

# Self-fulfilling Fire Sales: Fragility of Collateralised Short-term Debt Markets

John C.F. Kuong\*

INSEAD

## Abstract

This paper shows that collateralised short-term debt, although privately optimal for reducing borrowers' moral hazard, can cause fragility (multiple equilibria) when the collateral market is illiquid. A new form of coordination failure between borrowers' ex ante margin and risk-taking decisions engenders a systemic run in the collateralised debt market: large changes in credit rationing, margins, repo spreads, etc. The model also captures the large (small) cross-sectional differences between safe and risky collateral in bad (good) times. Finally, I show that asset price guarantees could improve welfare and promote stability but repealing repo contracts' "automatic stay" exemption might do the opposite.

---

\*I am grateful to Ulf Axelson, the late Sudipto Bhattacharya, Amil Dasgupta, Jason Donaldson, James Dow, Daniel Ferreira, Denis Gromb, Enisse Kharroubi, Kebin Ma, Igor Makarov, Bill Megginson, Uday Rajan, Dimitri Vayanos, David Webb, Kathy Yuan and Jing Zeng for useful comments and suggestions. All errors are mine. Corresponding email: john.kuong@insead.edu

# 1 Introduction

Financial firms' reliance on collateralised short-term funding, such as repurchase agreements (repo), is a prominent feature of the modern financial system but also a source of its fragility.<sup>1</sup> These secured loans in the so-called shadow banking system are in normal times nearly always rolled over automatically. Yet the 2008 financial crisis showed that these funding markets are not immune to “systemic runs”, whereby creditors collectively demand tougher borrowing terms or withdraw funding; the consequences are significant distress to the firms and sizeable liquidation of collateral assets at a discount. This phenomenon is commonly known as a fire sale.<sup>2</sup>

However, the apparent systemic runs in certain collateralised debt markets cannot be readily explained by classical bank-run models (e.g., [Diamond and Dybvig \(1983\)](#)) because the nature of bank debt is different. For example, the first-come-first-served nature of deposit contracts—which motivates depositors to front-run each other—is absent in repo contracts. As [Gorton \(2012, p. 2\)](#) concisely points out:

we know that crises are exits from bank debt. . . In this form of money (repo), each “depositor” receives a bond as collateral. There is no common pool of assets on which bank debt holders have a claim. So, strategic considerations about coordinating with other agents do not arise. This is a challenge for theory and raises issues concerning notions of liquidity and collateral, and generally of the design of trading securities—private money.

This paper can be viewed as a response to that challenge. It proposes a new form of coordination failure between *firms*—at the ex ante contracting stage—due to feedback between the risk-taking incentives of firms and the fire sales of collateral. Under certain conditions, self-fulfilling fire sales and systemic runs can arise.

I present a three-date, competitive equilibrium model of a continuum of firms— each

---

<sup>1</sup>[Adrian and Shin \(2011\)](#) call this a “market-based financial system”. See [Brunnermeier \(2009\)](#) and [Krishnamurthy \(2010b\)](#) for detailed reports on the use of repo and asset-backed commercial paper and on how these markets collapsed during the 2007–2009 global financial crisis.

<sup>2</sup>[Shleifer and Vishny \(2011\)](#) survey fire sales in the finance and macroeconomic literatures. [He et al. \(2010\)](#) show empirically that, from 2007Q4 to 2009Q1, hedge funds and broker dealers reduced holdings of securitised assets by \$800 billion; these assets were absorbed mainly by commercial banks (\$550 billion) and the government (\$350 billion). In terms of liabilities, repo finance shrank by \$1.5 trillion.

matched with a creditor — and an outside collateral buyer. Each firm is endowed with a divisible “asset in place” that pays a risky dividend at  $t = 2$ . This asset can be used as collateral to finance an independent, illiquid fixed-sized investment project that becomes successful with some probability and pays a verifiable cash flow at  $t = 2$ .<sup>3</sup> Firms are subject to moral hazard problems that at  $t = 0$ , after borrowing bilaterally from its creditor, each firm privately chooses the success probability of its project by incurring a non-pecuniary effort cost.<sup>4</sup> Pledging collateral to creditors lowers debt yields and thus mitigates firms’ incentives to shirk; equivalently, take on excessive project default risk in the classic [Stiglitz and Weiss \(1981\)](#) manner. The creditor is averse to the systematic risk associated with the dividend of the collateral asset and so effectively values the collateral less than both the firm and the collateral buyer do. She will seize and liquidate the collateral in a secondary market at  $t = 1$  when she knows her firm is insolvent. Finally, the secondary market for collateral is competitive yet the outside buyer is capital constrained; hence the market-clearing price of the collateral is decreasing in the amount of collateral liquidated.<sup>5</sup>

The key novelty of this paper is the feedback between firms’ moral hazard problems and the equilibrium collateral liquidation values, which generates a self-fulfilling fire-sale phenomenon. When agents expect a lower liquidation value ex post, creditors require a higher debt yield in order to break even. Firms must pledge more collateral, or initial margins, if they are to maintain incentives; when there is not enough collateral, they engage in more risk taking. In aggregate, both more pledged collateral and more defaults of firms lead to more collateral being liquidated in the market, resulting in a larger fire-sale discount ex post. Thus the anticipation of fire sales *causes* fire sales. [Figure 1](#) illustrates the dynamics at play.

The feedback just described can be strong enough to produce multiple rational expectation

---

<sup>3</sup>The model can be seen as a competitive equilibrium extension of the borrowers with non-project-related collateral model in [Tirole \(2006, Sec. 4.3.5\)](#) with multiple effort choices and a market for collateral.

<sup>4</sup>This moral hazard problem can also be modeled as risk-shifting in the [Jensen and Meckling \(1976\)](#) fashion, with an assumption that the cash flow difference between the safe project and the risky project in the case of success is not verifiable by the court. In that case, similar to [Acharya and Viswanathan \(2011\)](#), collateralised debt will emerge as the optimal contract.

<sup>5</sup>In this paper, as well as [Stein \(2012\)](#), the setup of these three classes of agents is exogenous. It aims to capture some stylised structure of the financial system by setting, respectively, firms as hedge funds/broker dealers, collateral buyer as the commercial banking sector, and creditors are money market funds. See [He et al. \(2010\)](#). A theory of endogenous co-existence of shadow banks and commercial banks is outside the scope of the paper. See [Plantin \(2014\)](#) for a model in which structured investment vehicles are set up by commercial banks to avoid tight capital requirement.

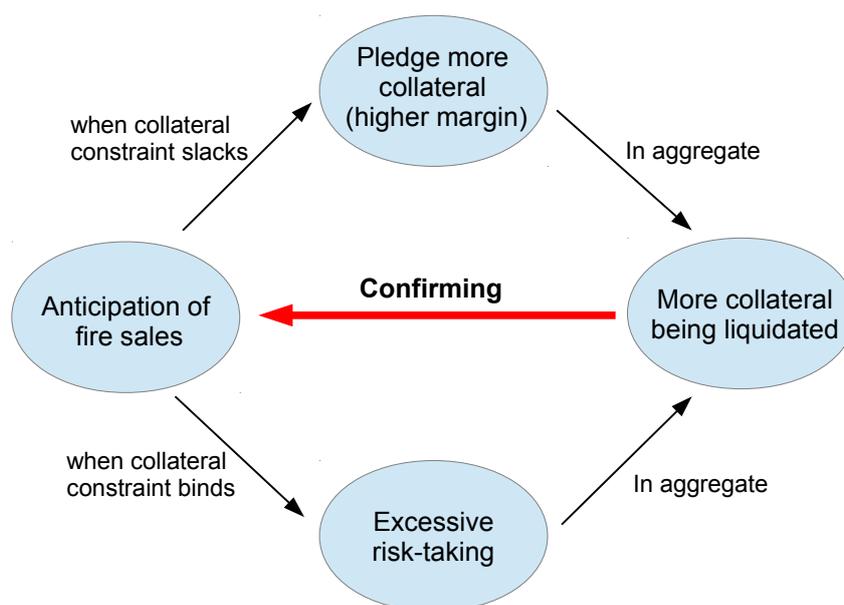


Figure 1: The self-fulfilling fire sale

equilibria featuring different collateral liquidation values. There are two (co-existing) channels through which multiple equilibria can arise. First, as mentioned previously, there is a threshold liquidation value below which there is not enough collateral to prevent risk taking. When the equilibrium liquidation value is just above this threshold, a pessimistic expectation that the liquidation value falling below this threshold triggers firms' risk taking. This results in a discrete jump in the amount of collateral liquidated as more firms default ex post, which in turn pushes the market-clearing collateral liquidation value below the threshold. I call this the *risk-taking channel*.

Self-fulfilling fire sales can also arise purely from firms' margin decisions. A lower expected collateral liquidation value requires firms to pledge more collateral; hence, in aggregate more collateral is supplied in the market even when firms' default risks remain unchanged. If the market-clearing price function is sensitive enough in the relevant range, then multiple equilibria emerge through this *margin channel*.<sup>6</sup>

<sup>6</sup>In a model with continuous effort choices, the risk-taking and the margin channels will operate *simultaneously*. It can be shown that the self-fulfilling fire sale is robust to such an extension. For a clear exposition of the distinct economic forces at play via these two channels, a model with multiple discrete effort choices is presented here.

To the best of my knowledge, the literature has not previously documented this self-fulfilling fragility with collateralised financing due to the feedback between endogenous risk taking and collateral fire sales in the *absence* of aggregate shock.<sup>7</sup> This mechanism generates a *systemic-run* phenomenon in the collateralised debt market that differs from those described by classic bank-run and financial market-run models. The source of fragility examined in this paper stems from a coordination failure between firms with their ex ante risk-taking and collateral margin decisions, as opposed to depositors' withdrawal decisions within a bank or traders' asset liquidation decisions in a market at the interim date under a de facto sequential service constraint.<sup>8</sup> The coordination failure here operates through the two channels just described: a firm that chooses either higher default risk or higher initial margin will thereby increase the expected amount of collateral liquidated in the market ex post. Given the limited liquidity in the secondary market, this extra supply of collateral marginally lowers the liquidation value; that reduction then tightens other firms' ex ante incentive constraints under rational expectation which requires them either to pledge more collateral or to take on excessive risk. As a result, firms' risk-taking and margin decisions in competitive equilibrium become *strategic complements* owing to the joint effect of the firms' incentive constraints and the fire-sale externality in the collateral market.

Although the model applies in general to any situation that involves multiple borrowing firms and a illiquid collateral market, the opaque operations of financial firms (e.g. hedge funds) and their reliance on collateralised borrowing make risk-taking concerns especially relevant.<sup>9</sup> In addition, the substantial and contemporaneous increase in debt yields, borrowers' counterparty risk, and collateral spreads during the recent crisis in wholesale funding markets is consistent with the model's proposed feedback mechanism between endogenous risk taking and collateral fire sales.<sup>10</sup>

---

<sup>7</sup>The margin channel here is similar to the margin spiral in [Brunnermeier and Pedersen \(2009\)](#) while in their model the margins are exogenous and the seed of fragility is an unanticipated, large aggregate shock on asset values.

<sup>8</sup>For instance, in [Morris and Shin \(2004\)](#) a market maker executes sellers' aggregate sell orders sequentially at decreasing prices and a seller's place in the queue for execution is randomly distributed. [He and Xiong \(2012\)](#) provides a recent dynamic bank run type model with coordination failure of roll-over decisions among asset-backed commercial paper holders.

<sup>9</sup>For evidence regarding risk-taking behavior of other financial firms, [Becker and Ivashina \(2014\)](#) and [Kacperczyk and Schnabl \(2013\)](#) document a 'reach-for-yield' phenomenon in insurance companies and money market mutual funds respectively.

<sup>10</sup>[Gorton and Metrick \(2012\)](#) and [Covitz et al. \(2013\)](#) find significant spikes and volatility in repo rates and ABCP yields in private-label asset-backed-securities markets during the recent crises which correlate positively

In terms of welfare and policy implications, equilibria with lower collateral liquidation values are less efficient—an outcome due to firms’ inefficient investment decisions, credit rationing, and the inefficient transfer of collateral from firms to creditors. The self-fulfilling nature of market fragility suggests that central banks can *reduce* firms’ risk-taking incentives, and thereby make the financial system more robust, through an ex ante commitment to intervening in the collateral market. Policies such as asset price guarantees can alter the agents’ pessimistic expectations and thus eliminate the inefficient equilibria. This proposal is in line with the idea that central banks should act as a “market maker of last resort” (Buiter and Sibert (2007)) to safeguard the proper functioning of certain key collateral and wholesale funding markets.<sup>11</sup>

The paper concludes with a discussion of possible unintended consequences of policies to limit post-default fire sales. In the U.S. when firms file for bankruptcy, an “automatic stay” provision prevents creditors from demanding repayments. Repo contracts in practice are usually exempted from automatic stay so that repo lenders can immediately access the collateral. Critics of this exemption from automatic stay have argued that it has precipitated fire sales of collateral during the recent crisis. Although this paper also features disorderly fire sales, I demonstrate that repealing the stay exemption may backfire. This is because, if repo debt is *not* exempted from these automatic stays, then defaulted firms can renegotiate with creditors ex post to lower the promised repayment amount of collateral by threatening to file for bankruptcy and thus delay the transfer of collateral. Since creditors value immediate access to and liquidation of the collateral, they would have to accept that offer. This renegotiation problem therefore reduces the amount of *credibly* pledgeable collateral and so worsens the firms’ ex ante moral hazard problem. In short, limiting post-default fire sales can actually exacerbate the ex ante risk-taking problem, leading to more fire sales ex post and to more drying up of debt markets collateralised by low-quality collateral.

**Related Literature** My paper is closely related to the recent literature on fragility and inefficiency of the collateralised debt market. [Martin et al. \(2014\)](#) build an infinite-horizon,

---

with proxies for counterparty risks such as the LIBOR-OIS spread.

<sup>11</sup>In a ‘longer term’ model with endogenous production of collateral, this asset price guarantee policy could encourage the over production of collateral with deteriorating quality. The usual moral hazard concern of government guarantee will hence kick in again. See [Acharya \(2009\)](#) and [Farhi and Tirole \(2012\)](#).

Diamond–Dybvig model with an asset market and characterise liquidity, collateral, and asset liquidation constraints under which in the steady-state banks can ward off an unexpected systemic run by depositors in all banks. This means that fragility in their model stems from an unanticipated aggregate shock to collateral value. In contrast, I show the anticipation of fire sales can interact with firms’ moral hazard problems to cause fragility.

Models of using collateral to mitigate borrowers’ moral hazard and adverse selection problems go back to [Chan and Thakor \(1987\)](#) and [Besanko and Thakor \(1987\)](#); See [Coco \(2000\)](#) for a survey. The main difference in my model is allowing the collateral fire-sale discount to be endogenous so that I can study the feedback between firms’ moral hazard and collateral fire sales. [Hombert \(2009\)](#) studies a similar feedback mechanism but assumes that solvency of firm is publicly observed at the interim date; that way, firms with successful projects can raise equity costlessly to purchase collateral from insolvent firms. In contrast to this paper, he shows that future fire sales discourage risk taking. I instead assume that a firm’s solvency is observed only by its creditor who has limited capital. This friction from information asymmetry is likely applicable to complex financial firms and can be micro-founded with, for instance, further moral hazard problem to operate newly acquired asset. The main result of this paper will survive provided that the cost of raising fresh capital at the interim date is not too small. [Acharya and Viswanathan \(2011\)](#) also study moral hazard and fire sales but focus on the ex ante endogenous entry decision of financial firms. They show that, because a benign credit environment in good times attracts more weak, highly leveraged financial firms to raise finance and invest, when a bad aggregate shock hits, these highly leveraged firms have to sell assets at discount to de-lever, deepening the illiquidity in the asset market. This paper studies the situation when financial firms (e.g. shadow banks) rely on an external collateral-buying sector (e.g. commercial banks), even if there is no aggregate uncertainty, fragility can still arise.

A recent paper by [Biais et al. \(2014\)](#) also studies the co-determination of optimal margins and equilibrium collateral prices in an insurance context. They consider a setting in which agents sign a contract to share risk before an aggregate uncertainty resolves; after the aggregate state is known, the insurance seller needs to exert private effort to mitigate his own default risk. The optimal contract in [Biais et al. \(2014\)](#) features variation margins call after the arrival of

bad news in order to maintain the insurance seller's incentive for exerting effort. They establish that there is excessive margining in any competitive equilibrium, and imposing a cap on margin can both eliminate the efficiency and restore equilibrium uniqueness. There are some important differences between [Biais et al. \(2014\)](#) and this paper. First, while aggregate uncertainty is crucial in their insurance motives, the main contribution of this paper is to demonstrate that even in the *absence* of aggregate uncertainty, a coordination failure between firms' initial margins and risk taking decisions can lead to aggregate risk and result in fragility endogenously. Second, this paper shows that, in addition to a similar destabilising margin result, fragility can also arise via a risk-taking channel. Therefore, a cap in margin does not necessarily reduce fragility.

My paper belongs to the literature on self-fulfilling financial crises. [Malherbe \(2014\)](#) shows how liquidity dry-up due to adverse selection can arise from the ex ante self-insurance motives to hoard liquidity. [Diamond and Rajan \(2005\)](#) demonstrate that, in a bad aggregate state, systemic failure in the banking system can arise because banks compete for deposits by raising their interest rates, which in turn leads to more bank failures and further liquidity shortage. In the context of financial market run, [Morris and Shin \(2004\)](#) show how loss-limit constraints on traders' positions can trigger coordinated liquidation.<sup>12</sup> My paper contributes to this literature by highlighting a new type of coordination failure resulting from firms' investment and contracting decisions.

The negative feedback spiral described here is similar in spirit to those in the literature on asset pricing with constraints. For example, [Brunnermeier and Pedersen \(2009\)](#) and [Danielsson et al. \(2011\)](#) establish the existence of an amplifying feedback loop between anticipated and realised asset price volatility when financial institutions operate under a value-at-risk constraint. [Gromb and Vayanos \(2002\)](#) and [Vayanos \(2004\)](#) study models with limits to arbitrage due to margin and agency constraint. Building on these insights, [Krishnamurthy \(2010a\)](#) proposes a policy of asset price guarantees to stabilise the asset market. Many of these papers take such constraints as given and focus on the consequences for asset pricing and portfolio allocation when an exogenous aggregate shock hits.<sup>13</sup> In contrast, this paper endogenises the collateralised

---

<sup>12</sup> For demand-deposit based bank runs models, see [Diamond and Dybvig \(1983\)](#) [Rochet and Vives \(2004\)](#), [Goldstein and Pauzner \(2005\)](#).

<sup>13</sup>For recent works on various aspects of repo market with exogenous margin constraints, [Infante \(2013\)](#) studies the effects of the exemption of automatic stay; [Lee \(2014\)](#) studies collateral circulation and shows that a feedback

debt contracts and margin constraints, and the source of risk comes from the endogenous risk-taking of firms.

This paper is also related to the vast literature on the consequences of short-term debt and asset fire sales. [Diamond and Rajan \(2011\)](#) demonstrate that distressed banks financed with deposits will “gamble for resurrection” and take the excessive risk of forced liquidation in the event of a future aggregate shock. Outside collateral buyers who anticipate this fire sale hoard liquidity for asset purchase, which results in reduced lending to the real sector. [Stein \(2012\)](#) assumes a money-like premium in lenders’ preferences for absolutely safe contracts and shows that firms tend to create too many safe assets by excessive short-term borrowing and fail to internalise the fire-sale externality when aggregate shocks hit. [Eisenbach \(2013\)](#) establishes that aggregate uncertainty distorts the disciplining effect of short-term debt, and creates inefficiency in both good and bad states. [Acharya et al. \(2011\)](#) show that the roll over risk of short-term debt can cause the credit market to freeze when bad news hits. My work complements this literature by showing that the expectation of fire sales can interact with borrowers’ risk-taking incentives to generate aggregate risk.

## **2 Model: Feedback between risk taking and fire sales**

This section begins with an overview of the model. I then proceed to analyse the firm–creditor contracting problem at the initial stage  $t = 0$  before describing creditors’ liquidation decisions and the collateral market at  $t = 1$ .

### **2.1 Overview of the model**

Consider a three-date ( $t = 0, 1, 2$ ) model with a continuum of borrowing firms, each matched with a corresponding creditor, and a representative outside collateral buyer. There is one good (i.e. money) in the economy and consumption takes place at  $t = 2$ . The risk-free rate of return is normalised to zero.

---

between repo spreads and collateral fire-sale discount can emerge; [Zhang \(2014\)](#) shows how contagion of illiquidity in banks’ balance sheets can occur in the bilateral repo market via borrowers’ defaults and repo haircuts.

**Firms and projects** Firms are risk neutral and identical ex ante. Each firm has a unit of asset in place (collateral) but does not have cash or debt. At  $t = 0$ , each firm has an opportunity to invest in a project requiring an initial investment of \$1 and returning, at  $t = 2$ , either a verifiable cash flow  $X$  in the event of success or  $X_f$  otherwise; without loss of generality,  $X_f$  is normalized to zero. Firms face a moral hazard problem of the effort provision type, as in [Holmström and Tirole \(1997\)](#). The project's likelihood of success depends on the unobservable effort exerted by the firm once the project has been financed. In effect, the firm can choose the success probability  $p_1 > p_2 > p_3$  of the project by incurring a private effort cost  $c(p_i) \geq 0$ . Shirking here is thus interpreted as (default-)risk taking. Project risk is idiosyncratic, so the realisation of projects is independent across firms.

**Collateral assets and financing** Aside from the investment opportunity, each firm has one divisible unit of an asset (e.g. financial securities) that pays a random, non negative dividend  $\tilde{v}$  with expected value  $v$  at  $t = 2$ . The dividend risk is uncorrelated with the project. The asset, which also is independent of the project's operation, can be used as collateral for borrowing. I assume that this collateral dividend  $\tilde{v}$  is non-verifiable to preclude further risk-sharing derivative contracts being written on the collateral. Therefore, the firm can effectively choose to pledge any  $k \in [0, 1]$  portion of the collateral to the creditor at the ex ante contracting stage while keeping the remainder  $(1 - k)$  unit beyond the creditor's reach. To fix ideas, one can imagine a shadow bank that can secretly move assets on and off the balance sheet unless they are explicitly pledged to the cash lenders. The flexibility to choose  $k$  is not crucial to this paper's main result, but it does allow me to endogenise the optimal amount of pledged collateral (initial margin) in the financing contract.

Firms borrow in the form of collateralised short-term debt contract. Specifically, a firm borrows \$1 from its creditor and promises either to repay  $r$  at  $t = 1$  or, failing that, to transfer immediately a portion  $k \in [0, 1]$  of the collateral to the creditor upon demand. This contract resembles a repurchase agreement (repo) as commonly used in practice, where  $r$  corresponds to the repo spreads and  $k$  to the initial margin. [Section 6](#) addresses the optimality of such a contract as well as its implementation.

I shall make the following assumptions about the project's net present value (NPV) and its extent of moral hazard:

**Assumption 1** (*Project's NPV and extent of moral hazard*) Define  $NPV_i \equiv p_i X - 1 - c(p_i)$ ,  $\Delta p_i \equiv p_i - p_{i+1}$ ,  $\Delta c_i \equiv c(p_i) - c(p_{i+1})$ , and  $A_i \equiv 1 - p_i(X - \frac{\Delta c_i}{\Delta p_i})$  for  $i = 1, 2$

$$(i) \quad NPV_1 \geq NPV_2 > 0 > NPV_3$$

$$(ii) \quad A_1 > A_2 > 0 \text{ and}$$

$$(iii) \quad (1 - p_1)A_1 \leq (1 - p_2)A_2$$

Assumption 1 is there to preserve the efficiency ranking of actions and also allows risk taking to arise in equilibrium. Assumption 1(i) implies that prudent investment ( $p_1$ ) is the efficient action yet that risk taking ( $p_2$ ) is also profitable. Part (ii) and (iii) concern the magnitude of the moral hazard, i.e. the absolute and relative size of  $\frac{\Delta c_i}{\Delta p_i}$ .  $A_i$  is the value of collateral required to induce action  $p_i$  when the firm and creditor value the collateral equally and part (ii) implies that the project cannot be funded without collateral because the firm will choose the negative-NPV action ( $p_3$ ) after financing ( $A_i > 0$ ) and more collateral is needed to induce prudent investment ( $A_1 > A_2$ ).<sup>14</sup> Finally, the collateral is transferred to the creditor when the project fails with probability  $(1 - p_i)$  and part (iii) implies that the expected value of collateral lost is weakly lower in the case of prudent investment. Although losing the collateral to the creditor in the case of symmetric valuation is costless, (iii) ensures that  $p_1$  is always the preferred and efficient action — even if the creditor values the collateral less — because  $p_1$  entails a higher NPV and a smaller expected collateral loss.

**Creditors' rollover and collateral liquidation decision** At  $t = 0$ , each firm is matched with a creditor who has cash \$1 to lend. Subsequent to financing, at  $t = 1$  each creditor receives a private and non-contractible signal about the success or failure of her borrowing firm's project — that is, whether cash flow  $X$  or 0 will be realised at  $t = 2$ . I assume the signal is perfect

---

<sup>14</sup>To see why the project cannot be funded without external collateral, the minimum repayment to the creditor is  $r_i = 1/p_i$  if  $p_i$  is chosen. However, the firm would privately choose the negative NPV action  $p_3$  after financing as  $p_3(X - r_i) - c_3 > p_i(X - r_i) - c_i$  when  $A_i > 0$

and so the creditor essentially observes the solvency of the firm she financed. If the project has succeeded, the creditor is willing to roll over her short-term debt to  $t = 2$  at the yield  $r$  because she is certain of repayment. When the project fails, the creditor demands repayment and, since the insolvent firm cannot repay, the creditor seizes the collateral asset and may sell it on the market.<sup>15</sup> I assume that creditors value the collateral less than do the firms and the collateral buyer, inducing the creditors to sell the collateral at a discount.

**Assumption 2** *Creditors' expected utility derived from holding the collateral to  $t = 2$  is  $\underline{l} \leq v$  — that is, less than the firms' and the collateral buyer's valuation.*

Assumption 2 can be understood as creditors are more risk-averse to the collateral dividend than the firms and the collateral are.  $\underline{l}$  can be micro-founded as the creditors' certainty equivalent of the risky dividend. Hence creditors prefer selling the collateral on the market provided the equilibrium collateral price  $l$  exceeds  $\underline{l}$ . The wedge between the creditors' and the collateral buyer's valuation of the collateral ( $v - \underline{l}$ ) can be motivated by the creditors' lack of expertise in managing the systematic risk associated with the collateral or (indirect) holding cost stemming from tougher regulatory constraints on creditors.<sup>16</sup> So from an ex post perspective, fire sales are in fact an efficient transfer of collateral.

I will interpret  $\underline{l}$  as the *collateral quality*. For example, safe collateral such as U.S. Treasuries will have a high  $\underline{l}$  (close to  $v$ ) and the creditor can hold such collateral to maturity with minimal cost or limitation. In Section 5, I discuss how collateral quality affects fragility and amplifies risk in the model.

The assumptions of a perfect signal and ex post efficient fire sales rule out other sources of inefficiency highlighted in the literature, such as wrongful liquidations of solvent firms due to imperfect signal and a coordination failure between creditors that results in excessive liquidation of collateral as in [Bernardo and Welch \(2004\)](#), [Morris and Shin \(2004\)](#) and [Oehmke \(2014\)](#). These assumptions are not crucial to the paper but allow me to focus on the inefficiency resulting

---

<sup>15</sup>Rolling over the debt of a failed firm and receiving the collateral risky dividend at  $t = 2$  is a weakly dominated strategy for the creditor because seizing the collateral at  $t = 1$  gives her the option to sell the collateral in the market.

<sup>16</sup>For example, money market mutual funds are typical lenders in the wholesale funding markets and they are subject to the regulation of Rule 2a-7 of the Investment Company Act of 1940 on the amount of holdings of assets with particular rating and maturity. See [Kacperczyk and Schnabl \(2013\)](#).

from the coordination failure of firms' ex ante investment and contracting decisions, which is the main result of this paper.

**Collateral buyer and endogenous fire-sale discount** The final element in the model is illiquidity in the collateral market. At  $t = 1$ , there is a competitive risk-neutral outside investor who clears the collateral market. Yet, he has limited capital in the sense that, instead of holding cash to purchase the collateral at  $t = 1$ , he could have invested in a productive technology that has decreasing returns to scale and pays off at  $t = 2$ . I assume the output of this productive technology is non-verifiable and thus creditors cannot directly lend to the collateral buyer. Similar assumptions of a patient investor or outside liquidity provider can be found in [Diamond and Rajan \(2011\)](#), [Stein \(2012\)](#), and [Bolton et al. \(2011\)](#).

As a result, the market-clearing price for the collateral offered by the buyer at  $t = 1$ , denoted by  $L(\phi; \theta)$ , decreases with  $\phi$  the amount of collateral sold but increases with  $\theta$  the amount of the buyer's available capital. Further discussion on the properties and micro-foundation of the function  $L(\phi; \theta)$  are advanced in [Section 2.3](#). The amount of the collateral buyer's capital  $\theta$  is an exogenous parameter and common knowledge in the model; it is therefore *not* a source of aggregate risk.

A time-line summarising the sequence of events is given in the Appendix.

## 2.2 Firms' investment problems: From fire sales to risk taking

In this section, I analyse the ex ante contracting problem between a firm and its creditor at  $t = 0$  while taking the equilibrium collateral liquidation value  $l$  as given. Each firm offers a collateralised short-term debt contract to its creditor in order to raise \$1 for investing in a project. More specifically, the firm promises to repay  $r$  at  $t = 1$  and if the firm fails to repay upon the creditor's request, a portion  $k \in [0, 1]$  of the collateral asset is transferred to the creditor. Creditors prefer a contract of this type to a long-term debt because it gives them the option to sell the collateral on the market at  $t = 1$  for  $l$ , which could be greater than the utility  $\underline{l}$  derived from holding it and receiving the risky dividend at  $t = 2$ . After signing a contract  $\{r, k\}$ , the firm privately chooses the project's success probability to maximise its expected net payoff from

investing:

$$p(r, k) \equiv \operatorname{argmax}_{p \in \{p_1, p_2, p_3\}} p(X - r) - (1 - p)kv - c(p) \quad (1)$$

which is the expected residual cash flow from the project minus the expected loss of collateral and effort cost. The incentive-compatible (IC) action  $p(r, k)$  for a given contract can be expressed as follows:

$$(IC) \quad p(r, k) = \begin{cases} p_1 & \text{for } r \leq \bar{r}_1(k) \\ p_2 & \text{for } r \in (\bar{r}_1(k), \bar{r}_2(k)] \\ p_3 & \text{otherwise} \end{cases} \quad (2)$$

$$\text{where } \bar{r}_i(k) \equiv X - \frac{\Delta c_i}{\Delta p_i} + kv \quad \text{for } i = 1, 2 \quad (3)$$

Equation (2) shows that when the promised repayment  $r$ , or debt yield, is higher than a threshold  $\bar{r}_i(k)$ , the firm chooses to take more risk. Pledging more collateral (higher  $k$ ) increases such thresholds, as seen in Equation (3), and thus discourages risk taking because then the firm loses more collateral if the project fails. Note that, in equilibrium  $p_3$  would not be chosen because in that case investing is a negative-NPV action.

The contract offered must satisfy the creditor's participation constraint (PC). For a given equilibrium collateral liquidation value  $l$ , the creditor accepts the contract when

$$(PC) \quad \hat{p}r + (1 - \hat{p})kl \geq 1 \quad (4)$$

where  $\hat{p}$  is the creditor's conjectured project success probability. In the case of failure, the creditor receives the portion  $k$  of the collateral which is worth  $l \in [L, v]$  to her in equilibrium.

Knowing the firm's incentive compatibility constraint, the creditor can rationally anticipate the firm's risk-taking decision by looking at the contractual terms  $\{r, k\}$ . Thus the creditor's conjectured probability  $\hat{p}$  is always correct in equilibrium:

$$(RE) \quad \hat{p} = p(r, k) \quad (5)$$

Finally, since pledging collateral to invest risks losing the collateral, the firm will undertake the project only if the expected net payoff from investing is positive. This project-taking (PT) constraint can be written as

$$(PT) \quad U(l) \equiv \max_{\{r,k\}} p(r, k)(X - r) - (1 - p(r, k))kv - c(p(r, k)) \geq 0 \quad (6)$$

where  $U(l)$  is the maximised (indirect) net utility from investing, for a given equilibrium collateral liquidation value  $l$ , when the firm offers the optimal collateralised short-term debt contract  $\{r, k\}$ .

Formally, the firm offers a contract  $\{r, k\}$  to the creditor that solves the following optimisation problem:

$$\begin{aligned} & \max_{\{r,k\}} p(X - r) - (1 - p)kv - c(p) \\ & \text{subject to } (IC), (PC), (RE) \text{ and } (PT) \end{aligned}$$

where  $k \in [0, 1]$  and  $r \geq 0$ . Where there is no solution, the firm chooses not to invest in any project.

Before proceeding to the firm's optimal investment decision and financing contract, I shall state some parameter assumptions about the expected value of the collateral  $v$  and the NPV of risk taking to ensure the analysis to be relevant. I will discuss the role of these parameter restrictions after presenting Proposition 1; detailed derivations are given in the Appendix.

**Assumption 3** (*Parameter assumptions on  $v$  and the NPV of risk-taking*)

$$(i) \quad v \in (A_1, \bar{v}) \text{ where } \bar{v} = \frac{A_1}{1 - [(1 - p_1)(NPV_2)] / [(1 - p_2)(A_2 + NPV_2)]}$$

$$(ii) \quad NPV_2 \leq \min\{v - A_2, \frac{1 - p_2}{p_2} A_2\}$$

**Proposition 1** (*Fire sales induce either a higher margin or more risk taking*) *If Assumptions 1 and 3 hold, then there exist two critical values  $l_{CR}$  and  $l_{RT}$  where  $0 \leq l_{CR} < l_{RT} < v$  such that for any given equilibrium collateral liquidation value  $l$ , the firm's optimal investment decision  $p^*(l)$  and the corresponding contract  $\{r(l), k(l)\}$  are as follows.*

1. for  $l \in [l_{RT}, v]$ , the firm invests prudently ( $p^*(l) = p_1$ ) and promises debt yield  $r_1(l)$  and pledges  $k_1(l)$  fraction of the collateral;
2. for  $l \in (l_{CR}, l_{RT})$ , the firm engages in risk-taking and promises debt yield  $r_2(l)$  and pledges  $k_2(l)$  fraction of the collateral;
3. for  $l = l_{CR}$ , the firm engages in risk-taking with probability  $\lambda \in [0, 1]$  and forgoes the project with probability  $(1 - \lambda)$ ;
4. for  $l < l_{CR}$ , the firm forgoes the investment project ( $p^*(l) = \emptyset$ ) (credit rationing)

The optimal margin and debt yield are, respectively,

$$k_i(l) = \frac{1 - p_i(X - \Delta c_i / \Delta p_i)}{p_i v + (1 - p_i)l}, \quad r_i(l) = \bar{r}_i(k_i(l)) = X - \frac{\Delta c_i}{\Delta p_i} + k_i(l)v \quad (7)$$

the terms  $l_{CR}$  and  $l_{RT}$  are implicitly defined in  $U(l_{CR}) = 0$  and  $k_1(l_{RT}) = 1$ .

**Proof:** See Appendix.

Proposition 1 demonstrates the first half of the feedback loop in Figure 1: anticipation of a lower collateral liquidation value requires the firm to pledge more collateral or, when there is not enough collateral, take on excessive risk. The intuition behind this result is that pledging collateral is costly to the firm yet good for incentives, but there is a finite amount of collateral. In general, the firm can repay the creditor in the form of either collateral or future cash generated from the project; however, cash is the preferred option because in equilibrium the creditor values the collateral less than does the firm ( $l \leq v$ ). As shown by Equation (2), the maximum repayment the firm can promise without triggering risk taking is  $r = \bar{r}_1(k)$ , which increases with the amount of collateral pledged  $k$ . As such, in order to satisfy the creditor's participation constraint under a given liquidation value  $l$ , the minimal amount of collateral required to be pledged is  $k_1(l)$  which satisfies

$$p_1 \bar{r}_1(k_1(l)) + (1 - p_1)k_1(l)l = 1$$

where  $k_1(l)$  and  $r_1(l) = \bar{r}_1(k_1(l))$  are as defined in Equation (7). When the liquidation

value  $l$  decreases,  $k_1(l)$  must increase in order to preserve incentives and satisfy the creditor's participation constraint.

When the liquidation value is high ( $l \geq l_{RT}$ ), the firm can pledge enough collateral  $k_1(l) \leq 1$  to induce prudent investment. When  $l$  falls below  $l_{RT}$  (as implicitly defined in  $k_1(l_{RT}) = 1$ ), even pledging all the collateral cannot simultaneously satisfy the creditor's participation constraint and induce prudent investment. That is, the debt yield required for the creditor to break even under prudent investment is too high:

$$r = \frac{(1 - p_1)l}{p_1} > \bar{r}_1(1)$$

Consequently, if  $l < l_{RT}$ , risk taking  $p_2$  is the only feasible action. In this case, the firm promises a higher debt yield  $r_2(l)$  but must still pledge  $k_2(l) < 1$  portion of collateral in order to commit to not privately choose negative-NPV action  $p_3$  after financing.

As compared with prudent investment, risk taking involves a smaller NPV and a larger expected fire-sales cost (due to a higher default risk). Hence the firm would choose to forgo the investment when  $l$  is too low, which I interpret as credit rationing. To see this, note that the firm's maximised net payoff from investing is

$$U(l) = \underbrace{p^*(l)X - c(p^*(l)) - 1}_{\text{NPV from investment}} - \underbrace{(1 - p^*(l))k(l)(v - l)}_{\text{Expected fire-sale cost}} \quad (8)$$

which is decreasing in  $l$ . Hence there exists a  $l_{CR}$  such that the surplus generated from the project equals the expected loss from a fire sale of collateral:  $U(l_{CR}) = 0$ . If  $l < l_{CR}$ , The firm thus optimally forgoes the investment. Finally, the firm is indifferent between no investment and risk taking at  $l_{CR}$  and therefore plays a mixed strategy. The probability of taking on the project is denoted by  $\lambda \in [0, 1]$  where the value of  $\lambda$  is pinned down in the competitive equilibrium.

Let me now discuss the role of Assumption 3. Part (i) bounds the expected value of the collateral  $v \in (A_1, \bar{v})$  to allow both risk taking and prudent investment to arise in equilibrium. When  $v$  is low enough, there is insufficient collateral to implement prudent investment. In contrast, if  $v$  is high enough, the collateral constraint binds *after* risk taking becomes unprofitable ( $l_{RT} < l_{CR}$ ), which rules out the possibility of risk taking. Assumption 3 (ii) ensures that risk

taking is not *too* profitable, otherwise credit rationing will not occur even when the expected fire-sale cost is maximal.

The firm's optimal investment decision  $p^*(l)$  is summarized graphically in Figure 2.

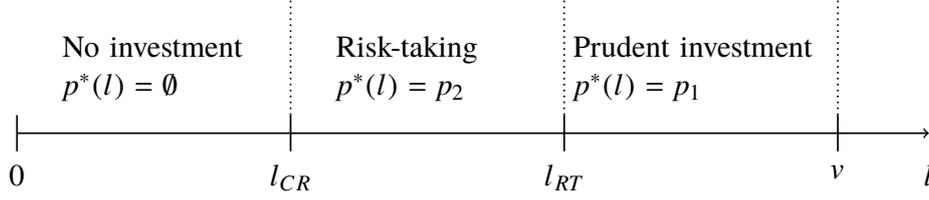


Figure 2: The firm's optimal investment decision at different collateral liquidation value  $l$ . Mixed strategies are played at the critical threshold  $l_{CR}$

### 2.3 Collateral market: From risk taking to fire sales

In this section I describe the supply of and demand for the repossessed collateral asset as well as how its market-clearing price is determined. There is a competitive collateral buyer with capital  $\theta \in [0, +\infty)$  who clears the collateral market at  $t = 1$ . At  $t = 0$ , this buyer also has an opportunity to invest in a productive technology with decreasing returns to scale that produces gross return  $F(\theta)$  at  $t = 2$ , where  $F(0) = 0$ ,  $F''(\theta) < 0$ ,  $\lim_{\theta \rightarrow 0^+} F'(\theta) \rightarrow +\infty$  and  $F'(\hat{\theta}) = 1$  for some  $\hat{\theta} > 0$ . Augmented with the storage technology which always returns 1, the investment opportunity gives  $F''(\theta) = 0$  and  $F'(\theta) = 1$  for  $\theta \geq \hat{\theta}$ . The output of this technology is assumed to be non-verifiable, so the buyer therefore cannot raise capital from creditors.

These conditions imply that the buyer cannot hoard liquidity  $I$  for asset purchase at  $t = 1$  without forgoing some productive investment; thus liquidity carries a premium when  $\theta - I < \hat{\theta}$ . Because the buyer behaves competitively, he takes the collateral liquidation value  $l$  as given and optimally hoards liquidity  $I$  so as to maximise his net payoff:

$$\Pi(l) \equiv \max_{I \in [0, \theta]} F(\theta - I) + I \frac{v}{l} - \theta \quad (9)$$

where the first order condition is

$$F'(\theta - I^*) \geq \frac{v}{l} \quad \text{with strict equality for } I^* > 0 \quad (10)$$

That is, the marginal return of investing in the productive technology must equal that of collateral purchase should the buyer decide to participate in the collateral market. For any given amount  $\phi \in [0, 1]$  of liquidated collateral in the market at  $t = 1$ , the market-clearing condition requires that  $I^* = \phi l$ . Thus, for  $\phi > 0$ , we have  $I^* > 0$ , and after substituting  $\phi l$  into the first order condition, one can rewrite the liquidation value  $l$  as a function of  $\phi$  and  $\theta$ :  $L(\phi; \theta) \in (0, v]$ . The following lemma summarises the properties of this market-clearing collateral liquidation value function.

**Lemma 1** (*Market-clearing collateral pricing function  $L(\phi; \theta)$* ) For a given collateral supply  $\phi \in (0, 1]$  and the collateral buyer's capital  $\theta \in [0, +\infty)$ ,  $L(\phi; \theta)$  satisfies the following statements

$$(i) \quad \frac{\partial L}{\partial \phi} \leq 0$$

$$(ii) \quad \frac{\partial L}{\partial \theta} \geq 0$$

$$(iii) \quad \lim_{\phi \rightarrow 0} L(\phi; \theta) \rightarrow 0 \text{ and for } \theta \geq \hat{\theta} + v, L(\phi; \theta) = v.$$

and  $L(0; \theta)$  is any value  $\in [\frac{v}{F'(\theta)}, v]$ .

**Proof:** The statements follow directly from total differentiating of the first-order condition Equation (10) and then applying the definition of  $\hat{\theta}$  where  $F'(\hat{\theta}) = 1$  for  $\theta \geq \hat{\theta}$ . □

Lemma 1 states that the collateral's market-clearing price is continuous, decreasing in  $\phi$ , and increasing in  $\theta$ . If the collateral buyer has enough capital is abundant enough, the collateral is always liquidated at the fundamental value  $v$ ; when capital is scarce, however he refuses to buy any collateral at any positive price.

Alternatively, one can view the collateral buyer as a competitive, risk-averse market maker whose risk tolerance is captured by  $\theta$ . This setup is commonly used in the financial market-run literature (e.g. [Morris and Shin \(2004\)](#) and [Bernardo and Welch \(2004\)](#)). The interpretation of an outside buyer with a productive investment technology is used again only in the welfare analysis offered in Section 4.<sup>17</sup>

---

<sup>17</sup>In the case of a competitive risk-averse market maker, the collateral buyer always breaks even in any equilibrium and his payoff thus does not play a role in the welfare analysis.

Next, I shall explain how the supply  $\phi$  of the collateral asset is determined. At  $t = 0$ , the firms and creditors form a conjecture about the collateral liquidation value  $l$  and all firms adopt their investment strategy as in Proposition 1. Owing to the independence of project realisation and the mixed strategy probability  $\lambda$ , hence the measure of firms with failed projects is deterministic and the measure of collateral repossessed by the creditors is

$$\lambda(l)(1 - p^*(l))k(l)$$

this term is a function of how many firms undertaking investment, the probability of default on their projects, and the amount of collateral pledged to creditors. Because the hold-to-maturity value of the collateral is worth  $\underline{l}$  to the creditors, they prefer liquidating the collateral when the liquidation value  $l$  is greater than  $\underline{l}$ . Denote the probability of selling the collateral by  $s(l)$ . Then the measure of collateral supplied in the market,  $\phi$ , is summarised in the following lemma.

**Lemma 2** (*Supply of collateral is affected by expected liquidation value via firms' investment*)  
For a given conjectured liquidation value  $l$ , the measure of collateral being liquidated at  $t = 1$  is given by

$$\phi(l) = s(l)\lambda(l)(1 - p^*(l))k(l) \quad (11)$$

$$\text{where } \lambda(l) = \begin{cases} 0 & \text{for } l < l_{CR} \\ \text{any } \lambda \in [0, 1] & \text{for } l = l_{CR} \\ 1 & \text{for } l > l_{CR} \end{cases} ; s(l) = \begin{cases} 0 & \text{for } l < \underline{l} \\ \text{any } s \in [0, 1] & \text{for } l = \underline{l} \\ 1 & \text{for } l > \underline{l} \end{cases} \quad (12)$$

**Proof:** The claim follows from the preceding discussion.

Figure 3 shows how the supply of collateral depends on the conjectured liquidation value. If the liquidation value is strictly below  $l_{CR}$  or  $\underline{l}$ , there is no collateral liquidated because, respectively, no firm undertakes the investment project or because creditors prefer to hold the collateral to maturity. At  $l_{CR}$  and  $\underline{l}$ , firms are indifferent between investing or not and creditors are indifferent between selling the collateral or not. Therefore, at  $\max\{l_{CR}, \underline{l}\}$ , any amount in  $[0, (1 - p_2)k_2(\max\{l_{CR}, \underline{l}\})]$  of collateral could be supplied. Beyond this critical value, all firms

invest and all creditors choose to sell the asset; hence the supply of collateral is  $(1 - p^*(l))k(l)$  which is decreasing and convex in  $l$ . Finally, there is a discrete jump at  $l_{RT}$ ; at this level firms invest prudently and the resulting decrease in defaults reduces the supply of collateral.<sup>18</sup>

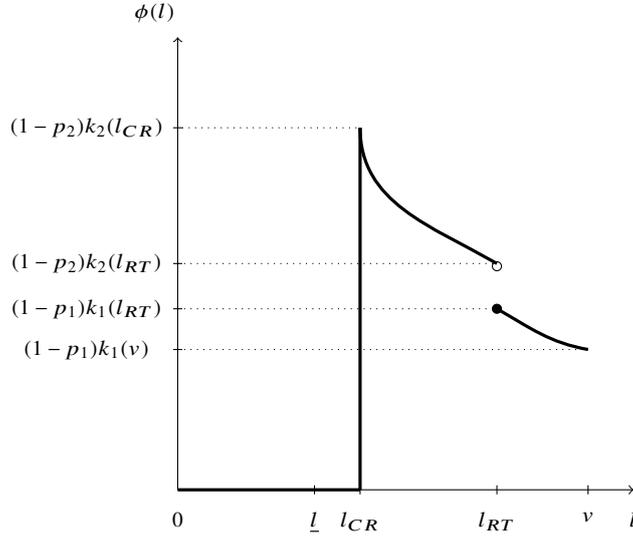


Figure 3: Supply of collateral asset  $\phi$  as a function of conjectured liquidation value  $l$

Since the market-clearing price of the collateral decreases with the amount of collateral supplied and since more collateral is supplied when firms engage in risk taking and pledge more collateral, the second half (and reverse direction) of the feedback loop in Figure 1 is completed: ex ante firms' risk-taking incentives deepen the fire-sale discount in the collateral market. As a result of this interdependence between moral hazard-type risk taking and collateral's equilibrium liquidation value, multiple rational expectation equilibria can arise. In the next section, I characterise these equilibria and discuss their implications for financial fragility.

### 3 Competitive equilibrium: Self-fulfilling fire sales

This section is devoted to characterising the equilibria and studying their features and implications for fragility.

**Definition 1** For any given amount of collateral buyer's available capital  $\theta \in [0, +\infty)$ , a symmetric, competitive rational expectation equilibrium consists of an equilibrium liquidation value  $\{l^*\}$  and mixed strategy probabilities  $\{s^*, \lambda^*\}$  such that

<sup>18</sup>The existence of the discrete jump,  $(1-p_1)k_1(l_{RT}) < (1-p_2)k_2(l_{RT})$ , is a consequence of Assumption 1(iii)

1. At  $t = 0$ , agents conjecture the equilibrium liquidation value to be  $l^*$ . Firms maximise their expected payoff by implementing the optimal investment strategy  $p^*(l^*)$  and offering the optimal contract  $\{r(l^*), k(l^*)\}$  as in Proposition 1;
2. At  $t = 1$ , creditors of insolvent firms seize the collateral and supply an amount  $\phi(l^*)$  of collateral in the market, as in Lemma 2;
3. The buyer with available capital  $\theta$  clears the collateral market at the market-clearing price  $L(\phi(l^*); \theta)$ ;
4. In equilibrium, agents' expectation of collateral liquidation value is correct; that is,  $l^* = L(\phi(l^*); \theta)$ .

I prove the existence of this rational expectation equilibrium in the next lemma.

**Lemma 3** (*Existence of equilibria*) For any  $\theta \in [0, +\infty)$ , there exists at least one equilibrium collateral liquidation value  $l^*$  that satisfies the following equation:

$$l^* = L(s(l^*)\lambda(l^*)(1 - p(l^*))k(l^*); \theta) \quad (13)$$

**Proof:** See Appendix.

Lemma 3 guarantees that equilibrium exists under any amount of the collateral buyer's capital  $\theta$ . Yet, there could be more than one equilibrium collateral liquidation value  $l^*$  that satisfies Equation (13).<sup>19</sup> The next proposition discusses the main result of this paper: how the parameter  $\theta$  affects the uniqueness and multiplicity of equilibria.

**Proposition 2** (*Fragility and collateral buyer's capital  $\theta$* ) Under Assumptions 1–3 and for collateral with  $\underline{l} < l_{RT}$ , there exists two distinct values  $\underline{\theta}, \bar{\theta} \in (0, +\infty)$  such that the following statements hold.

1. For  $\theta \in [\bar{\theta}, +\infty)$ , a unique equilibrium exists in which all firms invest prudently and the equilibrium collateral liquidation value is relatively high,  $l^*(\theta) \geq l_{RT}$ .

---

<sup>19</sup>I disregard the potential continuum of equilibria in which the collateral market clears without any supply or demand of the collateral. These equilibria are exactly the same economically except with a different no-trade price.

2. For  $\theta \in [0, \underline{\theta}]$ , a unique equilibrium exists in which firms either engage in risk taking or forgo investment; in both cases,  $l^*(\theta) < l_{RT}$ .
3. For  $\theta \in (\underline{\theta}, \bar{\theta})$ , there exist multiple values of  $l^* \in [0, v]$  satisfying Equation (13) and so multiple rational expectation equilibria exist.

- (a) If  $l^*(\theta) = l_{CR}$ , then a fraction  $1 - \lambda^*(\theta)$  of the firms are credit rationed where  $\lambda^*(\theta) \in [0, 1]$  uniquely satisfies

$$L(\lambda^*(\theta)(1 - p_2)k_2(l_{CR}); \theta) = l_{CR} \quad (14)$$

and complete credit rationing occurs for  $\theta$  such that  $L(0; \theta) \leq l_{CR}$ .

- (b) If  $l^*(\theta) = \underline{l}$ , then all firms are financed and a fraction  $1 - s^*(\theta)$  of the creditors in insolvent firms do not sell the collateral in the market and hold it to maturity where  $s^*(\theta) \in [0, 1]$  uniquely satisfies

$$L(s^*(\theta)(1 - p_2)k_2(\underline{l}); \theta) = \underline{l} \quad (15)$$

and no collateral is traded for  $\theta$  such that  $L(0; \theta) \leq \underline{l}$

**Proof:** See Appendix.

**Unique equilibrium under extreme  $\theta$**  Figure 4 plots the indirect collateral liquidation value function  $L(\phi(l))$  and the collateral liquidation value  $l$  against  $l$  itself. An intersection of these two graphs therefore constitutes an equilibrium (a fixed point  $l$  in Equation (13)). Figure 4 shows the two cases of unique equilibrium. Intuitively, when  $\theta$  is large, the competitive collateral buyer has sufficient capital to clear the market at a relatively high price. Consequently, even when all agents in the market are pessimistic that the collateral will be liquidated at a low price, and hence believe firms will take on excessive risk and the amount of collateral liquidated will be large, those beliefs are not vindicated in equilibrium because the collateral buyer has enough capital to clear the market at a price higher than the anticipated one. The same logic applies to the opposite case with  $\theta \leq \underline{\theta}$ . As a result, there can only be one equilibrium.

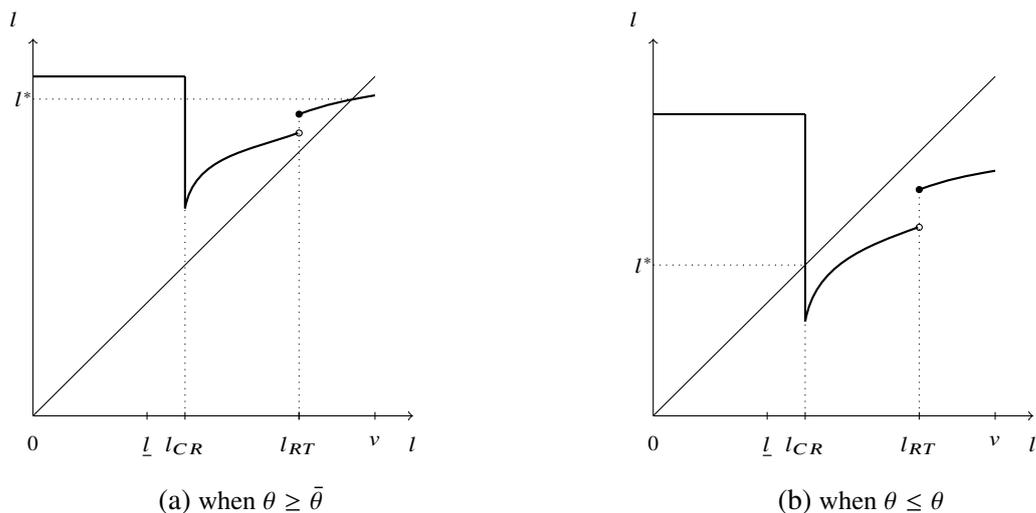


Figure 4: Cases of a unique equilibrium under extreme values of  $\theta$

If one interprets the amount of the collateral buyer's capital as a proxy for the strength of the aggregate economy, Proposition 2 suggests that the shadow banking system is pro-cyclical even when the fundamental value of the collateral ( $v$ ) and firms' investment profitability ( $pX - 1 - c$ ) do not correlate with  $\theta$ . In a capital-abundant (good) period ( $\theta \geq \bar{\theta}$ ), firms have low default risks, investment returns are high, the amount of credit granted by creditors to firms and by the collateral buyer to the real economy is large, debt yields are low and the collateral liquidation discount is small. In contrast, in a capital-constrained (bad) period firms are stuck in an equilibrium with high default risks, low returns, high borrowing costs, credit rationing and a large volume of collateral being liquidated at a substantial discount. This pro-cyclicality is explained by the effect that the aggregate capital available for collateral has on the liquidation value of that collateral, which in turn affects the firms' moral hazard problem. Thus the moral hazard problem becomes more severe in bad times, creating non-linear amplification in the system.

**Multiple equilibria and fragility** When the collateral buyer's capital is between the extreme amount  $\underline{\theta}$  and  $\bar{\theta}$ , the collateral's market-clearing price becomes more sensitive to changes in the amount of collateral being liquidated. In this case, there exist multiple rational expectation equilibria.

The multiple equilibria arise via two channels as shown in Figure 5.<sup>20</sup> The first of these is the *risk-taking channel* which is the case for switching equilibrium liquidation value from  $l_1^*$  to  $l_2^*$  where

$$l_1^* = L((1 - p_1)k_1(l_1^*); \theta) \geq l_{RT} > L((1 - p_2)k_2(l_2^*); \theta) = l_2^*$$

When the anticipated liquidation value changes from  $l_1^*$  to  $l_2^*$ , there is not enough collateral to maintain incentives at  $l_2^*$ ; that is,  $k_1(l_2^*) > 1$ . Hence firms thus can engage only in risk taking, which results in more defaults and a jump in the amount of collateral being liquidated,  $(1 - p_2)k_2(l_2^*) < (1 - p_1)k_1(l_1^*)$  — thus confirming the anticipated lower liquidation value  $l_2^*$ . This risk taking leads to market fragility when  $\theta$  is in the range for which  $l^*$  is sufficiently close to the risk-taking threshold  $l_{RT}$  and the discrete jump in market-clearing price leads to one equilibrium liquidation value above  $l_{RT}$  and the other below  $l_{RT}$ . Given that  $L(\phi(l); \theta)$  is continuously increasing in  $\theta$  from 0 to  $v$  for any given  $l$ , this range of  $\theta$  always exists, irrespective of the market-clearing price function's curvature or elasticity.

Multiple equilibria can arise also from a *margin channel*, as in the case of a change from  $l_2^*$  to  $l_3^*$  — where both values are below  $l_{RT}$  and thus the firms' default risks are the same. Note also that in this case there is some credit rationing in the equilibrium with  $l_3^*$ ; that is,

$$l_2^* = L((1 - p_2)k_2(l_2^*); \theta) > L(\lambda^*(\theta)(1 - p_2)k_2(l_3^*); \theta) = l_3^*$$

and  $\lambda^*(\theta) \in (0, 1)$ . When the anticipated liquidation decreases from  $l_2^*$  to  $l_3^*$ , firms have to pledge more collateral  $k_2(l_2^*) > k_2(l_3^*)$  to satisfy their own incentive constraints and their creditors' break even constraints. Hence more collateral is liquidated and, when the market-clearing price function is sensitive enough in the relevant range, the increase in collateral supply pushes the equilibrium liquidation value to  $l_3^*$ .

Both types of multiple equilibria discussed here are *self-fulfilling* and feature large variations in collateral asset prices, debt yields, the amount of credit rationed, and firms' profitability. There are also some differences in these two channels. Multiple equilibria that follow from risk taking have significant variations in firms' default risk but the change in margins is ambiguous

---

<sup>20</sup>I focus the discussion on stable equilibria only

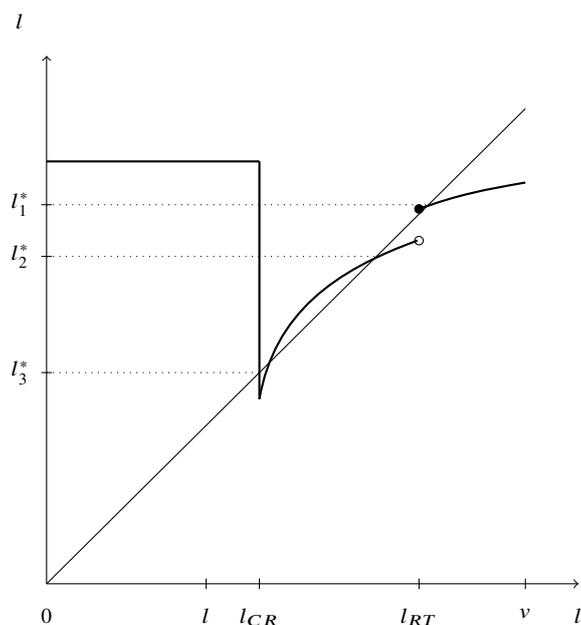


Figure 5: Multiple (stable) equilibria via different channels. *Risk-taking channel*:  $l_1^*$  to  $l_2^*$ ; *Margin channel*:  $l_2^*$  to  $l_3^*$

( $k_1(l_1^*) - k_2(l_2^*)$  can be positive or negative). Meanwhile, fragility via that margin channel leads to large changes in initial margins yet leaves firms' default risks unchanged. The different effect on margins from the two channels can help account for the mixed empirical findings on the behaviour of repo haircuts during the 2007–2009 subprime crisis. [Copeland et al. \(2014\)](#) and [Krishnamurthy et al. \(2012\)](#) report small variations in haircuts in the tri-party repo market, while [Gorton and Metrick \(2012\)](#) documented a substantial increase of haircut in the bilateral repo market.<sup>21</sup>

To conclude this section, Figure 6 summarises how the collateral buyer's capital affects the equilibrium characteristics and fragility of a collateral-based financial system.

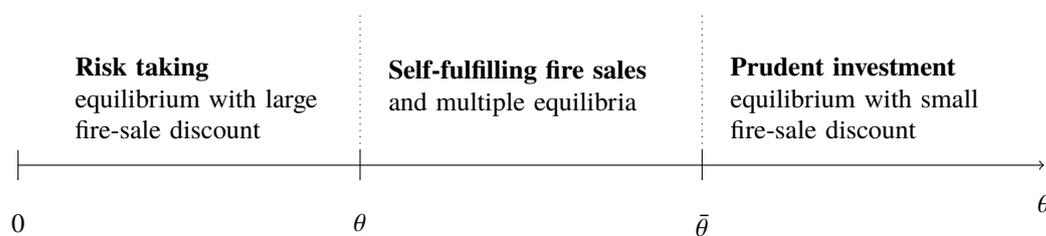


Figure 6: Characterisation of equilibria under various exogenous amounts  $\theta$  of collateral buyer's capital.

<sup>21</sup>[Copeland et al. \(2014\)](#) also finds that lenders in the tri-party repo market are more likely to withdraw funding than to increase haircuts to reduce risk exposure. Credit rationing in this model is analogous to fund withdrawal.

## 4 Welfare and policy implications: The case for central banks as market makers of last resort

In this section, I first discuss the welfare implications of multiple equilibria and show that an equilibrium with a lower collateral liquidation value is less efficient. Then I argue that this inefficiency creates a role for a social planner, or a central bank in this context, to intervene in and stabilize the collateral market thereby improving welfare. This role corresponds closely to the notion of “market maker of Last Resort” proposed by various academics and commentators, including Willem Buiter and Anne Sibert (see [Buiter and Sibert \(2007\)](#); [Buiter \(2012\)](#)).

I assume that the social planner’s objective is to maximise the total net utility of all agents. Creditors always break even in equilibrium so the social welfare function  $W(l^*)$  is defined as the sum of the net payoffs of the firms  $U(l^*)$  and the collateral buyer  $\Pi(l^*)$ , in equilibrium with collateral liquidation value  $l^*$ :<sup>22</sup>

$$W(l^*) = U(l^*) + \Pi(l^*) \quad (16)$$

where  $U(l^*)$  and  $\Pi(l^*)$  are as defined in Equations (8) and (9). Consider a collateral asset with quality  $\underline{l} < l_{RT}$  in a state  $\theta$  where multiple equilibria exist. The following proposition establishes that the equilibria with lower  $l^*$  are associated with lower social welfare.

**Proposition 3** (*Inefficiency*) *When multiple equilibria exist, social welfare  $W(l^*)$  is greater in the equilibrium with a higher  $l^*$ .*

**Proof:** *See Appendix.*

Let’s compare two equilibria with  $l_1^* > l_2^*$ . There are four potential sources of welfare loss in the equilibrium with  $l_2^*$ : (i) the crowding-out effect on the collateral buyer’s investment in productive technology, (ii) the inefficiency resulting from the firms’ risk-taking decision when  $l_1^* \geq l_{RT} > l_2^*$ , (iii) the credit rationing with regard to the firms’ positive-NPV investment when  $l_2^* \leq l_{CR}$ , and (iv) the creditors’ disutility from holding the collateral to maturity when  $l_2^* \leq \underline{l}$ .

---

<sup>22</sup>If the collateral buyer is alternatively modelled as a competitive risk-averse market maker, as suggested in Section 2.3, then the equilibria can be Pareto-ranked as both creditors and the collateral buyer always break even. Only firms have higher payoff in the equilibrium with a higher collateral value.

The self-fulfilling fragility and the inefficiency associated with the equilibria characterised by lower liquidation value call for welfare-improving policy intervention. In particular, a central bank can coordinate agents into the efficient equilibrium by committing to buy any amount of collateral at a certain price. Such a policy of asset price guarantees can alter agents' pessimistic (yet rational) expectations, thus precluding the inefficient equilibria.

**Asset Price Guarantee** Recall that there are two classes of multiple equilibria that can arise: one involves risk taking and the other acts through the change in margins. Consider the risk-taking case with two equilibrium liquidation values  $l_1^* \geq l_{RT} > l_2^*$ . If the social planner commits to buying any amount of collateral at a price  $l_{PG} \geq l_{RT}$ , the equilibrium with risk taking  $l_2^*$  ceases to exist. The reason is that, when agents know the collateral liquidation value will not fall below  $l_{RT}$ , firms can pledge enough collateral to induce prudent investment and thus no risk taking will occur in the first place.

For the case of multiple equilibria arising via the margin channel, there exists multiple  $l^*$  both below or above  $l_{RT}$ . Picking the equilibrium with the highest  $l^*$  requires only that the central bank to set the price guarantee  $l_{PG}$  strictly greater than the second highest  $l^*$ ; doing so eliminates all equilibria but the one with the highest  $l^*$ .

It is worth noting that, provided the price guarantee is strictly less than the highest  $l^*$ , the price guarantee facility will never be used because in equilibrium the price offered by the outside buyer is higher than that offered by the central bank. Thus the central bank can stabilise the market and improve welfare by simply promising to intervene. This is similar to the deposit insurance policy suggested in [Diamond and Dybvig \(1983\)](#).

As for the funding of an asset purchase programme, the central bank can issue bonds worth  $l_{PG}$  to finance the purchase— or, more accurately, give a riskless bond worth  $l_{PG}$  to creditors in exchange for collateral. These bonds could be backed by future taxes collected from the payoff of firms' projects. Note that firms cannot individually issue claims backed by the project to finance collateral purchase because of adverse selection: the creditors do not observe the solvency of other firms.

The credibility of such a commitment could be an issue in off-equilibrium because, at  $t = 1$ ,

the firms and collateral buyer have already made their investment decisions and the fire sale of collateral is simply a zero-sum transfer between the creditors and the buyer.<sup>23</sup> Thus the central bank has no interest in taxing and redistribution unless it assigns an increasingly greater weight to the welfare of creditors than to that of the buyer ex post when the collateral liquidation value decreases.

Although this model is highly stylised and does not deal with the collective moral hazard problem (c.f. Acharya (2009) and Farhi and Tirole (2012)), it does provide an economic rationale for the central bank to play an active role in stabilising certain important collateral markets in order to prevent systemic runs. I summarise the discussion of this policy in the following proposition.

**Proposition 4** *When multiple equilibria exist, an asset price guarantee can eliminate the inefficient equilibria at no cost.*

*Proof:* This statement follows from the preceding discussion.

## 5 Collateral quality and fragility

The purpose of this section is to show how collateral quality affects fragility. I interpret creditors' hold-to-maturity utility  $\underline{l}$  for a particular class of collateral as that collateral's quality. If one considers a setting where creditors are more risk averse to the underlying risk of the collateral dividend than are the firms and the collateral buyer, a lower quality collateral is a riskier collateral. The analysis to follow can be viewed as a comparison of equilibria supported by collateral of two types with different qualities (e.g. U.S. Treasuries and private-label asset-backed securities) or, alternatively, a comparison of equilibria supported by the same class of collateral before and after receiving an exogenous shock to its fundamental risk, (e.g. mortgage-backed securities around the 2007 subprime mortgage crisis).

**Lower quality collateral breeds fragility** Collateral quality reflects the eagerness of creditors to liquidate. For a given state  $\theta$ , collateral of different qualities can have a different number

---

<sup>23</sup>Except the case with  $l^* = \underline{l}$  in which creditors have to inefficiently hold some collateral to maturity.

of equilibria. Figure 7 provides an example: for lower-quality collateral  $\underline{l}'$ , there exist two stable equilibria  $l_1^* > l_2^*$ , whereas collateral with higher quality  $\underline{l}''$  supports only the equilibrium with the higher liquidation value. This is because a creditor's reservation price for the higher-quality collateral is greater than the market-clearing price in the equilibrium with lower liquidation value equilibrium ( $\underline{l}' > l_2^*$ ). The following proposition generalises this argument that low-quality collateral breeds fragility. In other words, if multiple equilibria exist in state  $\theta$  when the collateral quality is  $\underline{l}$ , then they exist also for a lower-quality collateral  $\underline{l}' < \underline{l}$ .

**Proposition 5** (*Low-quality collateral breeds fragility*) Denote  $\Theta^M(\underline{l})$  as the set of  $\theta \in [0, +\infty)$  that permit multiple equilibria to exist when collateral quality is  $\underline{l}$ . Then the set  $\Theta^M(\underline{l})$  is non-expanding in  $\underline{l}$ .

**Proof:** See Appendix.

Proposition 5 can explain why the market for high-quality collateral like the U.S. Treasuries and agency bonds remained fairly stable during the crisis even as substantial variation was observed in the borrower costs and the amount of borrowing of debt backed by lower-quality collateral such as private-label ABS and corporate bonds.

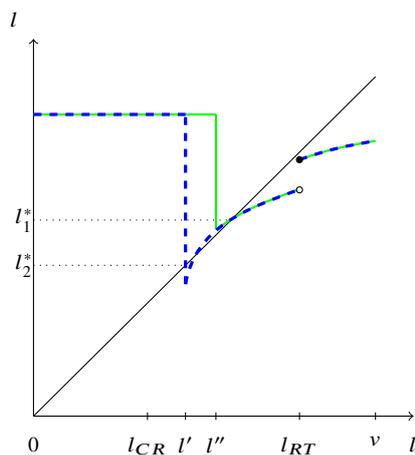


Figure 7: Fragility exists for lower-quality collateral (dashed blue plot) but not for higher-quality collateral (solid green plot) in state  $\theta'$ .

**Counter-cyclical credit spread** Another well-documented phenomenon during periods of economic distress is that the credit spreads between safe and relatively risky assets increase significantly. Consider again the two collateral assets discussed previously, with respective

reservation prices,  $\underline{l}'$  and  $\underline{l}''$ , but in the extreme states with unique equilibria. To make the comparison starker, let  $\underline{l}'' \geq l_{RT}$ . In the good state  $\theta \geq \bar{\theta}$ , there are minimal differences in terms of spreads and margins between a debt market backed by the higher-quality collateral and the one backed by the lower-quality collateral. This is because the competitive collateral-buying sector has abundant available for purchases; as a result, the difference in creditors' reservation prices for the two collateral assets does not matter at all in equilibrium.

The difference becomes apparent when the collateral-buying sector's capital is scarce; see Figure 8 for an illustration. The differences in quality are amplified by the existence of the moral hazard problem: the lower-quality collateral triggers risk-taking in the capital-constrained state, further compounding the problem of scarce capital. This result could explain why there are minimal spread and haircut differences for Treasuries and MBSs in capital-abundant periods whereas the two markets are markedly different during a crisis.

Along these lines, one can view the Federal Reserve's Large-Scale Asset Purchase program as injecting liquidity during the crisis and pushing the market from conditions depicted in the right panel to those in the left panel of Figure 8. That perspective suggests a new, moral hazard-based channel for interpreting the empirical results, reported in [Krishnamurthy and Vissing-Jorgensen \(2013\)](#), that the Fed's purchase of MBSs led to a much greater reduction in yields than did its purchase of Treasuries.

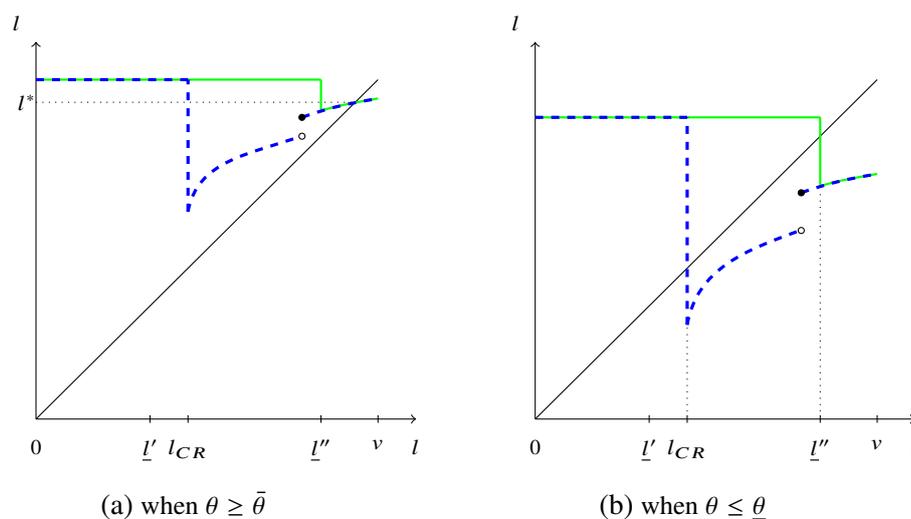


Figure 8: Spreads between the lower-quality (dashed blue plot) and higher-quality (solid green plot) collateral assets in (a) good time and (b) bad time respectively.

## 6 Repo as an optimal contract and the cost of automatic stay provisions

In Section 2.2 attention was restricted to collateralised short-term debt contracts with a promised repayment  $r$  and with a portion  $k$  of collateral being transferred to the lender at  $t = 1$  should the firm fail to repay as requested. This section addresses the optimality and implementation of such a contract. I show in particular that a repurchase agreement, with the exemption from automatic stay provision, can implement the optimal contract. The key friction here is that insolvent firms can threaten to file for bankruptcy protection and thus delay the transfer of collateral to creditors until  $t = 2$ . Such behaviour creates a hold-up problem similar to the one described in the literature on incomplete contracts (e.g. [Aghion and Bolton \(1992\)](#); [Hart and Moore \(1994\)](#); [Diamond and Rajan \(2001\)](#)).<sup>24</sup> I conclude this section with a discussion of the potential negative consequences of repealing the stay exemption.

A general contract consists of a pair of values  $\{r_s, k_s\}$  (resp.,  $\{r_f, k_f\}$ ) specifying cash repayment ( $r$ ) and the amount of collateral transfer ( $k$ ) in the event of project success (resp., failure). Timing of the payment is irrelevant for now as the information is fully revealed to both parties at  $t = 1$ . Recall that project cash flow is  $X$  and  $X_f$  when the project succeeds or fails respectively. The standard moral hazard result shows that  $k_s = 0$  and  $r_f = X_f$  are optimal. Intuitively, leaving some returns to the firm in the case of failure and giving collateral to the lender in the case of success worsen the incentive problems. Thus the optimal contract will be a debt contract with promised repayment  $r = r_s \geq r_f$  and with the portion  $k = k_f$  of collateral given to the lender if the project fails.

Furthermore, the firm prefers committing to transfer collateral to the lender at  $t = 1$  because that transfer allows the lender to liquidate the collateral in the market for price  $l^*$ , which is greater than  $\underline{l}$  (the lender's valuation of the collateral at maturity  $t = 2$ ). Improving the lender's payoff in the case of failure allows the firm to promise less repayment, relaxes the incentive constraint, and increases the firm's payoff.

---

<sup>24</sup>The analysis of the optimal contract here is also similar to that in [Acharya and Viswanathan \(2011\)](#) with a difference that the hold-up problem there is caused by borrower's ex post asset-substitution problem; assigning control rights to the lender can thus solve the problem.

I now address the implementation of the optimal contract. First note that, because the creditor's signal about her debtor's solvency is non-contractible, the court cannot enforce any payment that is contingent on the signal. That means that the collateralised debt contract must be "demandable" at  $t = 1$ . However, a general secured short-term debt contract will not be enough if the firm files for bankruptcy protection at  $t = 1$  and can then delay liquidation until  $t = 2$ . Formally, I make the following assumption.

**Assumption 4** (*Time-consuming bankruptcy and liquidation procedure*) Suppose that a firm files for bankruptcy protection at  $t = 1$ . Then the court needs time to verify the firm's bankruptcy and liquidate its assets, so the repayment to creditors cannot be made until  $t = 2$ .

Assumption 4 is broadly in line with the automatic stay provision in the United States that inhibits creditors from collecting debt after a firm has filed for Chapter 11 bankruptcy protection.<sup>25</sup> In practice, the bankruptcy and liquidation of complex securities firms is both time consuming and costly.<sup>26</sup> In the context of this paper, bankruptcy is costly because the collateral is worth only  $\underline{l}$  to creditors  $t = 2$  (because of their aversion to the collateral dividend risk). So at  $t = 1$ , once the firm fails to repay as requested, it can threaten to file for bankruptcy and make a take-it-or-leave-it offer to the lender: immediate transfer of  $k' \leq k$  units of collateral such that  $k'l^* = k\underline{l}$ . In other words, the firm ex ante cannot credibly commit to transfer  $k$  units of collateral to the creditor at  $t = 1$  in the event of insolvency.

The source of this renegotiation problem is delay of the liquidation procedure. Therefore, a short-term repurchase agreement with the exemption from automatic stay provisions avoids this problem by allowing the repo lender to seize the collateral immediately when the borrower defaults.<sup>27</sup> The following proposition summarises these ideas.

---

<sup>25</sup>For instance on the US Federal Courts website, automatic stay is defined as "an injunction that automatically stops lawsuits, foreclosures, garnishments, and all collection activity against the debtor the moment a bankruptcy petition is filed." See: <http://www.uscourts.gov/FederalCourts/Bankruptcy/BankruptcyBasics/Glossary.aspx>

<sup>26</sup>For example, Lehman Brothers filed for Chapter 11 in September 2008, exited from it in March 2012, and only made the first payment to creditors in April 2012. See "Lehman Exits Bankruptcy, Sets Distribution to Creditors", *Wall Street Journal*, March 06, 2012.

<sup>27</sup>In principle, an independent sale and repurchase transaction means the collateral rests on the balance sheet of the buyer (repo lender) and thus the automatic stay provision from the default of the seller (repo borrower) should not be applied to the collateral. In practice, nonetheless, repo in the U.S. is treated as *secured loans* and the repo securities are on the balance sheet of the borrower. See Acharya and Öncü (2010) for details and the historical development of the repo market in the U.S.

**Proposition 6** (*Repo with stay exemption as the optimal contract*) *The optimal contract is the collateralised short-term debt contract with promised repayment  $r$  at  $t = 1$  and immediate transfer of  $k$  units of collateral to the creditor at  $t = 1$  in the case of default. When Assumption 4 holds and so insolvent firms can renegotiate the debt contract by threatening to file for bankruptcy, a short-term repurchase agreement that is exempt from automatic stay provisions avoids this renegotiation problem and therefore implements the optimal contract.*

**Proof:** *The proposition is a consequence of the preceding discussion.*

**Cost of automatic stay** Critics of repo contracts' exemption from automatic stay provisions (e.g. [Roe \(2011\)](#)) argue that it could cause the disorderly liquidation of collateral assets when some borrowers default, which in turn would drive down the price of the collateral and thus lead to systemic risk. Some have proposed reforms that make also the repo lenders subject to an automatic stay of some duration in order to prevent the negative spiral just described. However this paper highlights another destabilising negative spiral because of *ex ante* contracting frictions and suggests that imposing automatic stay may induce *more* fire sales.

Here the main friction resulting from automatic stay provisions is that it allows insolvent firms to advantage of their creditors' preference of early liquidation of the collateral. Thus, a firm can renegotiate down the promised portion  $k$  of collateral to  $k' = k \frac{l}{l^*}$  ex post by threatening to file for bankruptcy, which implies that the maximum amount of collateral firms can *credibly* pledge is  $\frac{l}{l^*} \leq 1$ . Hence the collateral constraint becomes more binding and firms are more prone to take excessive risk, resulting in more fire sales and fragility in aggregate. Observe that even though lenders constrained by automatic stay provisions cannot seize and liquidate the collateral in the market at  $t = 1$ , it is optimal for firms to liquidate some assets to (partially) repay their lenders. As a result, constraining post-default fire sales by creditor worsens firms' moral hazard problem and thus could *increase* the total amount of collateral being liquidated via pre-default fire sales by the firms.<sup>28</sup>

---

<sup>28</sup>[Begalle et al. \(2013\)](#) make a similar distinction between pre-default and post-default fire sales and discuss how they can affect each other.

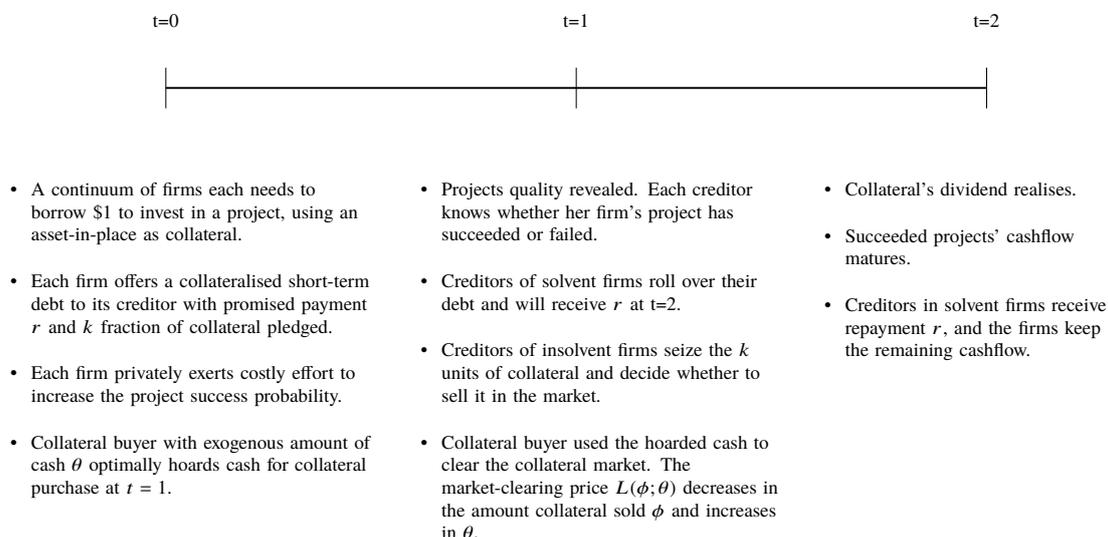
## 7 Concluding remarks

This paper describes a novel form of financial fragility stemming from a feedback effect between the risk-taking incentives of borrowing firms and illiquidity in the collateral asset market. This feedback mechanism offers a theory of systemic runs in the modern, market-based financial system, where traditional strategic considerations involving the depositors of a financial institution may not arise. I show that, when firms collateralise their assets to borrow in the form of short-term debt such as repo, a new kind of coordination failure among firms can arise because firms' margin and risk-taking decisions become *strategic complements* owing to the interaction between firms' moral hazard and the fire-sale externality in the collateral market. Fire sales can occur in a self-fulfilling manner, in which case aggregate default risk is endogenously chosen by individual firms.

In terms of policy, this paper provides an economic rationale for central banks to intervene in the collateral market. When the market is moderately illiquid, asset price guarantees can eliminate market participants' rational fear of fire sales and thereby the inefficient crisis equilibrium at no cost. I also find that repealing the stay-exemption status of repo contracts, which aims to limit post-default fire sales by creditors, may backfire because this could worsen the ex ante incentive problems of borrowing firms.

# Appendix

## Time-line of events



## Parametric restrictions in Assumption 3

Parametric restrictions in Assumption 3 are made to ensure  $0 \leq l_{CR} < l_{RT} < v$  so that prudent investment, risk-taking and credit rationing can arise in equilibrium. From the implicit definition of  $l_{RT}$  and  $l_{CR}$ ,  $k_1(l_{RT}) = 1$  and  $U(l_{CR}) = 0$ , one can show

$$l_{RT} = \frac{A_1 - p_1 v}{1 - p_1} \text{ and } l_{CR} = v \frac{(1 - p_2)A_2 - p_2 NPV_2}{(1 - p_2)A_2 + (1 - p_2)NPV_2}$$

It is immediate to check that  $l_{RT} < v$  and  $l_{CR} < l_{RT}$  require  $v > A_1$  and  $v < \bar{v}$  respectively. To have  $U(l_{CR}) = 0$  in equilibrium, one needs  $l_{CR} \geq 0$  and  $k_2(l_{CR}) \leq 1$  which together give the condition  $NPV_2 \leq \min\{v - A_2, \frac{1 - p_2}{p_2} A_2\}$ . □

## Proof

### Proof of Proposition 1:

First, both (IC) and (PC) are binding at optimal. If (PC) slacks, the firm can decrease  $r$  by a small amount to increase profit while (IC) still holds; If (IC) slacks, the firm can reduce  $k$

and increase  $r$  by a small amount to keep (PC) binding and (IC) still satisfied while a smaller  $k$  increases expected payoff due to lower fire-sale cost. To see this, suppose the contrary that  $\{r, k\}$  is optimal but (IC) slacks, that is

$$r < X - \frac{\Delta c_i}{\Delta p_i} + kv$$

Plugging the binding (PC)  $r = [1 - (1 - p_i)kl]/p_i$  into the above (IC), one can show  $k > \frac{A_i}{p_i v + (1 - p_i)l}$ . Consider another contract  $\{r', k'\}$  such that  $k' = k - \epsilon$  and  $r' = r + (1 - p_i)\epsilon l/p_i$ , (PC) still binds and for a small  $\epsilon > 0$  (IC) also holds. However the firm's expected payoff is strictly higher in the case of  $\{r', k'\}$ , as  $NPV_1 - (1 - k')(v - l) > NPV_1 - (1 - k)(v - l)$ , contradicting the optimality of  $\{r, k\}$ .

By binding (PC) and (IC), the optimal contract  $\{r(l), k(l)\}$  is described as in Equation 7. Note that for a given  $l$ , it could be both  $\{r_1(l), k_1(l)\}$  and  $\{r_2(l), k_2(l)\}$  satisfy the remaining (RE) and (PT) constraints, that is, both prudent investment and risk-taking are feasible choices. Since prudent investment is always superior by Assumption 1, the firm optimally chooses  $p_1(l)$  and the contract  $\{r_1(l), k_1(l)\}$ . Hence the firm chooses prudent investment whenever feasible, that is, when  $l \geq l_{RT}$ . If not, risk-taking is chosen as long as it is profitable, when  $l \geq l_{CR}$ .  $\square$

### **Proof of Lemma 3:**

For a fixed  $\theta$ ,  $L(s(l)\lambda(l)(1 - p(l))k(l); \theta)$  is a mapping from  $[0, v] \rightarrow [0, v]$ . Notice that the function  $L(l; \theta)$  is upper semi-continuous from the left and closed from the right. The existence of fixed-point follows from the Lemma in [Roberts and Sonnenschein \(1976\)](#).  $\square$

### **Proof of Proposition 2:**

There are three steps in this proof: I first show the existence of extreme regions of  $\theta$  that only exactly one equilibrium exists. Then I show multiple equilibria must exist under some regions of  $\theta$  and finally, I characterise the bounds of multiple equilibria regions  $\underline{\theta}, \bar{\theta}$  for different possible shapes of the market-clearing price function  $L(\phi(l); \theta)$ .

*Step 1: non-empty regions of  $\theta$  with unique equilibrium*

For  $\theta \in [\hat{\theta} + v, +\infty)$ ,  $L(\phi(l); \theta) = v$  for all  $l \in [0, v]$  according to Lemma 1. Thus there is only prudent investment equilibrium in this region as  $l_{RT} < v$ . On the other hand, for  $l < \max\{\underline{l}, l_{CR}\}$ ,  $\phi(l) = 0$  while the maximum price the collateral buyer willing to pay for the first unit is  $\frac{v}{F'(\theta)}$ . As  $\lim_{\theta \rightarrow 0^+} F'(\theta) \rightarrow +\infty$  and  $F''(\theta) < 0$ , there exists a  $\theta' > 0$  such that  $\frac{v}{F'(\theta')} = \max\{\underline{l}, l_{CR}\}$ . Then for  $\theta \in [0, \theta')$ , there is unique equilibrium with complete credit rationing (when  $\underline{l} < l_{CR}$ ) or risk-taking and no collateral traded (when  $\underline{l} > l_{CR}$ ).

*Step 2: non-empty set of  $\theta$  with multiple equilibria*

The key of this step is the upward jump of  $L(\phi(l); \theta)$  from  $l \rightarrow l_{RT}$ . At  $l = l_{RT}$ ,  $\phi(l_{RT}) = (1 - p_1)k_1(l_{RT}) = (1 - p_1) > (1 - p_2)k_2(l_{RT})$ , where the strict inequality is implied by Assumption 1(iii). By continuity of  $L(\cdot; \theta)$ , there exists a  $\theta''$  such that  $L(\phi(l_{RT}); \theta'') = l_{RT}$  hence  $l^* = l_{RT}$  is an equilibrium with prudent investment at  $\theta''$ . I am going to show that there also exists at least another equilibrium in the region  $l \in [\max\{\underline{l}, l_{CR}\}, l_{RT})$  at this  $\theta''$ . Due to the discontinuity of  $\phi(l)$  at  $l_{RT}$ ,  $L((1 - p_2)k_2(l_{RT}); \theta'')$  is strictly below  $l_{RT}$  and then  $L(\phi(l); \theta'')$  must cross the 45-degree line at some  $l^* \in [\max\{\underline{l}, l_{CR}\}, l_{RT})$ . To reduce notation, I will discuss the case with  $l_{CR} > \underline{l}$ . If  $L((1 - p_2)k_2(l_{CR}); \theta'') \geq l_{CR}$ , then by Intermediate Value Theorem, there exists a  $l^* \in [l_{CR}, l_{RT})$  such that  $L((1 - p_2)k_2(l^*); \theta'') = l^*$  because  $L(\phi(l); \theta'')$  is continuous in  $l$  and  $L((1 - p_2)k_2(l_{RT})) < l_{RT}$ ; If  $L((1 - p_2)k_2(l_{CR}); \theta'') < l_{CR} < L(0; \theta'')$ , then there exist a  $\lambda^* \in (0, 1)$  such that  $L(\lambda^*(1 - p_2)k_2(l_{CR}); \theta'') = l_{CR}$  as at  $l_{CR}$ ,  $L(\cdot; \theta'')$  can take any value between  $L((1 - p_2)k_2(l_{CR}); \theta'')$  and  $L(0; \theta'')$  due to Lemma 2. In conclusion, there exists multiple equilibria at  $\theta''$ .

*Step 3: Characterise the bounds of  $\underline{\theta}$  and  $\bar{\theta}$*

Let's start with the upper bound  $\bar{\theta}$ . For  $\theta > \theta''$ , multiple equilibria can exist because Equation 13 has multiple solutions in the region  $[l_{RT}, v]$  or at least one solution in  $[\max\{\underline{l}, l_{CR}\}, l_{RT})$  or both. Denote  $\theta_1$  and  $\theta_2$  as the smallest  $\theta > \theta''$  that  $L((1 - p_1)k_1(l); \theta_1) = l$  has exactly one solution in  $[l_{RT}, v]$  and  $L((1 - p_2)k_2(l); \theta_2) = l$  has no solution in  $[\max\{\underline{l}, l_{CR}\}, l_{RT})$  respectively. Both  $\theta_1$  and  $\theta_2$  exist as members in the non-empty set of  $\theta$  with unique prudent investment equilibrium satisfy these properties. Define  $\bar{\theta} = \max\{\theta_1, \theta_2\}$  and as  $L(\cdot; \theta)$  increases in  $\theta$ , there is a unique

equilibrium with prudent investment for any  $\theta \in [\bar{\theta}, +\infty)$ . Note that by construction  $\theta'' < \bar{\theta}$ . Similarly for  $\underline{\theta}$ . Denote  $\theta_3$  and  $\theta_4$  as the largest  $\theta < \theta''$  that  $L((1 - p_2)k_2(l); \theta_3) = l$  has exactly one solution in  $[\max\{\underline{l}, l_{CR}\}, l_{RT})$  and  $L((1 - p_1)k_1(l); \theta_4) = l$  has no solution in  $[l_{RT}, v]$  respectively. Define  $\underline{\theta} = \min\{\theta_3, \theta_4\}$  and as  $L(; \theta)$  increases in  $\theta$ , there is unique equilibrium with risk-taking (and credit rationing) for any  $\theta \in [0, \underline{\theta}]$ . Note that by construction,  $\underline{\theta} < \theta''$ . Finally by the fact that  $L(\phi(l); \theta)$  is continuous and strictly increases in  $\theta$  for  $\phi(l) > 0$ , any  $\theta \in (\underline{\theta}, \bar{\theta})$  contains multiple equilibria and this region is non-empty as  $\theta'' \in (\underline{\theta}, \bar{\theta})$ .  $\square$

### Proof of Proposition 3:

By the definition of  $U(l^*)$  and  $\Pi(l^*)$  in Equation (8) and (9) and the market-clearing condition  $F'(\theta - \phi(l^*)l^*) = v/l^*$ , the social welfare function  $W(l^*)$  can be expressed as

1. When  $l^* > \max\{l_{CR}, \underline{l}\}$ ,  $\phi(l^*) = (1 - p(l^*))k(l^*)$

$$W(l^*) = NPV(l^*) + \int_0^{\theta - \phi(l^*)l^*} [F'(x) - 1]dx \quad (17)$$

2. When  $l^* = l_{CR}$ ,  $\phi(l_{CR}) = \lambda(1 - p(l_{CR}))k(l_{CR})$

$$W(l_{CR}) = \lambda NPV(l_{CR}) + \int_0^{\theta - \phi(l_{CR})l_{CR}} [F'(x) - 1]dx \quad (18)$$

3. When  $l^* = \underline{l}$ ,  $\phi(\underline{l}) = s(1 - p(\underline{l}))k(\underline{l})$

$$W(\underline{l}) = NPV(\underline{l}) + \int_0^{\theta - \phi(\underline{l})\underline{l}} [F'(x) - 1]dx - (1 - s)(1 - p(\underline{l}))k(\underline{l})(v - \underline{l}) \quad (19)$$

It is then immediate to see that a higher  $l^*$  will increase  $W(l^*)$  in all cases.  $NPV(l^*) = p(l^*)X - 1 - c(p(l^*))$  increases in  $l^*$ ;  $\int_0^{\theta - \phi(l^*)l^*} [F'(x) - 1]dx$  is the net return from collateral buyer's productive investment and increases in  $l^*$  as  $\phi(l^*)l^*$  decreases in  $l$  by market-clearing condition. In case 2,  $(1 - \lambda)$  of firms do not invest and in case 3,  $(1 - s)$  of creditors could not sell the collateral to the buyer and both entail welfare loss. Therefore equilibria with lower  $l^*$  has a lower  $W(l^*)$ .  $\square$

**Proof of Proposition 5:**

Suppose  $\theta \in \Theta^M(\underline{l})$ ,  $L(\phi(l; \underline{l}); \theta) = l$  has multiple solutions  $\{l^*\}$ . What I need to show is that when  $\underline{l}$  decreases to any  $\underline{l}' < \underline{l}$ , there are as least as many solutions. First note that as  $\phi$  is only affected by  $\max\{l_{CR}, \underline{l}\}$ , changes  $\underline{l}$  below  $l_{CR}$  will not have any effect in equilibrium. Every member in the set  $\{l^*\}$  is at least as large as  $\underline{l}$  and when they are strictly larger than  $\underline{l}$ , they will still be part of the solution for any  $\underline{l}' < \underline{l}$ . When  $\underline{l}$  is one of the solutions and is changed to  $\underline{l}'$ , there are two cases: If  $L((1 - p(\underline{l}'))k(\underline{l}'); \theta) < \underline{l}'$ , the original solution  $\underline{l}$  changes to  $\underline{l}'$  with  $L(s^*(1 - p(\underline{l}'))k(\underline{l}'); \theta) = \underline{l}'$  for some  $s^* \in [0, 1]$ . This is because at  $\underline{l}'$ ,  $L()$  is a correspondence taking any value from  $L((1 - p(\underline{l}'))k(\underline{l}'); \theta)$  to  $L(0; \theta) \geq \underline{l} > \underline{l}'$ . On the other hand, if  $L((1 - p(\underline{l}'))k(\underline{l}'); \theta) > \underline{l}'$ , the original solution  $\underline{l}$  changes to some  $\underline{l}'' \in [\underline{l}', \underline{l}]$  where  $L((1 - p(\underline{l}''))k(\underline{l}''); \theta) = \underline{l}''$ . This follows from Intermediate Value Theorem as  $L((1 - p(l))k(l); \theta)$  is a continuous function in  $l$  and  $L((1 - p(\underline{l}))k(\underline{l}); \theta) \leq \underline{l}$ . Therefore, decreasing  $\underline{l}$  to  $\underline{l}'$  does not reduce the number of solutions  $\{l^*\}$ .  $\square$

## References

- Acharya, V. V. (2009). A theory of systemic risk and design of prudential bank regulation. *Journal of Financial Stability*, 5(3):224–255.
- Acharya, V. V., Gale, D., and Yorulmazer, T. (2011). Rollover risk and market freezes. *Journal of Finance*, 66(4):1177–1209.
- Acharya, V. V. and Öncü, S. (2010). The repurchase agreement (repo) market. *Regulating Wall Street: The Dodd-Frank Act and the New Architecture of Global Finance*, pages 319–350.
- Acharya, V. V. and Viswanathan, S. (2011). Leverage, moral hazard, and liquidity. *Journal of Finance*, 66(1):99–138.
- Adrian, T. and Shin, H. (2011). Financial intermediary balance sheet management. *FRB of New York Staff Report*, (532).
- Aghion, P. and Bolton, P. (1992). An incomplete contracts approach to financial contracting. *Review of Economic Studies*, 59(3):473–494.
- Becker, B. and Ivashina, V. (2014). Reaching for yield in the bond market. *Journal of Finance*.
- Begalle, B. J., Martin, A., McAndrews, J., and McLaughlin, S. (2013). The risk of fire sales in the tri-party repo market. *FRB of New York Staff Report*, (616).
- Bernardo, A. E. and Welch, I. (2004). Liquidity and financial market runs. *Quarterly Journal of Economics*, 119(1):135–158.
- Besanko, D. and Thakor, A. V. (1987). Collateral and rationing: sorting equilibria in monopolistic and competitive credit markets. *International Economic Review*, 28(3):671–689.
- Biais, B., Heider, F., and Hoerova, M. (2014). Optimal margins and equilibrium prices. *working paper*.
- Bolton, P., Santos, T., and Scheinkman, J. A. (2011). Outside and inside liquidity. *Quarterly Journal of Economics*, 126(1):259–321.

- Brunnermeier, M. K. (2009). Deciphering the liquidity and credit crunch 2007-2008. *Journal of Economic Perspectives*, 23(1):77–100.
- Brunnermeier, M. K. and Pedersen, L. H. (2009). Market liquidity and funding liquidity. *Review of Financial Studies*, 22(6):2201–2238.
- Buiter, W. (2012). The role of central banks in financial stability: How has it changed? *CEPR Discussion Paper no. 8780*.
- Buiter, W. and Sibert, A. (2007). The central bank as the market maker of last resort: From lender of last resort to market maker of last resort. *Voxeu.org*.
- Chan, Y.-S. and Thakor, A. V. (1987). Collateral and competitive equilibria with moral hazard and private information. *Journal of Finance*, 42(2):345–363.
- Coco, G. (2000). On the use of collateral. *Journal of Economic Surveys*, 14(2):191–214.
- Copeland, A., Martin, A., and Walker, M. (2014). Repo runs: Evidence from the tri-party repo market. *Journal of Finance*, 69(6):2343–2380.
- Covitz, D., Liang, N., and Suarez, G. A. (2013). The evolution of a financial crisis: Collapse of the asset-backed commercial paper market. *Journal of Finance*, 68(3):815–848.
- Danielsson, J., Shin, H. S., and Zigrand, J.-P. (2011). Balance sheet capacity and endogenous risk. FMG Discussion Papers dp665, Financial Markets Group.
- Diamond, D. W. and Dybvig, P. H. (1983). Bank runs, deposit insurance, and liquidity. *Journal of Political Economy*, 91(3):401–19.
- Diamond, D. W. and Rajan, R. G. (2001). Liquidity risk, liquidity creation, and financial fragility: A theory of banking. *Journal of Political Economy*, 109(2):287–327.
- Diamond, D. W. and Rajan, R. G. (2005). Liquidity shortages and banking crises. *Journal of Finance*, 60(2):615–647.
- Diamond, D. W. and Rajan, R. G. (2011). Fear of fire sales, illiquidity seeking, and credit freezes. *Quarterly Journal of Economics*, 126(2):557–591.

- Eisenbach, T. M. (2013). Rollover risk as market discipline: A two-sided inefficiency. *FRB of New York Staff Report*, (597).
- Farhi, E. and Tirole, J. (2012). Collective moral hazard, maturity mismatch, and systemic bailouts. *American Economic Review*, 102(1).
- Goldstein, I. and Pauzner, A. (2005). Demand–deposit contracts and the probability of bank runs. *Journal of Finance*, 60(3):1293–1327.
- Gorton, G. and Metrick, A. (2012). Securitized banking and the run on repo. *Journal of Financial Economics*, 104(3):425–451.
- Gorton, G. B. (2012). Some reflections on the recent financial crisis. Working Paper 18397, National Bureau of Economic Research.
- Gromb, D. and Vayanos, D. (2002). Equilibrium and welfare in markets with financially constrained arbitrageurs. *Journal of Financial Economics*, 66(2-3):361–407.
- Hart, O. and Moore, J. (1994). A theory of debt based on the inalienability of human capital. *Quarterly Journal of Economics*, 109(4):841–879.
- He, Z., Khang, I. G., and Krishnamurthy, A. (2010). Balance sheet adjustments during the 2008 crisis. *IMF Economic Review*, 58(1):118–156.
- He, Z. and Xiong, W. (2012). Dynamic debt runs. *Review of Financial Studies*, 25(6):1799–1843.
- Holmström, B. and Tirole, J. (1997). Financial intermediation, loanable funds, and the real sector. *Quarterly Journal of economics*, 112(3):663–691.
- Hombert, J. (2009). Optimal financial structure and asset prices. *Working Paper*.
- Infante, S. (2013). Repo collateral fire sales: The effects of exemption from automatic stay. *working paper*.
- Jensen, M. C. and Meckling, W. H. (1976). Theory of the firm: Managerial behavior, agency costs and ownership structure. *Journal of Financial Economics*, 3(4):305–360.

- Kacperczyk, M. and Schnabl, P. (2013). How safe are money market funds? *Quarterly Journal of Economics*.
- Krishnamurthy, A. (2010a). Amplification mechanisms in liquidity crises. *American Economic Journal: Macroeconomics*, 2(3).
- Krishnamurthy, A. (2010b). How debt markets have malfunctioned in the crisis. *Journal of Economic Perspectives*, 24(1).
- Krishnamurthy, A., Nagel, S., and Orlov, D. (2012). Sizing up repo. NBER Working Papers 17768, National Bureau of Economic Research, Inc.
- Krishnamurthy, A. and Vissing-Jorgensen, A. (2013). The ins and outs of lsaps. In *Federal Reserve Bank of Kansas City's Jackson Hole Symposium on the Global Dimensions of Unconventional Monetary Policy, Jackson Hole, Wyoming, August*.
- Lee, J. (2014). Collateral circulation and repo spreads. *working paper*.
- Malherbe, F. (2014). Self-fulfilling liquidity dry-ups. *Journal of Finance*, 69(2):947–970.
- Martin, A., Skeie, D., and von Thadden, E.-L. (2014). The fragility of short-term secured funding markets. *Journal of Economic Theory*, 149:15–42.
- Morris, S. and Shin, H. S. (2004). Liquidity black holes. *Review of Finance*, 8(1):1–18.
- Oehmke, M. (2014). Liquidating illiquid collateral. *Journal of Economic Theory*, 149:183–210.
- Plantin, G. (2014). Shadow banking and bank capital regulation. *Review of Financial Studies*.
- Roberts, J. and Sonnenschein, H. (1976). On the existence of cournot equilibrium without concave profit functions. *Journal of Economic Theory*, 13(1):112–117.
- Rochet, J.-C. and Vives, X. (2004). Coordination failures and the lender of last resort: Was bagehot right after all? *Journal of the European Economic Association*, 2(6):1116–1147.
- Roe, M. J. (2011). The derivatives market's payment priorities as financial crisis accelerator. *Stanford Law Review*, 63:539.

- Shleifer, A. and Vishny, R. (2011). Fire sales in finance and macroeconomics. *Journal of Economic Perspectives*, 25(1):29–48.
- Stein, J. C. (2012). Monetary policy as financial stability regulation. *Quarterly Journal of Economics*, 127(1):57–95.
- Stiglitz, J. E. and Weiss, A. (1981). Credit rationing in markets with imperfect information. *American Economic Review*, 71(3):393–410.
- Tirole, J. (2006). *The Theory of Corporate Finance*. Princeton University Press.
- Vayanos, D. (2004). Flight to quality, flight to liquidity, and the pricing of risk. *working paper*, LSE.
- Zhang, S. (2014). Collateral risk, repo rollover and shadow banking. *working paper*.