities. Here, I allow the debts with different maturities to have the different interest rates. So, r in equation (21) needs to be replaced by r_i. A necessary condition for C1 is

$$(\theta, \sigma) \notin \{(\theta, \sigma) : \ \theta \text{ and } \sigma \text{ are sufficiently large with } \theta = K\sigma^2 \text{ for some } K\}. \tag{23}$$

Sufficient conditions for C2 are

$$\rho < r_i < \rho + \phi, \quad \forall i, \tag{24}$$

$$\mu < \rho + \phi, \tag{25}$$

$$L + l = \frac{\alpha r}{\rho + \phi} + \frac{\alpha \phi}{\rho + \phi - \mu} < 1, \tag{26}$$

$$r_i > \alpha r, \quad \forall i. \tag{27}$$

Let me first discuss condition (23). Intuitively, when θ is sufficiently large and all the other creditors are running unconditionally, the firm will be more or less liquidated immediately at time 0. But, the present value of debt is not merely equal to $\min\{L + ly_0, 1\}$ because

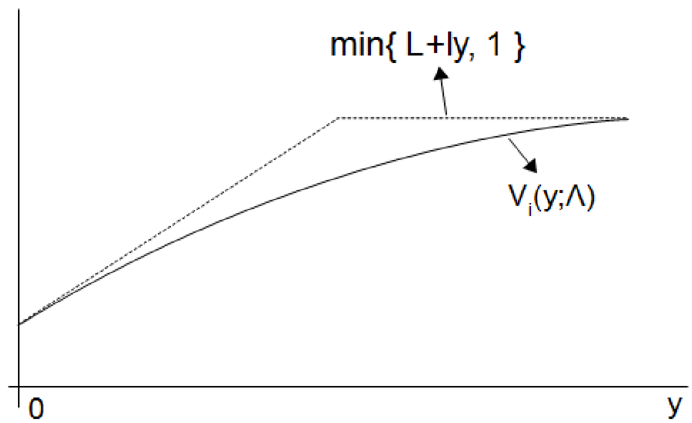

Figure 8: When θ and σ are sufficiently large and all the other creditors are unconditionally running, the present value of debt $V(y;\infty)$ becomes lower than the immediate liquidation payoff.

the volatility of the fundamental is also very large. As a result, the debt value must be asymptotically equal to[10]

$$V_i(y_0;\Lambda) = E[\min\{L + ly_\tau, 1\}], \tag{28}$$

where τ is a randomly arrived stopping time with an arrival rate $\theta\Lambda$ and y_t follows a geometric Brownian motion with *zero* drift and volatility σ (asympotically). But, since the liquidation payoff $\min\{L+ly, 1\}$ is concave, the creditors must hate any risks from the firm's asset as if they are risk averse agents. So, it must hold

$$V_i(y_0) \leq \min\{L + ly_0, 1\}. \tag{29}$$

Alternatively, observe

$$V_i(y_0) \leq \min\{E[L + ly_\tau], 1\} = \min\{L + ly_0, 1\},$$

where the first inequality uses Jensen's inequality and the second equality uses the fact that y_t is asymptotically a martingale. Moreover, since $\min\{L + ly, 1\}$ is not a linear function, at least one y_0 satisfies (29) with strict inequality. Therefore, condition C1 is violated. See Figure 8 (in fact, from (28) it is easy to see that $V(y)$ is increasing and concave in y.). Hence, in order to exclude this situation, either θ or σ has to be moderate.

With respect to conditions (24) through (26), it is almost straightforward to prove those conditions are sufficient to imply inequalities (9) to (12), because one can explicitly obtain the formulas for the boundary values of V_i. See Appendix A.9.

[10]Note that $V_i(y;\Lambda)$ in general satisfies

$$0 = r_i + \phi\min\{y, 1\} + \theta\Lambda(\min\{L + ly, 1\} - V_i) + \lambda_i \max\{1 - V_i, 0\} - (\rho + \phi)V_i + \mu y V_{iy} + \frac{\sigma^2}{2}y^2 V_{iyy}.$$

When θ and σ are sufficiently large with $\theta = K\sigma^2$, the above equation asymptotically reduces to

$$0 = \theta\Lambda(\min\{L + ly, 1\} - V) + \frac{\sigma^2}{2}y^2 V_{iyy}.$$

By the Feynman-Kac theorem, probabilistic representation of V is given by (28). Conducting a precise asymptotic analysis is beyond the scope of this paper.

Remind that HX (2012) does not impose condition C1 or equivalently condition (23). But, they impose the following condition:

$$\theta > \frac{\phi}{\lambda(1 - L - l)},$$

which is used to show that the creditor's value function exhibits a certain single crossing property; see Lemma 3 there. So, the scope of economies covered by this paper is not entirely included by that in HX (2012).

For the other example economies, one can also obtain appropriate parameter restrictions because it is straightforward to compute the boundary values of the value functions. For the example with creditors of heterogeneous beliefs, one can simply replace μ by μ_i in (24) to (27). For the example with senior and junior debts, the conditions are a little messy.

6 Conclusion

This paper studies a dynamic coordination problem between ex-ante heterogeneous creditors of a single firm. Unlike HX (2012), I show that this economy has a unique equilibrium without restricting attention to a specific type of equilibrium. This paper also analyzes three motivating examples, where creditors differ in their maturity, seniority, and beliefs about the economic status. I find that ceteris paribus long-term/junior/pessimistic creditors run earlier than short-term/senior/optimistic creditors.

It would be interesting for future research if one studies 1) the effect of a change in the share of a certain group of creditors on the equity value of the firm, 2) endogenous bailout decisions of the credit-line provider, and 3) a firm's dynamic decisions for an optimal maturity, interest rate, seniority, and so on.

A Appendix

In this Appendix, a positive (negative) number indicates a number ≥ 0 (≤ 0). A strictly positive (negative) number indicates a number > 0 (< 0).

A.1 Proof of Theorem 3.1

I will first prove that

$$V_i(u; m_2) \geq V_i(u; \Lambda), \quad \forall u. \tag{30}$$

By the way of contradiction, suppose that $V_i(u; m_2) < V_i(u; \Lambda)$ for some u. Let $G(u) = V_i(u; m_2) - V_i(u; \Lambda)$. Then from condition (10), there must exist some u_1 such that[11]

$$G(u_1) < 0, \quad G_u(u_1) = 0, \quad \text{and} \quad G_{uu}(u_1) > 0. \tag{31}$$

Subtracting equation (6) for $V_i(u; \Lambda)$ from that for $V_i(u; m_2)$, I have

$$0 = -\theta_i(\Lambda - m_2(u))(f_{\theta,i}(u) - V_i(u; \Lambda)) + \lambda_i \max\{1 - V_i(u; m_2), 0\} - \lambda_i \max\{1 - V_i(u; \Lambda), 0\} - (\rho_i + \phi_i + \theta_i m_2(u))G(u) + \mu_i G_u(u) + \frac{\sigma_i^2}{2} G_{uu}(u).$$

Then from (31),

$$0 > -\theta_i(\Lambda - m_2(u_1))(f_{\theta,i}(u_1) - V_i(u_1; \Lambda)) + \lambda_i \max\{1 - V_i(u_1; m_2), 0\} - \lambda_i \max\{1 - V_i(u_1; \Lambda), 0\} - (\rho_i + \phi_i + \theta_i m_2(u_1))G(u_1).$$

I consider the following three cases:

- Case 1: If $V_i(u_1; m_2) \geq 1$ and $V_i(u_1; \Lambda) \geq 1$, then

$$0 > -\theta_i(\Lambda - m_2(u_1))(f_{\theta,i}(u_1) - V_i(u_1; \Lambda)) - (\rho_i + \phi_i + \theta_i m_2(u_1))G(u_1),$$

which is a contradiction because $\Lambda \geq m_2(u_1)$, $f_{\theta,i}(u_1) \leq V_i(u_1; \Lambda)$, and $G(u_1) < 0$.

- Case 2: If $V_i(u_1; m_2) \leq 1$ and $V_i(u_1; \Lambda) \geq 1$, then

$$0 > -\theta_i(\Lambda - m_2(u_1))(f_{\theta,i}(u_1) - V_i(u_1; \Lambda)) + \lambda_i(1 - V_i(u_1; m_2)) - (\rho_i + \phi_i + \theta_i m_2(u_1))G(u_1),$$

which is a contradiction by the same reason above plus $V_i(u_1; m_2) \leq 1$.

- Case 3: If $V_i(u_1; m_2) \leq 1$ and $V_i(u_1; \Lambda) \leq 1$, then

$$0 > -\theta_i(\Lambda - m_2(u_1))(f_{\theta,i}(u_1) - V_i(u_1; \Lambda)) - (\rho_i + \phi_i + \theta_i m_2(u_1) + \lambda_i)G(u_1),$$

which is a contradiction by the same reason. Therefore, it must hold that $V_i(u; m_2) \geq V_i(u; \Lambda)$ for all u.

Together with (8), the above result (30) implies

$$V_i(u; m_2) \geq f_{\theta,i}(u), \quad \forall u.$$

Thus, I can apply exactly the same argument above with m_1 and m_2 in place of m_2 and Λ, respectively, to show the main result (13).

[11] If $G(u)$ is not twice differentiable at u_1, apply the above argument to $u_1 + \epsilon$ for sufficiently small ϵ.

A.2 Proof of Theorem 3.3

Let

$$m(u;w) = \sum_{j=1}^{N} \lambda_j \xi_j 1_{u<w_j}.$$

Differentiating (6) with respect to u via the envelope theorem, I have

$$\begin{aligned} 0 = r_i'(u) + \phi_i f_{\phi,i}'(u) + \theta_i m(u,w) f_{\theta,i}'(u) - \theta_i \sum_j \xi_j \lambda_j \delta(u-w_j)(f_{\theta,i}(u) - V_i(u)) - \\ (\lambda_i 1_{1>V_i(u)} + \rho_i + \phi_i + \theta_i m(u,w)) V_{iu}(u) + \mu_i V_{iuu}(u) + \frac{\sigma_i^2}{2} V_{iuuu}(u), \quad (32) \end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function centered at zero; see chapter 3 in Stein and Shakarchi (2011). I view (32) as a second order differential equation for $V_{iu}(u)$ with boundary conditions given by (11). In this sense, the term $V_i(u)$ in the first line of (32) should be regarded as some given function. Moreover, this equation holds in the sense of distribution because $m_u(u,w)$ is involved with the Dirac delta function; see Stein and Shakarchi (2011) again.

The first three terms in (32) determine the fundamental driven marginal effect of u and the fourth term in (32) determines the externality driven marginal effect of u. By the assumption on r_i, $f_{\phi,i}$ and $f_{\theta,i}$, the fundamental driven marginal effect is positive. Also, by Corollary 3.2, the externality driven marginal effect is also positive. Hence, V_{iu} satisfies (32) with positive flow terms, strictly positive discount rate, and positive boundary terms. Then, the probabilistic representation of V_{iu}, derived from the Feynman-Kac theorem, implies

$$V_{iu}(u) \geq 0, \quad \forall u.$$

But $r_i(\cdot)$ and $f_{\phi,i}(\cdot)$ are assumed to be strictly increasing over a nontrivial region and the volatility σ is nonzero. It implies that the fundamental u_t spends a nontrivial amount of time over that nontrivial region. So the Feynman-Kac theorem says the following stronger property:

$$V_{iu}(u) > 0, \quad \forall u,$$

implying $V_i(u;w)$ is strictly increasing in u. In fact, the proof for the fact that both $V_i(u;0)$ and $V_i(u;\Lambda)$ are strictly increasing in u is much simpler than the above because $m'(u)$ for those cases are merely zero.

Remark 4. In the differential equations literature, this type of argument is called the comparison principle; see Theorem 8.1 in Fleming and Soner (2006). I will keep using this method below.

A.3 Proof of Theorem 3.6

Let $G(u;w) = V_{iu}(u;w) + \sum_j V_{iw_j}(u;w)$. Differentiating (6) with respect to u and w_j for all j, respectively, and summing up all the resulting equations, I have

$$0 = r_i'(u) + \phi_i f_{\phi,i}'(u) - (\lambda_i 1_{\{1>V_i(u)\}} + \rho_i + \phi_i + \theta_i m(u;w))G(u) + \mu_i G_u(u) + \frac{\sigma_i^2}{2} G_{uu}(u). \tag{33}$$

Also boundary condition (12) says

$$\lim_{u\to-\infty} G(u) \geq 0 \quad \text{and} \quad \lim_{u\to\infty} G(u) \geq 0. \tag{34}$$

Thus, $G(u)$ satisfies HJB equation (33) with the positive flow term and the positive boundary values. Also, the flow term is strictly positive over a nontrival region. Therefore, the comparison principle implies

$$G(u;w) > 0, \quad \forall u, w. \tag{35}$$

From (18) and state monotonicity together, I have

$$\sum_j k_{iw_j}(w) < 1.$$

A.4 Proof of Theorem 3.7

Let $X = \prod_{j=1}^N [A, B]$. To prove that k is a contraction mapping from X to X, I will use the supreme norm because all norms on $\mathbb{R}^N$ are equivalent. Observe that for any $w^1, w^2 \in X$,

$$\begin{aligned}
\max_i |k_i(w^1) - k_i(w^2)| &= \max_i |\int_0^1 \frac{\partial}{\partial\epsilon} k_i(w^2 + \epsilon(w^1 - w^2))d\epsilon| \\
&= \max_i |\int_0^1 \sum_j k_{iw_j}(w^2 + \epsilon(w^1 - w^2))(w_j^1 - w_j^2)d\epsilon| \\
&\leq \max_i \max_l |w_l^1 - w_l^2| \int_0^1 \sum_j k_{iw_j}(w^2 + \epsilon(w^1 - w^2))d\epsilon \quad (\text{by } k_{iw_j} \geq 0) \\
&< \max_l |w_l^1 - w_l^2| \quad (\text{by } \sum_j k_{iw_j} < 1).
\end{aligned}$$

But since X is compact, k must be a contraction mapping.

A.5 Proof of Theorem 4.1

As usual, let $k_i(z)$ be the best response function of an individual creditor in group i against the other creditors' rollover strategies represented by z. Let $G(y;z) = V_j(y;z) - V_i(y;z)$. Subtracting equation (21) for i from that for j, I have

$$0 = \lambda_j 1_{y<k_j(z)}(1 - V_j) - \lambda_i 1_{y<k_i(z)}(1 - V_i) - (\rho + \phi + \theta m(y,z))G + \mu y G_y + \frac{\sigma^2}{2} y^2 G_{yy}. \tag{36}$$

It is easy to check

$$\lim_{y\to 0} G(y) = \frac{(\lambda_j - \lambda_i)(\rho + \phi - r + \theta\Lambda(1-L))}{A}$$

where

$$A = (\rho + \phi + \theta\Lambda + \lambda_i)(\rho + \phi + \theta\Lambda + \lambda_j)$$

and

$$\lim_{y\to\infty} G(y) = 0.$$

According to the parameter restrictions given in (24) through (27), I have

$$\lim_{y\to 0} G(y) > 0.$$

Now by way contradiction suppose $k_i(z) \leq k_j(z)$. Then I have

$$\lambda_j 1_{y<k_j}(1-V_j) - \lambda_i 1_{y<k_i}(1-V_i) = 1_{y<k_i}(\lambda_j - \lambda_i)(1-V_j) + 1_{k_i<y<k_j}\lambda_j(1-V_j) - 1_{y<k_i}\lambda_i G.$$

Thus, I can rewrite (36) as

$$0 = 1_{y<k_i}(\lambda_j - \lambda_i)(1-V_j) + 1_{k_i<y<k_j}\lambda_j(1-V_j) - (\rho + \phi + \theta m(y,z) + 1_{\{y<k_i\}}\lambda_i)G + \mu y G_y + \frac{\sigma^2}{2}y^2 G_{yy}. \quad (37)$$

Here, observe that the flow terms are positive and strictly positive over a nontrivial region because $\lambda_i < \lambda_j$ and $V_j(y) < 1$ for $y < k_j$. So by the comparison principle,

$$G(y) > 0, \quad \forall y. \quad (38)$$

It contradicts the assumption $k_i(z) \leq k_j(z)$. So I must have $k_i(z) > k_j(z)$, which implies

$$z_i^* > z_j^*.$$

In fact, if $k_i(z) > k_j(z)$ is the case, then I have

$$\lambda_j 1_{y<k_j}(1-V_j) - \lambda_i 1_{y<k_i}(1-V_i) = 1_{y<k_j}(\lambda_j - \lambda_i)(1-V_j) - 1_{k_j<y<k_i}\lambda_i(1-V_j) - 1_{y<k_i}\lambda_i G.$$

Thus, I can rewrite (36) as

$$0 = 1_{y<k_j}(\lambda_j - \lambda_i)(1-V_j) - 1_{k_j<y<k_i}\lambda_i(1-V_j) - (\rho + \phi + \theta m(y,z) + 1_{y<k_i}\lambda_i)G + \mu y G_y + \frac{\sigma^2}{2}y^2 G_{yy}. \quad (39)$$

Again, it is easy to see that the flow terms are positive and strictly positive over a nontrivial region. Therefore, I have

$$G(y) > 0, \quad \forall y.$$

A.6 Proof of Theorem 4.2

Let $G(y;z) = V_S(y;z) - V_J(y;z)$. Subtracting equation (22) for J from that for S, I have

$$0 = \phi(f_{\phi,S} - f_{\phi,J}) + \theta m(y,z)(f_{\theta,S} - f_{\theta,J}) + \lambda 1_{\{y<k_S\}}(1 - V_S) - \lambda 1_{\{y<k_J\}}(1 - V_J) - \\ (\rho + \phi + \theta m(y;z))G + \mu y G_y + \frac{\sigma^2}{2} y^2 G_{yy}, \quad (40)$$

with

$$\lim_{y \to 0} G(y) = \frac{\theta\lambda(\frac{\min\{L,\xi\}}{\xi} - \frac{L-\min\{L,\xi\}}{1-\xi})}{\rho + \phi + \theta\lambda + \lambda} > 0$$

and

$$\lim_{y \to \infty} G(y) = 0.$$

First, suppose $k_J(z) \leq k_S(z)$. Then I have

$$1_{y<k_S}(1 - V_S) - 1_{y<k_J}(1 - V_J) = 1_{k_J<y<k_S}(1 - V_S) - 1_{y<k_J} G.$$

Then, I can rewrite (40) as

$$0 = \phi(f_{\phi,S} - f_{\phi,J}) + \theta m(y,z)(f_{\theta,S} - f_{\theta,J}) + \lambda 1_{k_J<y<k_S}(1 - V_S) - \\ (\rho + \phi + \theta m(y,z) + 1_{y<k_J})G + \mu y G_y + \frac{\sigma^2}{2} y^2 G_{yy}.$$

It is easy to see that the flow terms are positive and strictly positive over a nontrivial region. So I have

$$G(y) > 0, \quad \forall y.$$

It contradicts the assumption $k_J(z) \leq k_S(z)$. So I have $k_J(z) > k_S(z)$, implying

$$z_J^* > z_S^*.$$

In fact, if $k_S(z) < k_J(z)$ is the case, then

$$1_{y<k_S}(1 - V_S) - 1_{y<k_J}(1 - V_J) = -1_{k_S<y<k_J}(1 - V_S) - 1_{\{y<k_J\}} G.$$

Then, I can rewrite (40) as

$$0 = \phi(f_{\phi,S} - f_{\phi,J}) + \theta m(y,z)(f_{\theta,S} - f_{\theta,J}) - \lambda 1_{k_S<y<k_J}(1 - V_S) - \\ (\rho + \phi + \theta m(y,z) + 1_{y<k_J})G + \mu y G_y + \frac{\sigma^2}{2} y^2 G_{yy}.$$

As the flow terms are positive and strictly positive over a nontrivial region, I have

$$G(y) > 0, \quad \forall y.$$

A.7 Proof of Theorem 4.3

Let $G(y;z) = V_j(y;z) - V_i(y;z)$. Subtracting the HJB equation for i from that for j, I have

$$0 = (\mu_j - \mu_i)yV_{jy} + \lambda 1_{y<k_j(z)}(1 - V_j) - \lambda 1_{y<k_i(z)}(1 - V_i) - (\rho + \phi + \theta m(y,z))G + \mu_i y G_y + \frac{\sigma^2}{2}y^2 G_{yy}. \quad (41)$$

First, observe that $(\mu_j - \mu_i)yV_{jy} > 0$ due to state monotonicity. Then, from now on I can apply exactly the same argument used in Theorem 4.2 to prove the desired result. Let me omit the details.

A.8 Proof of Theorem 4.4

Suppose that arbitrary z_1 and z_2 such that $z_1 \leq z_2$ are given. But, I will use the following change of variables: $u = \log y$ and $w_i = \log z_i$ as before. Then, the individual creditor's problem becomes

$$0 = r_i + \phi_i f_{\phi,i} + \theta_i m(u,w)(f_{\theta,i} - V_i) + \lambda \max\{1 - V_i, 0\} - (\rho_i + \phi_i)V_i + (\mu_i - \frac{\sigma_i^2}{2})V_{iu} + \frac{\sigma_i^2}{2}V_{iuu}, \quad (42)$$

where

$$m(u,w) = \lambda(\xi 1_{u<w_1} + (1-\xi)1_{u<w_2}).$$

Differentiating (42) with respect to ξ, I have

$$0 = -\theta_i 1_{w_1 \leq u < w_2}(f_{\theta,i} - V_i) - (\rho_i + \phi_i + \theta_i m(u,w) + \lambda 1_{u<k_i(w)})V_{i\xi} + (\mu_i - \frac{\sigma_i^2}{2})V_{i\xi u} + \frac{\sigma_i^2}{2}V_{i\xi uu}.$$

Here, since $\lambda_1 = \lambda_2 = \lambda$ and r_i, $f_{\phi,i}$, and $f_{\theta,i}$ are all independent of ξ, it is obvious to see that

$$\lim_{u\to\pm\infty} V_{i\xi}(u) = 0.$$

Then, as the flow term $-\theta_i 1_{w_1 \leq u < w_2}(f_{\theta,i} - V_i)$ is positive, the comparison principle implies

$$V_{i\xi}(u) \geq 0, \quad \forall u.$$

Together with state monotonicity, I have

$$k_{i\xi}(w;\xi) \leq 0, \quad \forall w \text{ such that } w_1 \leq w_2. \quad (43)$$

Now recall that in equilibrium, it holds

$$k_i(w_1^*(\xi), w_2^*(\xi); \xi) = w_i^*(\xi).$$

Differentiating it with respect to ξ, I have

$$\begin{bmatrix} 1 - k_{1w_1} & -k_{1w_2} \\ -k_{2w_1} & 1 - k_{2w_2} \end{bmatrix} \begin{bmatrix} w_1^{*\prime}(\xi) \\ w_2^{*\prime}(\xi) \end{bmatrix} = \begin{bmatrix} k_{1\xi} \\ k_{2\xi} \end{bmatrix}.$$

Then from Corollary 3.5, Theorem 3.6, and inequality (43), it is straightforward to see

$$w_i^{*\prime}(\xi) \leq 0, \quad \forall i.$$

This result certainly implies

$$z_i^{*\prime}(\xi) \leq 0, \quad \forall i.$$

A.9 Discussions on Parameter Restrictions

I will prove boundary conditions from (9) to (12) hold whenever (24) through (27) holds true. Since I used the log scale in section 2.2, I need to replace V_{iu} and V_{iwj} by yV_{iy} and $z_jV_{iz_j}$, respectively, in (12) to (12). First, for convenience, let me write the HJB equation for $V_i(y;m)$, where $m \in \mathcal{M}$, again below.

$$0 = r_i + \phi\min\{y,1\} + \theta m(y)(\min\{L+ly,1\} - V_i(y)) + \lambda_i\max\{1-V_i(y),0\} - (\rho+\phi)V_i(y) + \mu y V_{iy} + \frac{\sigma^2}{2}y^2V_{iyy}.$$

Assuming the debt value does not blow up at $y=0$ and ∞, one can easily check that

$$\lim_{y\to 0} V_i(y;m) = \frac{r_i + \theta m_0 L + \lambda_i}{\rho + \phi + \theta m_0 + \lambda_i} \quad \text{and} \quad \lim_{y\to\infty} V_i(y;m_\infty) = \frac{r_i + \phi + \theta m_\infty}{\rho + \phi + \theta m_\infty},$$

where $m_0 = m(0)$ and $m_\infty = \lim_{y\to\infty} m(y)$. For $m \equiv 0$ and $m \equiv \Lambda$, I have

$$\lim_{y\to 0} V_i(y;0) = \frac{r_i + \lambda_i}{\rho + \phi + \lambda_i} < 1,$$

$$\lim_{y\to\infty} V_i(y;\Lambda) = \frac{r_i + \phi + \theta\Lambda}{\rho + \phi + \theta\Lambda} > 1.$$

So, condition (9) is satisfied.

Now consider m_1 and $m_2 \in \mathcal{M}$ such that $m_1(y) \le m_2(y)$ for all y. Let $m_{i0} = m_i(0)$ and $m_{i\infty} = \lim_{y\to\infty} m_i(y)$ for each i=1 and 2. Then, it is obvious to show

$$\lim_{y\to 0} V_i(y;m_1) = \frac{r_i + \theta m_{10} L + \lambda_i}{\rho + \phi + \theta m_{10} + \lambda_i} \ge \frac{r_i + \theta m_{20} L + \lambda_i}{\rho + \phi + \theta m_{20} + \lambda_i} = \lim_{y\to 0} V_i(y;m_2),$$

$$\lim_{y\to\infty} V_i(y;m_1) = \frac{r_i + \phi + \theta m_{1\infty}}{\rho + \phi + \theta m_{1\infty}} \ge \frac{r_i + \phi + \theta m_{2\infty}}{\rho + \phi + \theta m_{2\infty}} = \lim_{y\to\infty} V_i(y;m_2).$$

So condition (10) is satisfied.

With respect to conditions (11) and (12), for any $z \in (0,\infty)^N$, it is easy to see that for sufficiently small y it holds

$$V_i(y;z) = \frac{r_i + \theta\Lambda L + \lambda_i}{\rho + \phi + \theta\Lambda + \lambda_i} + \frac{\phi + \theta\Lambda l}{\rho + \phi + \theta\Lambda + \lambda_i - \mu} y + Ay^\eta$$

for some A and $\eta > 1$. Here, A depends on z but not y. Thus,

$$\lim_{y\to 0} yV_{iy}(y;z) = \lim_{y\to 0} z_jV_{iz_j}(y;z) = 0.$$

Similarly, for sufficiently large y,

$$V_i(y) = \frac{r_i + \phi}{\rho + \phi} + By^\gamma$$

for some B and $\gamma < 0$. Again, B depends on z but not y. Thus,

$$\lim_{y\to\infty} yV_{iy}(y;z) = \lim_{y\to\infty} z_jV_{iz_j}(y;z) = 0.$$

Hence, conditions (11) and (12) are satisfied.

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