

Back-Running: Seeking and Hiding Fundamental Information in Order Flows*

Liyan Yang[†] Haoxiang Zhu[‡]

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Abstract

We study the strategic interaction between fundamental informed trading and order-flow informed trading. In a standard two-period Kyle (1985) model, we add a “technical trader” who observes, *ex post* and potentially with noise, the order flow of the fundamental informed investor in the first period. Hence the technical trader is “back-running,” instead of “front-running.” Learning from order-flow information, the technical trader competes with the fundamental investor in the second period. If order-flow information is accurate, the fundamental investor hides her information by endogenously adding noise into her first-period trade, resulting in a mixed strategy equilibrium. As order-flow information becomes less precise, the equilibrium switches to a pure strategy one. The presence of order-flow informed trading harms price discovery in the first period but improves it in the second period; its impact on market liquidity is mixed.

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[†]Rotman School of Management, University of Toronto; liyan.yang@rotman.utoronto.ca.

[‡]MIT Sloan School of Management; zhuh@mit.edu.

1 Introduction

This paper studies the strategic interaction between fundamental informed trading and order-flow informed trading, as well as its implications for market equilibrium outcomes. By order-flow informed trading, we refer to strategies that begin with no innate trading motives—be it fundamental information or liquidity needs—but instead first learn about other investors’ order flows and then act accordingly.

A primary example of order-flow informed trading is “order anticipation” strategies. According to the Securities and Exchange Commission (SEC, 2010, p. 54–55), order anticipation “involves any means to ascertain the existence of a large buyer (seller) that does not involve violation of a duty, misappropriation of information, or other misconduct. Examples include the employment of sophisticated pattern recognition software to ascertain from *publicly available information* the existence of a large buyer (seller), or the sophisticated use of orders to ‘ping’ different market centers in an attempt to locate and trade in front of large buyers and sellers [emphasis added].”

Always been controversial,¹ order anticipation strategies have recently attracted intense attention and generated heated debates in the context of high-frequency trading (HFT). In a colorful account of today’s U.S. equity market, Lewis (2014) argues that high-frequency traders observe part of investors’ orders on one exchange and “front-run” the remaining orders before they reach other exchanges.² Although most (reluctantly) agree that such strategies are legal in today’s regulatory framework, many investors and regulators have expressed severe concerns that they could harm market quality and long-term investors.³ For example, in its influential Concept Release on Equity Market Structure, SEC (2010, p. 56) asks: “Do commenters believe that order anticipation significantly detracts from market quality and harms institutional investors?”

To address important policy questions like this, we need to first address the broader

¹For example, Harris (2003) writes “Order anticipators are *parasitic traders*. They profit only when they can prey on other traders [emphasis in original].”

²In its original sense, front-running refers to the illegal practice that a broker executes orders on his own account before executing a customer order. In recent discussions of market structure, this term is often used more broadly to refer to any type of trading strategy that takes advantage of order flow information, including some academic papers that we will discuss shortly. When discussing these papers we use “front-running” to denote the broader meaning, as the original authors do.

³It should be noted that order anticipation strategies are not restricted to HFT; they also apply to other market participants such as broker-dealers who have such technology. “The successful implementation of this strategy (order anticipation) depends less on low-latency communications than on high-quality pattern-recognition algorithms,” remarks Harris (2013). “The order anticipation problem is thus not really an HFT problem.”

questions of market equilibrium. For example, how do order anticipators take advantage of their superior order-flow information of fundamental investors (such as mutual funds and hedge funds)? How do these fundamental investors, in turn, respond to potential information leakage? How does the strategic interaction between these two types of traders affect market equilibrium and associated market quality?

In this paper, we take up this task. Our analysis builds on a simple theoretical model of strategic trading. We start from a standard two-period Kyle (1985) model, which has a “fundamental investor” who is informed of the true asset value, noise traders, and a competitive market maker. The novel part of our model is that we add a “technical trader” who begins with no fundamental information nor liquidity needs, but who receives a signal of the fundamental investor’s order flow after that order is executed by the market maker. We emphasize that it is the fundamental investor’s order flow, not her information,⁴ that is partially observed by the technical trader (otherwise, the technical trader would be conceptually indistinguishable from a fundamental investor); and this order-flow information is observed *ex post*, not *ex ante*. This important feature is directly motivated by the “publicly available information” part in the SEC’s definition above. In this sense, the technical trader engages in what we call “back-running,” which is trading on information about *past* order flows, instead of “front-running,” which is trading on information about *future* order flows.

The trading mechanism of this market is the same as the standard Kyle (1985) model. In the first period, only the fundamental investor and noise traders submit market orders, which are filled by the market maker at the market-clearing price. Only after the period-1 market clears does the technical trader observe the fundamental investor’s period-1 order flow. In the second period, the fundamental investor, the technical trader, and noise traders all submit market orders, which are filled at a new market-clearing price.

It may be tempting to conjecture that the order flow of the fundamental investor would completely reveal her private information to the technical trader. We show that the fundamental investor can do better than this. In particular, we identify a mixed strategy equilibrium in which the fundamental investor injects an endogenous, normally-distributed noise into her period-1 order flow to hide her information. This garbled order flow, in turn, confuses the technical trader in his inference of the asset fundamental value. This mixed strategy equilibrium obtains if the technical trader’s order-flow signal is sufficiently precise, in which case playing a pure strategy is suboptimal as it costs the fundamental investor

⁴Throughout this paper, we will use “her”/“she” to refer to the fundamental investor and use “his”/“he” to refer to the technical trader.

too much of her information advantage. As a result, the mixed strategy equilibrium is the unique one among linear equilibria. In other words, if order-flow information is precise, then randomization is the best defense. Conversely, if the technical trader’s order-flow information is sufficiently noisy, he has a hard time inferring the fundamental investor’s information anyway. In this case, the fundamental investor does not need to inject additional noise; she simply plays a pure strategy.

Our analysis points out a new channel—i.e., the amount of noise in the technical trader’s signal—that determines whether a mixed strategy equilibrium or a pure strategy one should prevail in a Kyle-type auction game. In particular, if there is less *exogenous* noise in the technical trader’s signal, the fundamental investor *endogenously* injects more noise into her own period-1 order flow. As a result, as the amount of noise in the technical trader’s signal increases from 0 to ∞ , the unique linear equilibrium switches from a mixed strategy equilibrium, which involves randomization by the fundamental investor in period 1, to a pure strategy equilibrium. Characterizing the endogenous switch between a mixed strategy equilibrium and a pure strategy one is the first, theoretical contribution of our paper.

The second, applied contribution of our paper is to investigate the implications of order-flow informed trading for market quality and traders’ welfare. Our results reveal that order-flow informed trading has ambiguous impacts on price discovery and market liquidity. The natural comparison is with a standard two-period Kyle model without the order-flow informed trader. In the presence of order-flow informed trading, the fundamental investor trades less aggressively in the first period, harming price discovery. Price discovery is improved in the second period, however, since the technical trader also has value-relevant information and trades with the fundamental investor.

Market liquidity, measured by the inverse of Kyle’s lambda, is also affected differently in the two periods by order-flow informed trading. The first-period liquidity is generally improved because the more cautious trading of the fundamental investor weakens the market maker’s adverse selection problem. However, introducing order-flow informed trading can either improve or harm the second-period market liquidity. It will harm liquidity if the technical trader’s order-flow signal is sufficiently precise, which means that his trading will inject more private information into the second-period order flow, aggravating the market maker’s adverse selection problem.

Unsurprisingly, the fundamental investor suffers from the presence of order-flow informed trading, but noise traders benefit from it. Because institutional investors like mutual funds, pension funds, and ETFs employ a wide variety of investment strategies, they may act as

either fundamental investors or liquidity (noise) traders, depending on contexts. Nonetheless, since the order-flow informed trader makes a positive expected profit, the net result is that the other two trader types suffer collectively. We thus confirm the suspicion by regulators that order-flow informed trading tends to harm institutional investors on average.⁵

1.1 Related Literature

The model of our paper is closest to that of Huddart, Hughes, and Levine (2001), which is also an extension of the Kyle (1985) model. Motivated by the mandatory disclosure of trades by firm insiders, they assume that the insider’s orders are disclosed *publicly and perfectly* after being filled. They show that the only equilibrium in their setting is a mixed strategy one, for otherwise the market maker would always perfectly infer the asset value, preventing any further trading profits of the insider. The mandatory public disclosure unambiguously improves price discovery and market liquidity in each period of their model.

Our results differ from those of Huddart, Hughes, and Levine (2001) in at least two important aspects. First, while their model applies to public disclosure of insider trades, our model is much more suitable to analyze the *private* learning of order-flow information by proprietary firms such as HFTs. As we have shown, private learning of order-flow information has mixed effects on price discovery and market liquidity, in sharp contrast to the unambiguous improvement in both measures in Huddart, Hughes, and Levine (2001). Second, a theoretical contribution of our analysis relative to Huddart, Hughes, and Levine (2001) is that we identify the endogenous switching between the mixed strategy equilibrium and the pure strategy one, depending on the precision of the order-flow information. To the best of our knowledge, ours is the first model that presents a pure-mix strategy equilibrium switch among many extensions of Kyle (1985).

Other studies that identify a mixed strategy equilibrium in microstructure models include Back and Baruch (2004) and Baruch and Glosten (2013). In a continuous-time extension of Glosten and Milgrom (1985) model, Back and Baruch (2004) show that there is a mixed strategy equilibrium in which the informed trader’s strategy is a point process with stochastic intensity. Baruch and Glosten (2013) show that “flickering quotes” and “fleeting orders” can arise from a mixed strategy equilibrium in which quote providers repeatedly undercut each other. Neither paper explores the question of trading on order-flow information, and more

⁵We caution against interpreting noise traders as retail orders. In today’s U.S. equity market, retail market orders rarely make their way to stocks exchanges; instead, they are filled predominantly by broker-dealers acting as agent or principal.

importantly, neither identifies a switch between pure and mixed strategy equilibria.

Our paper is broadly related to the literature on strategic trading with information about order flows from informed or liquidity traders. For instance, Li (2014) models high-frequency trading “front-running,” whereby multiple HFTs with various speeds observe the aggregate order flow *ex ante* with noise and front-run it before it reaches the market maker. In his model the informed trader has one trading opportunity and does not counter information leakage by adding noise.

A few other earlier models explore information of liquidity-driven order flows. In the two-period model of Bernhardt and Taub (2008), for example, a single informed speculator observes liquidity trades *ex ante* in both periods. In period 1, the speculator front-runs the period-2 liquidity trades and later reverses the position. Attari, Mello, and Ruckes (2005) study a setting in which a strategic investor observes the initial position of an arbitrageur who faces capital constraint. They find that the strategic investor can benefit from the predictable price deviations caused by the arbitrageur’s trades. Brunnermeier and Pedersen (2005) model predatory trading whereby some arbitrageurs take advantage of others that are subject to liquidity shock. Predators first trade in the same direction as distressed investors and later reverse the position. Carlin, Lobo, and Viswanathan (2007) consider repeated interaction among traders and characterize conditions under which they predate each other or provide liquidity to each other. Neither Brunnermeier and Pedersen (2005) nor Carlin, Lobo, and Viswanathan (2007) addresses information asymmetry or price discovery.

Relative to the prior literature on order flow information, our model differs in two important aspects. First, in most existing models, except Attari, Mello, and Ruckes (2005), strategic traders explore information about incoming order flows (“front-running”). By contrast, our order-flow informed trader learns from *past* order flow (“back-running”), which is much more realistic in today’s market than observing incoming orders.⁶ Second, to hide her information, the fundamental investor in our model optimally injects noise into her orders to confuse the technical trader. This endogenous response is absent in other studies.

⁶A typical example that involves information leakage of incoming order flows is the in-person observation on the trading floor of exchanges. Harris (2003) gives an example of “legal front running,” in which an “observant trader” on an options trading floor notices the subtle behavior difference of another trader working on the same floor and front-runs him. In today’s electronic markets, gaining information by physical proximity is no longer possible.

2 A Model of Order-Flow Informed Trading

This section provides a model of order-flow informed trading, based on the standard Kyle (1985) model. For ease of reference, main model variables are tabulated and explained in Appendix A. All proofs are in Appendix B.

2.1 Setup

There are two trading periods, $t = 1$ and $t = 2$. The timeline of the economy is described by Figure 1. A risky asset pays a liquidation value $v \sim N(p_0, \Sigma_0)$ at the end of period 2, where $p_0 \in \mathbb{R}$ and $\Sigma_0 > 0$. A single “fundamental investor” learns v at the start of the first period and places market orders x_1 and x_2 at the start of periods 1 and 2, respectively. Noise traders’ net demands in the two periods are u_1 and u_2 , both distributed $N(0, \sigma_u^2)$, with $\sigma_u > 0$. Random variables v , u_1 and u_2 are mutually independent. Asset prices p_1 and p_2 are set by a competitive market maker who observes the total order flow at each period, y_1 and y_2 , and sets the price equal to the posterior expectation of v given public information.

The main difference from a standard Kyle model is that there is a “technical trader” who can extract private information from public order flows and trades on this private information in period 2. Specifically, after seeing the aggregate period-1 order flow y_1 , which is public information in period 2, the technical trader is able to extract a signal about the fundamental investor’s period-1 trades x_1 as follows:

$$s = x_1 + \varepsilon, \tag{1}$$

where $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, where $\sigma_\varepsilon \in [0, \infty]$, is independent of all other random variables (v , u_1 and u_2). (Since y_1 is public, observing a signal of u_1 would be equivalent to observing a signal of x_1 .) Parameter σ_ε controls the information quality of the signal s —a larger σ_ε means less accurate information about x_1 . In particular, we deliberately allow σ_ε to take values of 0 and ∞ , which respectively corresponds to the case in which s perfectly reveals x_1 and the case in which s reveals nothing about x_1 .

After receiving the signal s , the technical trader places a market order d_2 in period 2. As a result, the market maker receives an aggregate order flow

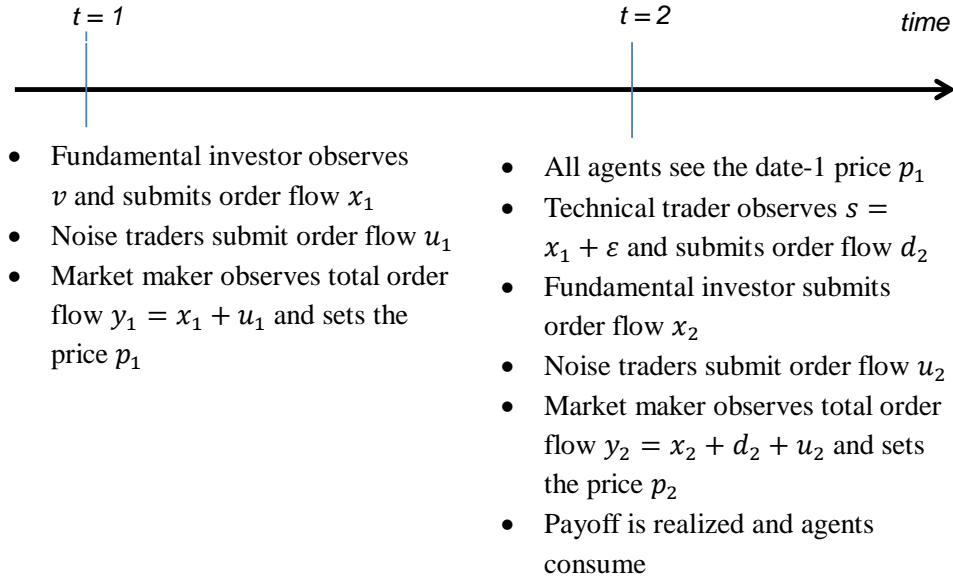
$$y_2 = x_2 + d_2 + u_2. \tag{2}$$

Of course, in period 1, the aggregate order flow is

$$y_1 = x_1 + u_1, \tag{3}$$

since during period 1 the technical trader has no private information and does not send any

Figure 1: Model Timeline



order. The weak-form-efficiency pricing rule of the market maker implies

$$p_1 = E(v|y_1) \text{ and } p_2 = E(v|y_1, y_2). \quad (4)$$

At the end of period 2, all agents receive their payoffs and consume, and the economy ends.

As mentioned in the introduction, the practical interpretation of this technical trader is that he uses advanced order-parsing technology that processes public information better than the market maker does. Our objective is to explore how the presence of such traders affects the trading strategies of the fundamental investor (such as pension funds, mutual funds, or hedge funds) as well as the resulting market equilibrium outcomes.

2.2 Equilibrium Definitions

A *perfect Bayesian equilibrium* of the trading game is given by a strategy profile

$$\{x_1^*(v), x_2^*(v, p_1, x_1), d_2^*(s, p_1), p_1^*(y_1), p_2^*(y_1, y_2)\},$$

such that:

1. Profit maximization:

$$x_2^* \in \arg \max_{x_2} E [x_2 (v - p_2) | v, p_1, x_1],$$

$$d_2^* \in \arg \max_{d_2} E [d_2 (v - p_2) | s, p_1],$$

$$\text{and } x_1^* \in \arg \max_{x_1} E [x_1 (v - p_1) + x_2^* (v - p_2) | v].$$

2. Market efficiency: p_1 and p_2 are determined according to equation (4).

We consider linear equilibria in which the trading strategies and pricing functions are linear. Specifically, a *linear equilibrium* is defined as a perfect Bayesian equilibrium in which there exist constants

$$(\beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1, \lambda_2) \in \mathbb{R}^8 \quad \text{and} \quad \sigma_z \geq 0,$$

such that

$$x_1 = \beta_{v,1} (v - p_0) + z \text{ with } z \sim N(0, \sigma_z^2), \quad (5)$$

$$x_2 = \beta_{v,2} (v - p_1) - \beta_{x_1} x_1 + \beta_{y_1} y_1, \quad (6)$$

$$d_2 = \delta_s s - \delta_{y_1} y_1, \quad (7)$$

$$p_1 = p_0 + \lambda_1 y_1 \text{ with } y_1 = x_1 + u_1, \quad (8)$$

$$p_2 = p_1 + \lambda_2 y_2 \text{ with } y_2 = x_2 + d_2 + u_2, \quad (9)$$

where z is independent of all other random variables $(v, u_1, u_2, \varepsilon)$.

Equations (5)–(9) are intuitive. Equations (5)–(7) simply say that the fundamental investor and the technical trader trade on their information advantage. In particular, our specification (5) allows the fundamental investor to play a mixed strategy in period 1. We have followed Huddart, Hughes, and Levine (2001) and restricted attention to normally distributed z in order to maintain tractability. If $\sigma_z = 0$, the fundamental investor plays a pure strategy in period 1, and we refer to the resulting linear equilibrium as a *pure strategy equilibrium*. If $\sigma_z > 0$, the fundamental investor plays a mixed strategy in period 1, and we refer to the resulting linear equilibrium as a *mixed strategy equilibrium*. As we show shortly, by adding noise into her orders, the fundamental investor limits the technical trader's ability to infer x_1 and hence v . To an outside observer, the endogenously added noise z may look like exogenous noise trading.

Although in principle the fundamental investor and the technical trader can play mixed strategies in period 2, we show later that using mixed strategies in period 2 is suboptimal in equilibrium. Thus, the linear period-2 trading strategies specified in equations (6) and (7) are without loss of generality. They are also the most general linear form, as each equation

spans the information set of the relevant trader in the relevant period. Note that at this stage we do not require that β_{x_1} , β_{y_1} , δ_s or δ_{y_1} be positive, although in equilibrium they will be positive.

Equation (6) has three terms. The first term $\beta_{v,2}(v - p_1)$ captures how aggressively the fundamental investor trades on her information advantage about v . The other two terms $-\beta_{x_1}x_1$ and $\beta_{y_1}y_1$ say that the fundamental investor potentially adjusts her period-2 market order by using lagged information x_1 and y_1 . Because the technical trader generally uses y_1 and his signal s about x_1 to form his period-2 order (see equation (7)), the fundamental investor takes advantage of this predictive pattern by using x_1 and y_1 in her period-2 order as well.

In equilibrium characterized later, the conjectured strategy in equation (7) can also be written alternatively as:

$$d_2 = \alpha [E(v|s, y_1) - E(v|y_1)], \quad (10)$$

for some constant $\alpha > 0$ (see Appendix B.1 for a proof). That is, the technical trader's order is proportional to his information advantage relative to the market maker's. By the joint normality of s and y_1 , this alternative form implies that d_2 is linear in s and y_1 . We nonetheless start with (7) because it is the most general and does not impose any structure as (10) does. We start with equation (6) for a similar reason.

The pricing equations (8) and (9) state that the price in each period is equal to the expected value of v before trading, adjusted by the information carried by the new order flow. Although the conjectured p_2 may in principle depend on y_1 , in equilibrium p_1 already incorporates all information of y_1 .⁷ Thus, we can start with (9).

2.3 Equilibrium Derivation

We now derive by backward induction all possible linear equilibria. Along the derivations, we will see that the distinction between pure strategy and mixed strategy equilibria lies only in the conditions characterizing the fundamental investor's period-1 decision. Explicit statements of the equilibria and their properties are presented in the next subsection.

Fundamental investor's date-2 problem In period 2, the fundamental investor has information $\{v, p_1, x_1\}$. Given $\lambda_1 \neq 0$, which holds in equilibrium, the fundamental investor

⁷Strictly speaking the most general form is $p_2 = p_1 + \lambda_2 [y_2 - E(y_2|y_1)]$. But in equilibrium we can show that $E(y_2|y_1) = 0$, so the more general form reduces to (9).

can infer y_1 from p_1 by equation (8). Using equations (7) and (9), we can compute

$$E[x_2(v - p_2) | v, p_1, x_1] = -\lambda_2 x_2^2 + [v - p_1 - \lambda_2(\delta_s x_1 - \delta_{y_1} y_1)] x_2. \quad (11)$$

Taking the first-order-condition (FOC) results in the solution as follows:

$$x_2 = \frac{v - p_1}{2\lambda_2} - \frac{\delta_s}{2} x_1 + \frac{\delta_{y_1}}{2} y_1. \quad (12)$$

The second-order-condition (SOC) is⁸

$$\lambda_2 > 0. \quad (13)$$

Equation (12) also implies that the fundamental investor optimally chooses to play a pure strategy in equilibrium, which verifies our conjectured pure strategy specification (6).

Comparing equation (12) with the conjectured strategy (6), we have

$$\beta_{v,2} = \frac{1}{2\lambda_2}, \beta_{x_1} = \frac{\delta_s}{2} \text{ and } \beta_{y_1} = \frac{\delta_{y_1}}{2}. \quad (14)$$

Let $\pi_2 = x_2(v - p_2)$ denote the fundamental investor's profit that is directly attributable to her period-2 trade. Inserting (12) into (11) yields

$$E(\pi_2 | v, p_1, x_1) = \frac{[v - p_1 - \lambda_2(\delta_s x_1 - \delta_{y_1} y_1)]^2}{4\lambda_2}. \quad (15)$$

Technical trader's date-2 problem In period 2, the technical trader chooses d_2 to maximize $E[d_2(v - p_2) | s, p_1]$. Using (6) and (9), we can compute the FOC, which delivers

$$d_2 = \frac{(1 - \lambda_2 \beta_{v,2}) E(v - p_1 | s, y_1) - \lambda_2 \beta_{y_1} y_1 + \lambda_2 \beta_{x_1} E(x_1 | s, y_1)}{2\lambda_2}. \quad (16)$$

The SOC is still $\lambda_2 > 0$, as given by (13) in the fundamental investor's problem. Again, equation (16) means that the technical trader optimally chooses to play a pure strategy in a linear equilibrium.

We then employ the projection theorem and equations (1), (3) and (5) to find out the expressions of $E(v - p_1 | s, y_1)$ and $E(x_1 | s, y_1)$, which are in turn inserted into (16) to express d_2 as a linear function of s and y_1 . Finally, we compare this expression with the conjectured strategy (7) to arrive at the following two equations:

$$\begin{aligned} \delta_s &= \frac{\left[(1 - \lambda_2 \beta_{v,2}) \frac{\beta_{v,1} \Sigma_0}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2} + \lambda_2 \beta_{x_1} \right] \frac{\sigma_\varepsilon^{-2}}{(\beta_{v,1}^2 \Sigma_0 + \sigma_z^2)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}}{2\lambda_2}, \\ \delta_{y_1} &= -\delta_s \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2}} + \frac{\lambda_1 (1 - \lambda_2 \beta_{v,2}) + \lambda_2 \beta_{y_1}}{2\lambda_2}. \end{aligned}$$

⁸The SOC cannot be $\lambda_2 = 0$, because otherwise, we have $p_2 = p_1 = p_0 + \lambda_1 y_1 = p_0 + \lambda_1(x_1 + u_1)$, and thus $E(p_2 | v) = p_0 + \lambda_1 x_1$, which means that the fundamental trader can choose x_1 and x_2 to make infinite profit in period 2. Thus, in any linear equilibrium, we must have $\lambda_2 > 0$.

Using (14), we can further simplify the above two equations as follows:

$$\delta_s = \frac{\frac{\sigma_\varepsilon^{-2}}{(\beta_{v,1}^2 \Sigma_0 + \sigma_z^2)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}}{4 - \frac{\sigma_\varepsilon^{-2}}{(\beta_{v,1}^2 \Sigma_0 + \sigma_z^2)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}} \frac{\beta_{v,1} \Sigma_0}{\lambda_2 (\beta_{v,1}^2 \Sigma_0 + \sigma_z^2)}, \quad (17)$$

$$\delta_{y_1} = \frac{\lambda_1}{3\lambda_2} - \delta_s \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}. \quad (18)$$

Market maker's decisions In period 1, the market maker sees the aggregate order flow y_1 and sets $p_1 = E(v|y_1)$. Accordingly, we have $\lambda_1 = \frac{Cov(v, y_1)}{Var(y_1)}$. By equation (5) and the projection theorem, we can compute

$$\lambda_1 = \frac{Cov(v, y_1)}{Var(y_1)} = \frac{\beta_{v,1} \Sigma_0}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2 + \sigma_u^2}. \quad (19)$$

Similarly, in period 2, the market maker sees $\{y_1, y_2\}$ and sets $p_2 = E(v|y_1, y_2)$. By equations (6), (7) and (14) and applying the projection theorem, we have

$$\begin{aligned} \lambda_2 &= \frac{Cov(v, y_2|y_1)}{Var(y_2|y_1)} \\ &= \frac{\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2}\beta_{v,1}\right) \Sigma_0 - \frac{\beta_{v,1} \Sigma_0 \left[\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2}\beta_{v,1}\right) \beta_{v,1} \Sigma_0 + \frac{\delta_s}{2} \sigma_z^2\right]}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2 + \sigma_u^2}}{\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2}\beta_{v,1}\right)^2 \Sigma_0 + \frac{\delta_s^2}{4} \sigma_z^2 + \delta_s \sigma_\varepsilon^2 + \sigma_u^2 - \frac{\left[\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2}\beta_{v,1}\right) \beta_{v,1} \Sigma_0 + \frac{\delta_s}{2} \sigma_z^2\right]^2}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2 + \sigma_u^2}}. \end{aligned} \quad (20)$$

Fundamental investor's date-1 problem We denote by $\pi_1 = x_1(v - p_1)$ the fundamental investor's profit that comes from her period-1 trade. In period 1, the fundamental investor chooses x_1 to maximize

$$E(\pi_1 + \pi_2|v) = x_1 E(v - p_1|v) + E \left[\frac{[v - p_1 - \lambda_2(\delta_s x_1 - \delta_{y_1} y_1)]^2}{4\lambda_2} \middle| v \right],$$

where the equality follows from equation (15). Using (8), we can further express $E(\pi_1 + \pi_2|v)$ as follows:

$$\begin{aligned} E(\pi_1 + \pi_2|v) &= - \left[\lambda_1 - \frac{(\lambda_1 + \lambda_2 \delta_s - \lambda_2 \delta_{y_1})^2}{4\lambda_2} \right] x_1^2 \\ &\quad + \left[1 - \frac{\lambda_1 + \lambda_2 \delta_s - \lambda_2 \delta_{y_1}}{2\lambda_2} \right] (v - p_0) x_1 \\ &\quad + \frac{(v - p_0)^2 + \sigma_u^2 (\lambda_1 - \lambda_2 \delta_{y_1})^2}{4\lambda_2}. \end{aligned} \quad (21)$$

Depending on whether the fundamental investor plays a mixed or a pure strategy (i.e., whether σ_z is equal to 0), we have two cases:

Case 1. Mixed Strategy ($\sigma_z > 0$)

For a mixed strategy to sustain in equilibrium, the fundamental investor has to be indifferent between any realized pure strategy. This in turn means that coefficients on x_1^2 and x_1 in (21) are equal to zero, that is,

$$\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1})^2}{4\lambda_2} = 0 \text{ and } 1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}}{2\lambda_2} = 0.$$

These two equations, together with equation (18), imply

$$\lambda_1 = \lambda_2 \text{ and } \delta_s = \frac{\frac{4}{3}}{1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}}. \quad (22)$$

Case 2. Pure Strategy ($\sigma_z = 0$)

When the fundamental investor plays a pure strategy, $z = 0$ (and $\sigma_z = 0$) in the conjectured strategy, and thus (5) degenerates to $x_1 = \beta_{v,1}(v - p_0)$. The FOC of (21) yields

$$x_1 = \frac{\left(1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}}{2\lambda_2}\right)}{2 \left[\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1})^2}{4\lambda_2}\right]} (v - p_0),$$

which, compared with the conjectured pure strategy $x_1 = \beta_{v,1}(v - p_0)$, implies

$$\beta_{v,1} = \frac{1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1}}{2\lambda_2}}{2 \left[\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1})^2}{4\lambda_2}\right]}. \quad (23)$$

The SOC is

$$\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1})^2}{4\lambda_2} > 0. \quad (24)$$

2.4 Equilibrium Characterization and Properties

A mixed strategy equilibrium is characterized by equations (14), (17), (18), (19), (20) and (22), together with one SOC, $\lambda_2 > 0$ (given by (13)). These conditions jointly define a system that determine nine unknowns, $\sigma_z, \beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1$ and λ_2 . The following proposition formally characterizes a linear mixed strategy equilibrium.

Proposition 1 (Mixed Strategy Equilibrium). *Let $\gamma \equiv \frac{\sigma_\varepsilon}{\sigma_u}$. If and only if $\gamma < \frac{\sqrt{\sqrt{17}-4}}{2} \approx 0.175$, there exists a linear mixed strategy equilibrium, and it is specified by equations (5)–(9),*

where

$$\begin{aligned}
\sigma_z &= \sigma_u \sqrt{\frac{(1+4\gamma^2)(1-32\gamma^2-16\gamma^4)}{(3+4\gamma^2)(13+40\gamma^2+16\gamma^4)}}, \\
\beta_{v,1} &= \frac{\sigma_u}{\sqrt{\Sigma_0}} \sqrt{\frac{1-4\gamma^2-(3+4\gamma^2)\frac{\sigma_z^2}{\sigma_u^2}}{3+4\gamma^2}}, \\
\lambda_1 &= \lambda_2 = \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2\Sigma_0 + \sigma_z^2 + \sigma_u^2} > 0, \\
\beta_{v,2} &= \frac{1}{2\lambda_2}, \delta_s = \frac{4}{3+4\gamma^2}, \delta_{y_1} = \frac{1-4\gamma^2\delta_s}{3}, \\
\beta_{x_1} &= \frac{\delta_s}{2} \text{ and } \beta_{y_1} = \frac{\delta_{y_1}}{2}.
\end{aligned}$$

When it exists, this equilibrium is the unique linear mixed strategy equilibrium.

To illustrate the intuition of the equilibrium strategies, it is useful to explicitly decompose d_2 as follows:

$$\begin{aligned}
d_2 &= \delta_s(x_1 + \varepsilon) - \delta_{y_1}(x_1 + u_1) = (\delta_s - \delta_{y_1})x_1 + \delta_s\varepsilon - \delta_{y_1}u_1 \\
&= x_1 + \delta_s\varepsilon - \delta_{y_1}u_1 = \beta_{v,1}(v - p_1) + (\delta_s\varepsilon + z) - \delta_{y_1}u_1,
\end{aligned} \tag{25}$$

where we have used the fact that $\delta_s - \delta_{y_1} = 1$ in equilibrium. Equation (25) says that the technical trader's order d_2 consists of three parts. The first part is the fundamental trader's order x_1 in period 1. The second part, $\delta_s\varepsilon + z$, reflects the imprecision of his signal, caused by both the exogenous noise ε in his signal-processing technology and the endogenous noise z added by the fundamental investor. The third part, $-\delta_{y_1}u_1 < 0$, says that the technical trader trades against the period-1 noise demand u_1 , which is profitable in expectation because the technical trader can tell x_1 from u_1 better than the market maker does. Note that equation (25) should be read as purely as a decomposition but not the strategy used by the technical trader, as v , ε , z and u_1 are not separately observable to him.

We can do a similar decomposition for x_2 :

$$\begin{aligned}
x_2 &= \frac{v - p_1}{2\lambda_2} - \frac{\delta_s}{2}x_1 + \frac{\delta_{y_1}}{2}y_1 = \frac{v - p_1}{2\lambda_2} - \frac{\delta_s - \delta_{y_1}}{2}(\beta_{v,1}(v - p_1) + z) + \frac{\delta_{y_1}}{2}u_1 \\
&= \frac{v - p_1}{2\lambda_2} - \frac{1}{2}(\beta_{v,1}(v - p_1) + z - \delta_{y_1}u_1) \\
&= \left(\frac{1}{2\lambda_2} - \frac{1}{2}\beta_{v,1}\right)(v - p_1) - \frac{1}{2}E[d_2 - \beta_{v,1}(v - p_1)|v - p_1, z, u_1].
\end{aligned} \tag{26}$$

The fundamental investor's order consists of two parts. The first part, $\left(\frac{1}{2\lambda_2} - \frac{1}{2}\beta_{v,1}\right)(v - p_1)$, is driven by fundamental information, as usual. The second part, written as

$$-\frac{1}{2}E[d_2 - \beta_{v,1}(v - p_1)|v - p_1, z, u_1],$$

says that the fundamental investor trades against the *non-information*, or noise, component of the expected order submitted by the technical trader, given all her information, with half the intensity. Trading against the noise component is intuitive as this noise is a “mistake” of the technical trader. Overall, we expect x_2 and d_2 to be unconditionally positively correlated, although conditional on $v - p_1$ the correlation is negative:

$$Cov(x_2, d_2 | v - p_1) = Cov\left(-\frac{1}{2}(z - \delta_{y_1} u_1), \delta_s \varepsilon + z - \delta_{y_1} u_1\right) < 0.$$

In the mixed strategy equilibrium, x_1 , x_2 , $v - p_0$ and $v - p_1$ need not have the same sign. For example, if $v > p_0$ but z is sufficiently negative, the fundamental trader ends up selling in period 1 (with $x_1 < 0$), before purchasing in period 2 ($x_2 > 0$). While such a pattern in the data may raise red flags of potential “manipulation” (trading in the opposite direction of the true intention), it could simply be part of an optimal execution strategy that involves randomizing.

Proposition 1 reveals that a mixed strategy equilibrium exists if and only if the size σ_ε of the noise in the technical trader’s signal is sufficiently small relative to σ_u . This result is natural and intuitive. A small σ_ε implies that the technical trader can observe x_1 relatively accurately. The technical trader will in turn compete aggressively with the fundamental investor in period 2, which reduces the fundamental investor’s profit substantially. Worried about information leakage, the fundamental investor optimally plays a mixed strategy in period 1 by injecting an endogenous noise z into her order x_1 , with σ_z uniquely determined in equilibrium. This garbled x_1 limits the technical trader’s ability to learn about v . In other words, if the technical trader’s order-parsing technology is accurate enough, randomization is the fundamental investor’s best camouflage.

Conversely, if σ_ε is sufficiently large already, the fundamental investor retains much of her information advantage, and further obscuring x_1 is unnecessary. In this case a linear pure strategy equilibrium, characterized shortly, would be more natural.

Looked another way, all else equal, the mixed strategy equilibrium obtains if and only if σ_u is sufficiently *large*. Traditional Kyle-type models would not generate this result, as noise trading provides camouflage for the informed trader. In our model, however, a large σ_u confuses only the market maker, not the technical trader. Thus, more noise trading implies a higher profit for the fundamental investor and hence a stronger incentive to retain her proprietary information by adding noise. A natural implication of this observation is that the exogenous noise σ_u reinforces the endogenous noise σ_z .

The threshold value for the existence of the mixed strategy equilibrium is $\sigma_\varepsilon/\sigma_u \approx 17.5\%$; whether it is large or small is an empirical question. We speculate that in a setting with

more than two periods or a two-period setting with $Var(u_1) < Var(u_2)$,⁹ this threshold value is likely to increase. The intuition is that in these extended settings, the fundamental investor will have a higher incentive to add noise to her period-1 trading, for otherwise she will give up more profits in the future.

Now we turn to pure strategy equilibria. In a pure strategy equilibrium, we have $\sigma_z = 0$. This type of equilibrium is characterized by equations (14), (17), (18), (19), (20) and (23), together with two SOC's, (13) and (24). These conditions jointly define a system that determine eight unknowns, $\beta_{v,1}, \beta_{v,2}, \beta_{x_1}, \beta_{y_1}, \delta_s, \delta_{y_1}, \lambda_1$ and λ_2 . The following proposition formally characterizes a linear pure strategy equilibrium.

Proposition 2 (Pure Strategy Equilibrium). *A linear pure strategy equilibrium is characterized by equations (5)–(9) with $\sigma_z = 0$ as well as the following two conditions on $\beta_{v,1} \in \left(0, \frac{\sigma_u}{\sqrt{\Sigma_0}}\right]$:*

(1) $\beta_{v,1}^2$ solves the 7th order polynomial:

$$f(\beta_{v,1}^2) = A_7\beta_{v,1}^{14} + A_6\beta_{v,1}^{12} + A_5\beta_{v,1}^{10} + A_4\beta_{v,1}^8 + A_3\beta_{v,1}^6 + A_2\beta_{v,1}^4 + A_1\beta_{v,1}^2 + A_0 = 0,$$

where the coefficients A 's are given by equations (B14)–(B21) in Appendix B; and

(2) The following SOC (i.e., (24)) is satisfied:

$$\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_{y_1})^2}{4\lambda_2} > 0,$$

where $\lambda_1, \lambda_2, \delta_s$ and δ_{y_1} are expressed as functions of $\beta_{v,1}$ as follows:

$$\begin{aligned} \lambda_1 &= \frac{\beta_{v,1}\Sigma_0}{\beta_{v,1}^2\Sigma_0 + \sigma_u^2}, \\ \lambda_2 &= \sqrt{\frac{\Sigma_0(2\sigma_u^4 + 4\sigma_\varepsilon^4 + 5\sigma_u^2\sigma_\varepsilon^2)\Sigma_0^2\beta_{v,1}^4 + (8\sigma_u^2\sigma_\varepsilon^4 + 5\sigma_u^4\sigma_\varepsilon^2)\Sigma_0\beta_{v,1}^2 + 4\sigma_u^4\sigma_\varepsilon^4}{(\beta_{v,1}^2\Sigma_0 + \sigma_u^2)(3\sigma_u^2\Sigma_0\beta_{v,1}^2 + 4\sigma_\varepsilon^2\Sigma_0\beta_{v,1}^2 + 4\sigma_u^2\sigma_\varepsilon^2)^2}}, \\ \delta_s &= \frac{\beta_{v,1}\Sigma_0\sigma_u^2}{\lambda_2[(3\sigma_u^2 + 4\sigma_\varepsilon^2)\Sigma_0\beta_{v,1}^2 + 4\sigma_u^2\sigma_\varepsilon^2]}, \\ \delta_{y_1} &= \frac{\lambda_1}{3\lambda_2} - \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\delta_s. \end{aligned}$$

Propositions 1 and 2 respectively characterize mixed strategy and pure strategy equilibria. The following proposition provides sufficient conditions under which either equilibrium prevails as the unique one among linear equilibria.

Proposition 3 (Mixed vs. Pure Strategy Equilibria). *If the technical trader has a sufficiently precise signal about x_1 (i.e., σ_ε^2 is sufficiently small), there is no pure strategy equilibrium,*

⁹The setting with $Var(u_1) < Var(u_2)$ is natural if the second period corresponds to a longer trading interval (e.g., one hour versus one minute) or a larger market (multiple exchanges versus a single exchange).

and the unique linear strategy equilibrium is the mixed strategy equilibrium characterized by Proposition 1. If the technical trader has a sufficiently noisy signal about x_1 (i.e., σ_ε^2 is sufficiently large), there is no mixed strategy equilibrium, and there is a unique pure strategy equilibrium characterized by Proposition 2.

Given Proposition 1 and the discussion of its properties, the mixed strategy part of Proposition 3 is relatively straightforward. The existence of a pure strategy equilibrium for a sufficiently large σ_ε is also natural, as in this case the technical trader's signal has little information and does not deter the fundamental investor from using a pure strategy. In fact, as $\sigma_\varepsilon \uparrow \infty$ our setting degenerates to a standard two-period Kyle (1985) setting, and the unique linear equilibrium in our model indeed converges to the pure strategy equilibrium of Kyle (1985). This result is shown in the following corollary.

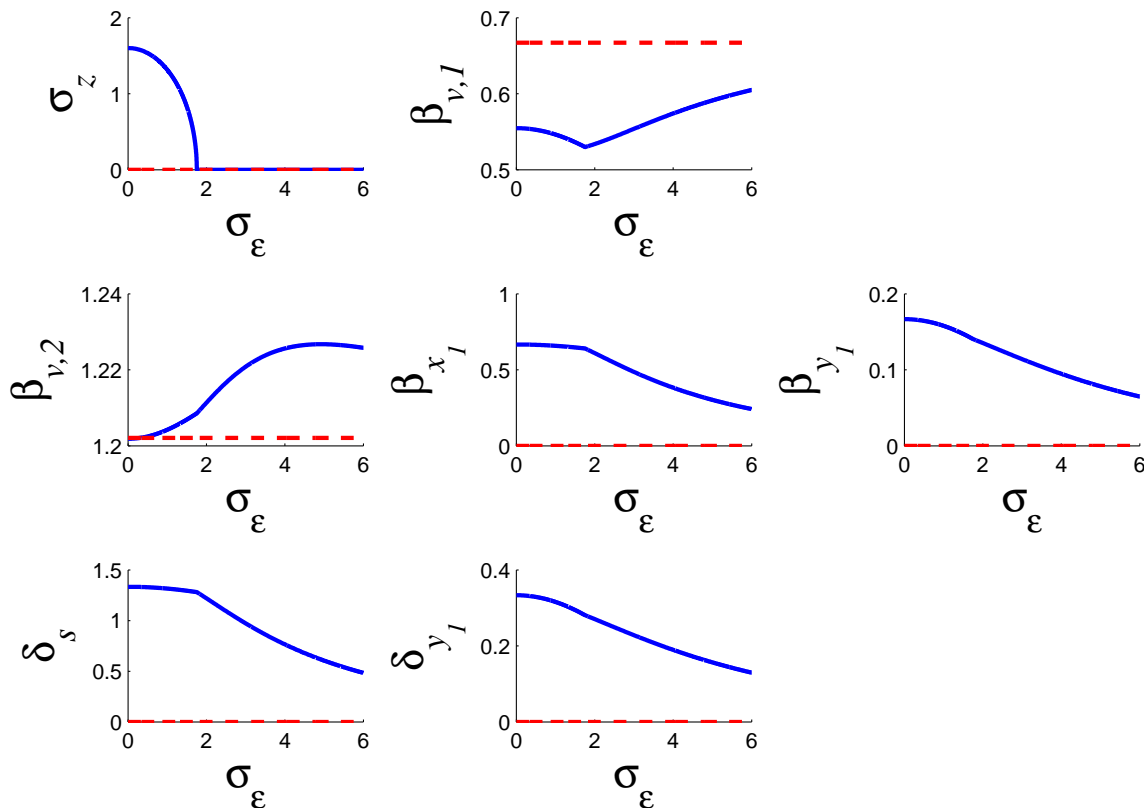
Corollary 1. *As $\sigma_\varepsilon \rightarrow \infty$, the linear equilibrium in the two-period economy with an order-flow informed trader converges to the linear equilibrium in the standard two-period Kyle model.*

Proposition 3 analytically proves the uniqueness of a linear equilibrium only for sufficiently small or sufficiently large values of σ_ε^2 . It would be desirable to generalize this uniqueness result to any value of σ_ε , but we have not managed to do so due to the complexity of the 7th order polynomial characterizing a pure strategy equilibrium in Proposition 2. In particular, given Proposition 1, a reasonable conjecture is that the boundary between pure and mix strategy equilibria is at $\frac{\sigma_\varepsilon}{\sigma_u} = \frac{\sqrt{\sqrt{17}-4}}{2}$. This conjecture, albeit not formally proven, seems to hold numerically. That is, if $\frac{\sigma_\varepsilon}{\sigma_u} < \frac{\sqrt{\sqrt{17}-4}}{2}$, only a mixed strategy linear equilibrium exists, and if $\frac{\sigma_\varepsilon}{\sigma_u} \geq \frac{\sqrt{\sqrt{17}-4}}{2}$, only a pure strategy linear equilibrium exists. Either way, the linear equilibrium seems unique for all parameter values.

Propositions 1 and 2 suggest the following three-step algorithm to compute a linear equilibrium:

- Step 1: Compute all the positive root of the polynomial $f(\beta_{v,1}^2) = 0$ in Proposition 2. Retain the values of $\beta_{v,1} \in \left(0, \frac{\sigma_u}{\sqrt{\Sigma_0}}\right]$ to serve as candidates for a pure strategy equilibrium.
- Step 2: For each $\beta_{v,1}$ retained in Step 1, check whether the SOC in Proposition 2 is satisfied. If yes, then it is a pure strategy equilibrium; otherwise, it is not.
- Step 3: If $\frac{\sigma_\varepsilon}{\sigma_u} < \frac{\sqrt{\sqrt{17}-4}}{2}$, employ Proposition 1 to compute a mixed strategy equilibrium.

Figure 2: Implications for Trading Strategies



This figure plots the implications of trading on order-flow information for trading strategies of the fundamental investor and the technical trader. In each panel, the blue solid line plots the value in the equilibrium of this paper, and the dashed red line plots the value in a standard Kyle economy (i.e., $\sigma_\varepsilon = \infty$). The horizontal axis in each panel is the standard deviation σ_ε of the noise in the technical trader’s private signal about the fundamental investor’s past order. The other parameters are: $\sigma_u = 10$ and $\Sigma_0 = 100$.

Figure 2 plots in solid lines the equilibrium trading strategies of the fundamental investor and the technical trader as functions of σ_ε , where we set $\sigma_u = 10$ and $\Sigma_0 = 100$. As a comparison, the dashed lines show corresponding strategies in the standard two-period Kyle model without the order-flow informed trader. The first panel confirms that $\sigma_z > 0$ if and only if $\sigma_\varepsilon < 0.175\sigma_u = 1.75$. Also, when $\sigma_\varepsilon < 1.75$, the equilibrium value of σ_z decreases with σ_ε . That is, when there is more *exogenous* noise in the technical trader’s signal, the fundamental investor *endogenously* injects less noise into her own period-1 orders. This result points out a new channel—i.e., the amount of noise in the technical trader’s signal—that determines whether a mixed strategy equilibrium or a pure strategy one should prevail in a

Kyle-type auction game.

The other panels in Figure 2 are also intuitive. For instance, $\beta_{v,1}$ decreases with σ_ε in the mixed strategy regime, but increases with σ_ε in the pure strategy regime. This is because in the mixed strategy regime, as σ_ε increases, the fundamental investor adds less noise z to her order; to avoid revealing too much information to the technical trader, she trades less aggressively on v in period 1. In contrast, in the pure strategy equilibrium, as σ_ε increases, the fundamental investor knows that the technical trader will learn less from her order due to the increased exogenous noise ε , and so she can afford to trade more aggressively in period 1. The intensity $\beta_{v,1}$ with order-flow information is smaller than its counterpart without order-flow information in a standard Kyle model.

An interesting observation is that $\beta_{v,2}$ is hump-shaped in σ_ε , but the peak obtains when σ_ε is substantially above $\sigma_u\sqrt{\sqrt{17}-4/2}$. This is a combination of two effects. First, $\beta_{v,2}$ should have a negative relation with $\beta_{v,1}$, as the fundamental investor smoothes her trades across the two periods. Thus, the U-shaped $\beta_{v,1}$ leads to a hump-shaped $\beta_{v,2}$. Second, the fundamental investor also faces competition from the technical trader in the second period, and as σ_ε increases, this competition is less severe, so that the fundamental investor can afford to trade more aggressively on her private information. The second competition effect, adding to the first smoothing effect, implies that the hump-shaped $\beta_{v,2}$ achieves its peak above $\sigma_u\sqrt{\sqrt{17}-4/2}$.

It is straightforward to understand that δ_s decreases with σ_ε : A higher value of σ_ε means that the technical trader's private information s is less precise, and so he trades less aggressively on this information.

3 Implications of Order-Flow Informed Trading for Market Quality and Welfare

In this section we discuss the positive and normative implications of order-flow informed trading. In particular we show that the size of σ_ε affects equilibrium market outcomes, including price discovery, market liquidity, and the trading profits (or losses) of various trader types. Because these measures are proxies for market quality and welfare, our analysis generates important policy implications regarding the use of order-flow informed trading strategies.

We first examine the behavior of positive variables that represent market quality. In the microstructure literature, two leading positive variables are price discovery and market

liquidity. Price discovery measures how much information about the asset value v is revealed in prices p_1 and p_2 . Given price functions (8) and (9), prices are linear transformations of aggregate order flows y_1 and y_2 , and thus the literature has measured price discovery by the market maker’s posterior variances of v in periods 1 and 2:

$$\Sigma_1 \equiv \text{Var}(v|y_1) \text{ and } \Sigma_2 \equiv \text{Var}(v|y_1, y_2).$$

A lower Σ_t implies a more informative period- t price about v , for $t \in \{1, 2\}$. Price discovery is important because it helps allocation efficiency by conveying information that is useful for real decisions (see, for example, O’Hara (2003) and Bond, Edmans, and Goldstein (2012)).

In Kyle-type models (including ours), market liquidity is measured by the inverse of Kyle’s lambda (λ_1 and λ_2), which are price impacts of trading. A lower λ_t means that the period- t market is deeper and more liquid. One important reason to care about market liquidity is that it is related to the welfare of noise traders, who can be interpreted as investors trading for non-informational, liquidity/hedging reasons that are decided outside the financial markets. In general, noise traders are better off in a more liquid market, because their expected trading loss is $(\lambda_1 + \lambda_2) \sigma_u^2$ in our economy.

Next, the normative variables are the payoffs of each group of players in the economy, that is, the expected profit $E(\pi_1 + \pi_2)$ of the fundamental investor, the expected profit $E[(v - p_2) d_2]$ of the technical trader, and the expected loss $(\lambda_1 + \lambda_2) \sigma_u^2$ of noise traders. This approach allows us to discuss who wins and who loses as a result of a particular policy. In practice, investors’ trading motives range from fundamental analysis to liquidity shocks (e.g., client withdrawal from mutual funds or hedge funds). Our fundamental investor can be viewed as investors trading for informational reasons, and noise traders those trading for liquidity reasons. The technical trader is more in line with broker-dealers or HFTs who employ sophisticated trading technology and may possess superior order flow information. If the regulator wishes to protect liquidity-driven traders, the welfare of noise traders would be the relevant measure. If the regulator wishes to protect investors who acquire fundamental information, then the informed profit $E(\pi_1 + \pi_2)$ would be a relevant measure.

The following proposition gives a comparison between two “extreme” economies: the economy with $\sigma_\varepsilon = 0$ and the one with $\sigma_\varepsilon = \infty$ (i.e. the standard Kyle setting). For instance, the first economy may correspond to one in which HFTs are able to extract very precise information about the past orders submitted by large institutions. The second economy may represent one in which institutional investors manage to hide order-flow information almost completely (or an economy in which order-flow informed trading is banned by the regulator). In the proposition, we have used superscripts “0” and “Kyle” to indicate these

two economies.

Proposition 4 (Perfect Order-Flow Information vs. Standard Kyle). *In the two-period setting, the following orderings apply:*

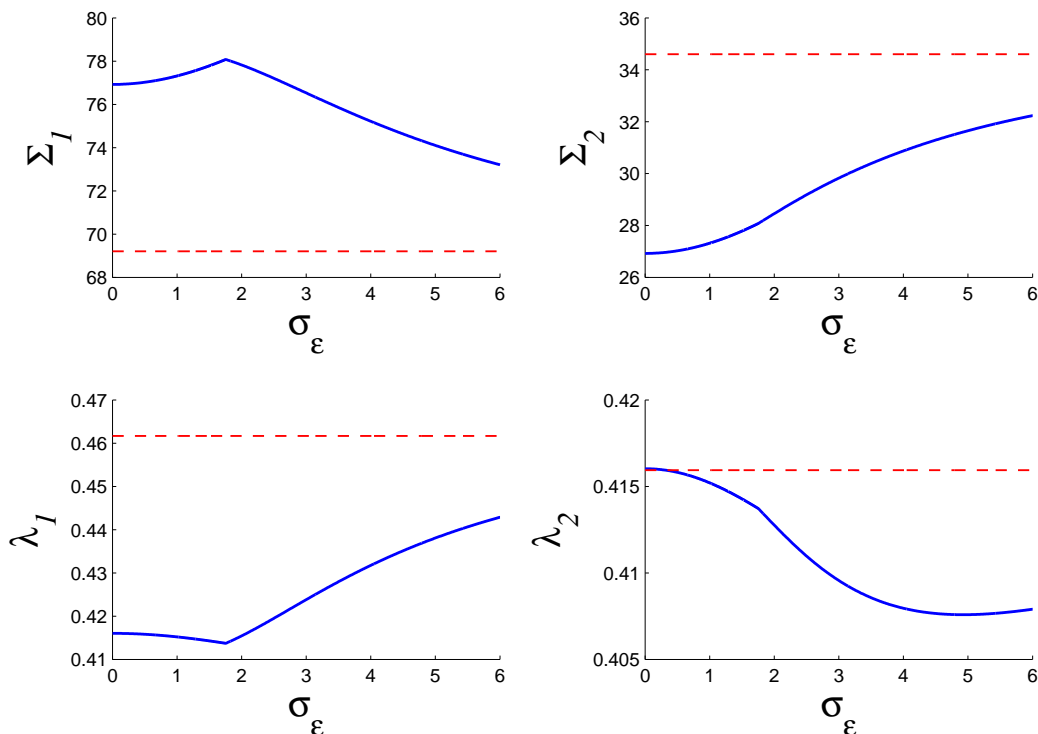
$$\begin{aligned} \Sigma_1^0 &> \Sigma_1^{Kyle}, \Sigma_2^0 < \Sigma_2^{Kyle}, \\ \lambda_1^0 &< \lambda_1^{Kyle}, \lambda_2^0 > \lambda_2^{Kyle}, \\ E(\pi_1^0) &< E(\pi_1^{Kyle}), E(\pi_2^0) < E(\pi_2^{Kyle}) \text{ and} \\ (\lambda_1^0 + \lambda_2^0) \sigma_u^2 &< (\lambda_1^{Kyle} + \lambda_2^{Kyle}) \sigma_u^2. \end{aligned}$$

The positive implications in Proposition 4 are in sharp contrast to those presented by Huddart, Hughes, and Levine (2001), although both studies consider a comparison between an economy featuring a mixed strategy equilibrium and a standard Kyle economy. In Huddart, Hughes, and Levine (2001), the market maker perfectly observes the past order placed by an informed trader. They find that market liquidity and price discovery unambiguously improve in both periods of their economy relative to a standard Kyle setting (i.e., their λ_1 , λ_2 , Σ_1 and Σ_2 are all smaller than the Kyle setting counterparts). In contrast, in our setting, the market maker does not observe the informed fundamental investor's past trade x_1 ; it is the technical trader who does, with some noise. As a result of the endogenous noise z placed by the fundamental investor in the mixed strategy and her more cautious trading on fundamental information (i.e., a smaller $\beta_{v,1}$), the first-period price discovery is harmed by order-flow information in our setting (i.e., $\Sigma_1^0 > \Sigma_1^{Kyle}$). The presence of *perfect* order-flow informed trading also worsens the second period market liquidity relative to the standard Kyle setting (i.e., $\lambda_2^0 > \lambda_2^{Kyle}$). This is again opposite to the effect of publicly revealing the informed orders in period 1, as in Huddart, Hughes, and Levine (2001).

Figures 3 and 4 respectively plot in solid lines the positive and normative implications as we continuously increase σ_ε from 0 to ∞ . The other two exogenous parameters are the same as those in Figure 2 ($\sigma_u = 10$ and $\Sigma_0 = 100$). The dashed lines plot the corresponding variables in a standard two-period Kyle model without the order-flow informed trader.

In Figure 3, we see that Σ_1 is hump-shaped in σ_ε , with the peak at the cutoff $\sigma_\varepsilon = \sigma_u \sqrt{\sqrt{17} - 4}/2$. The intuition is as follows. In the first period, only the fundamental investor's trade brings information about v into the market. Since her trading sensitivity $\beta_{v,1}$ on fundamental information is U-shaped in σ_ε (see Figure 2), Σ_1 should have the opposite pattern, i.e., hump-shaped. By contrast, Σ_2 monotonically increases with σ_ε in Figure 3. This is because in period 2, both the fundamental and the technical traders trade on value-relevant information, and as σ_ε increases, the technical trader's order brings less information

Figure 3: Implications for Positive Variables



This figure plots the market quality implications of trading on order-flow information. In each panel, the blue solid line plots the value in the equilibrium of this paper, and the dashed red line plots the value in a standard Kyle economy (i.e., $\sigma_\varepsilon = \infty$). The horizontal axis in each panel is the standard deviation σ_ε of the noise in the technical trader's private signal about the fundamental investor's past order. The other parameters are: $\sigma_u = 10$ and $\Sigma_0 = 100$.

about v into the price. Comparing the solid lines to dashed lines, we see that adding the order-flow informed trader harms price discovery in period 1 but improves price discovery in period 2.

The illiquidity measures in both periods, λ_1 and λ_2 , first decrease and then increase with σ_ε . Since adverse selection from the fundamental investor is the sole source of price impact in period 1, it is rather intuitive that λ_1 has a similar U-shape as $\beta_{v,1}$ (see equation (19)). The period-2 illiquidity measure λ_2 is also U-shaped and opposite to the humped-shaped $\beta_{v,2}$, by the first-order condition in period 2 (i.e., $\lambda_2 = \frac{1}{2\beta_{v,2}}$ by (14)).

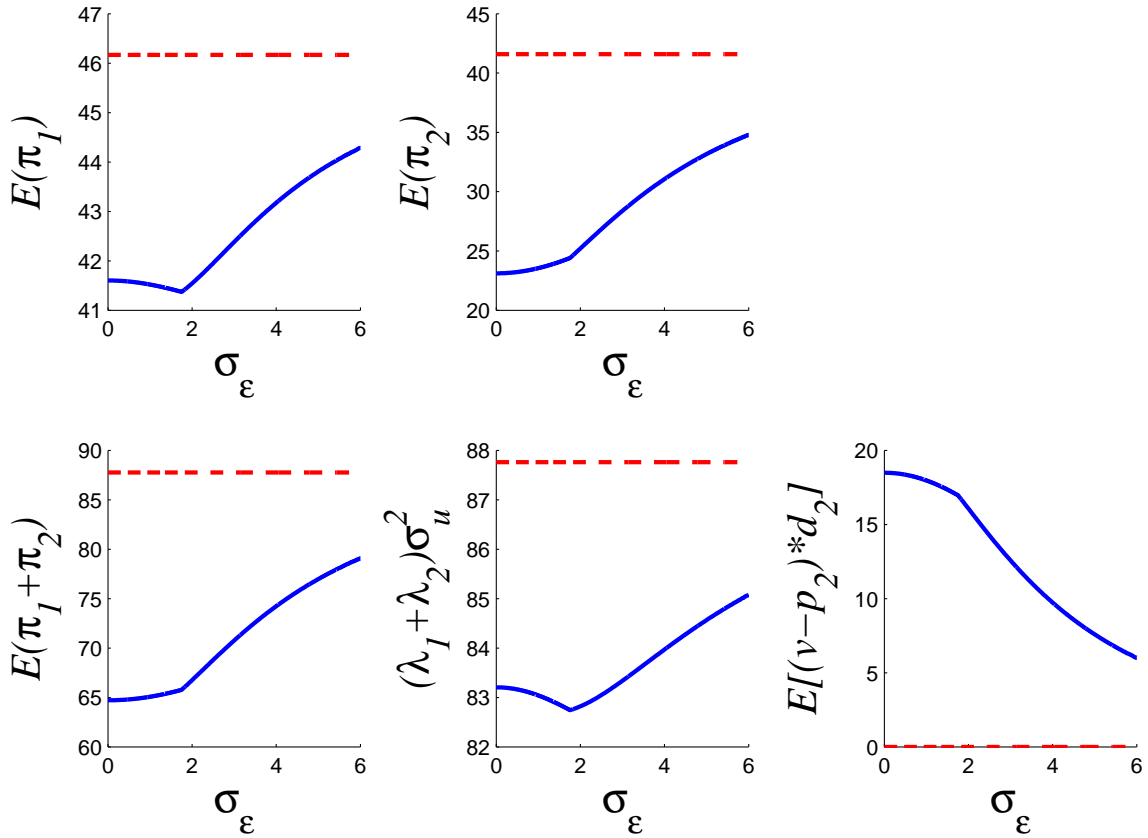
Comparing the solid lines to dashed lines, we find that order-flow informed trading generally improves the first-period market liquidity because the fundamental investor trades

less aggressively on her private information, but its impact on the second-period market liquidity is ambiguous. Consistent with Proposition 4, order-flow informed trading worsens the second-period liquidity relative to the standard Kyle setting, if and only if the technical trader's order-flow information is sufficiently precise. (In the neighborhood of $\sigma_\varepsilon = 0$, the solid line is strictly above the dashed line in the plot for λ_2 .) This is due to a combination of two effects. First, adding the technical trader introduces competition, which makes the period-2 aggregate order flow reflect more of the fundamental than noise trading. This tends to reduce λ_2 . Second, order-flow information also increases the amount of *private* information, which makes the adverse selection problem faced by the market maker more severe. This generally tends to increase λ_2 . When σ_ε is small, the technical trader has very precise private information and the second effect dominates, so that λ_2 is higher than its counterpart in a standard Kyle setting.

The top two panels of Figure 4 plot the fundamental investor's expected profits in the two periods, $E(\pi_1)$ and $E(\pi_2)$. We observe that $E(\pi_2)$ monotonically increases with σ_ε . This result is intuitive: A higher σ_ε means that the fundamental investor faces a less competitive technical trader in period 2, so her period-2 profit is higher on average. The period-1 profit $E(\pi_1)$ first decreases with σ_ε (in the mixed strategy regime) and then increases with σ_ε (in the pure strategy regime). This U-shaped profit pattern is natural given the U-shaped $\beta_{v,1}$ pattern in Figure 2. Clearly, the presence of order-flow informed trading lowers the profit of the fundamental investor.

The bottom three panels of Figure 4 present the total profit $E(\pi_1 + \pi_2)$ of the fundamental investor, the total loss $(\lambda_1 + \lambda_2) \sigma_u^2$ of noise traders, and the expected profit $E[(v - p_2) d_2]$ of the technical trader. All the results are as expected. As σ_ε increases, the technical trader has less precise private information, and thus $E[(v - p_2) d_2]$ decreases. Meanwhile, a higher σ_ε also implies that the fundamental investor faces less competition from the technical trader, and $E(\pi_1 + \pi_2)$ increases. The U-shaped total loss $(\lambda_1 + \lambda_2) \sigma_u^2$ of noise traders is a direct result of the U-shaped λ_1 and λ_2 in Figure 3. In general, noise traders lose less because of order-flow informed trading.

Figure 4: Implications for Normative Variables



This figure plots the profits of various groups of traders. In each panel, the blue solid line plots the value in the equilibrium of this paper, and the dashed red line plots the value in a standard Kyle economy (i.e., $\sigma_\varepsilon = \infty$). The horizontal axis in each panel is the standard deviation σ_ε of the noise in the technical trader's private signal about the fundamental investor's past order. The other parameters are: $\sigma_u = 10$ and $\Sigma_0 = 100$.

4 Conclusion

Order-flow informed trading is a salient part of modern financial markets. This type of trading strategies, such as order anticipation, often starts with no innate trading motive, but instead seeks and exploits information from other investors' order flows. While order-flow informed trading has long existed in financial markets, its latest incarnation in certain high-frequency trading strategies caused severe concerns among investors and regulators.

In this paper we study the strategic interaction between order-flow informed trading and fundamental informed trading. In our two-period model, which is based on Kyle (1985),

a technical trader observes, *ex post* and potentially with noise, the executed trades of the informed investor in period 1. The informed order flow thus provides a signal to the technical trader regarding the asset fundamental value. Using this information, the technical trader competes with the informed investor in period 2. If the technical trader's signal is sufficiently precise, the fundamental investor hides her information by endogenously adding noise into her period-1 order flow, leading to a mixed strategy equilibrium. The more precise is the order-flow signal, the more volatile is the added noise. As the technical trader's signal becomes sufficiently imprecise, the equilibrium switches to a pure strategy one, with no added noise in order flows. We prove uniqueness of equilibrium under natural conditions. The characterization of the equilibria, in particular the endogenous switch between a mixed strategy equilibrium and a pure strategy one, is the first main contribution of this paper.

Our second main contribution is to identify the effects of order-flow informed trading on market quality and welfare. Because the fundamental investor trades more cautiously and potentially adds noise into her period-1 orders, the presence of the order-flow informed trader harms price discovery in the first period. In the second period, however, price discovery is improved because of competition. Implications for market liquidity, measured by the inverse of Kyle's lambda, are mixed: Liquidity improves in the first period but it can either improve or worsen in the second period. The presence of order-flow informed trading harms the fundamental investor but benefits noise traders.

Appendix

A List of Model Variables

Variables	Description
<u>Random Variables</u>	
v	Asset liquidation value at the end of period 2, $N(p_0, \Sigma_0)$
x_1, x_2	Orders placed by the fundamental investor in periods 1 and 2
z	Noise component in the period-1 order x_1 of the fundamental investor
d_2	Order placed by the technical trader in period 2
s, ε	Signal observed by the technical trader, and its noise component
u_1, u_2	Noise trading in periods 1 and 2
y_1, y_2	Aggregate order flows in periods 1 and 2
p_1, p_2	Asset prices in periods 1 and 2
π_1, π_2	Fundamental investor's profits attributable to trades in periods 1 and 2
<u>Deterministic Variables</u>	
p_0, Σ_0	Prior mean and variance of the asset value
σ_u^2	Variance of noise trading in periods 1 and 2
σ_z	Standard deviation of the noise component z in the period-1 order x_1 placed by the fundamental investor
Σ_1, Σ_2	Posterior variance of the asset value in periods 1 and 2 (i.e., $Var(v y_1)$ and $Var(v y_1, y_2)$)
<u>Strategy Summary</u>	
$\beta_{v,1}$	$x_1 = \beta_{v,1}(v - p) + z$
$\beta_{v,2}, \beta_{x_1}, \beta_{y_1}$	$x_2 = \beta_{v,2}(v - p) - \beta_{x_1}x_1 + \beta_{y_1}y_1$
δ_s, δ_{y_1}	$d_2 = \delta_s s - \delta_{y_1}y_1$
λ_1	$p_1 = p_0 + \lambda_1 y_1$, with $y_1 = x_1 + u_1$
λ_2	$p_2 = p_1 + \lambda_2 y_2$, with $y_2 = x_2 + d_2 + u_2$

B Proofs

B.1 Proof of Equation (10)

Define $\sigma_x^2 \equiv Var(x_1) = \beta_{v,1}^2 \Sigma_0 + \sigma_z^2$. Direct computation shows

$$E(v|s, y_1) - E(v|y_1) = \frac{\beta_{v,1} \Sigma_0 \sigma_\varepsilon^{-2}}{\sigma_x^2 (\sigma_x^{-2} + \sigma_\varepsilon^{-2} + \sigma_u^{-2})} \left(s - \frac{\sigma_x^2}{\sigma_u^2 + \sigma_x^2} y_1 \right).$$

Thus, it suffices to show that

$$\frac{\delta_{y_1}}{\delta_s} = \frac{\sigma_x^2}{\sigma_u^2 + \sigma_x^2} \quad (\text{B1})$$

holds in equilibrium, in order for d_2 in equation (7) to admit a form given by equation (10).

By equation (17), we have:

$$\frac{\beta_{v,1} \Sigma_0}{\delta_s \lambda_2} = \frac{\sigma_x^2 [4(\sigma_x^{-2} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}) - \sigma_\varepsilon^{-2}]}{\sigma_\varepsilon^{-2}}. \quad (\text{B2})$$

Plugging the expression of $\lambda_1 = \frac{\beta_{v,1} \Sigma_0}{\sigma_z^2 + \sigma_u^2}$ (i.e. equation (19)) into equation (18) yields

$$\frac{\delta_{y_1}}{\delta_s} = \frac{\beta_{v,1} \Sigma_0}{\delta_s \lambda_2} \frac{1}{3(\sigma_x^2 + \sigma_u^2)} - \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}. \quad (\text{B3})$$

Inserting equation (B2) into (B3) and simplifying, we have equation (B1).

B.2 Proof of Proposition 1

A mixed strategy equilibrium is characterized by nine parameters, $\sigma_z, \beta_{v,1}, \beta_{v,2}, \beta_{y_1}, \beta_{x_1}, \delta_{y_1}, \delta_s, \lambda_1$ and λ_2 . These parameters are jointly determined by a system consisting of nine equations (given by (14), (17), (18), (19), (20) and (22)) as well as one SOC ($\lambda_2 > 0$ given by (13)). Note that by equation (22), δ_s is already known, and also $\lambda_1 = \lambda_2$ degenerates to one parameter, denoted by λ . So, the system characterizing a mixed strategy equilibrium essentially has six unknowns. To solve this system, we first simplify it to a 3-equation system in terms of $(\lambda, \beta_{v,1}, \sigma_z)$ and then solve this new system analytically.

Given that δ_s is known, parameter δ_{y_1} is also known by (18). Also, once λ is solved, the three equations in (14) will yield solutions of $\beta_{v,2}, \beta_{x_1}$ and β_{y_1} . Thus, the three equations left to compute $(\lambda, \beta_{v,1}, \sigma_z)$ are given by equations (17), (19) and (20). To solve this 3-equation system, we first express $\beta_{v,1}$ and λ as functions of σ_z , and then solve the single equation of σ_z .

By (17) and noting that $\lambda \equiv \lambda_1 = \lambda_2$, we have

$$\lambda = \frac{1}{\delta_s} \frac{\frac{\sigma_\varepsilon^{-2}}{(\beta_{v,1}^2 \Sigma_0 + \sigma_z^2)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}}{4 - \frac{\sigma_\varepsilon^{-2}}{(\beta_{v,1}^2 \Sigma_0 + \sigma_z^2)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}} \frac{\beta_{v,1} \Sigma_0}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2}.$$

Combining the above equation with (19) and the expression $\delta_s = \frac{\frac{4}{3}}{1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}}$, we can compute

$$\beta_{v,1}^2 = \frac{\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2}{3\Sigma_0\sigma_u^2 + 4\Sigma_0\sigma_\varepsilon^2}. \quad (\text{B4})$$

Equation (B4) puts an restriction on the endogenous value of σ_z , i.e., $\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2 > 0$, which can be shown to hold in equilibrium.

By (19) and (B4), we can express λ^2 as a function of σ_z^2 as follows:

$$\lambda^2 = \frac{\frac{\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2}{3\Sigma_0\sigma_u^2 + 4\Sigma_0\sigma_\varepsilon^2} \Sigma_0^2}{\left(\frac{\sigma_u^4 - 3\sigma_u^2\sigma_z^2 - 4\sigma_u^2\sigma_\varepsilon^2 - 4\sigma_z^2\sigma_\varepsilon^2}{3\Sigma_0\sigma_u^2 + 4\Sigma_0\sigma_\varepsilon^2} \Sigma_0 + \sigma_z^2 + \sigma_u^2 \right)^2}. \quad (\text{B5})$$

Inserting $\delta_s = \frac{\frac{4}{3}}{1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}}$ into equation (20) and further simplification yield

$$\left(\begin{aligned} & (52\Sigma_0\sigma_u^6 + 160\Sigma_0\sigma_u^4\sigma_\varepsilon^2 + 64\Sigma_0\sigma_u^2\sigma_\varepsilon^4) \beta_{v,1}^2 \\ & + (36\sigma_u^8 + 52\sigma_u^6\sigma_z^2 + 160\sigma_u^6\sigma_\varepsilon^2 + 160\sigma_u^4\sigma_z^2\sigma_\varepsilon^2 + 64\sigma_u^4\sigma_\varepsilon^4 + 64\sigma_u^2\sigma_z^2\sigma_\varepsilon^4) \end{aligned} \right) \lambda^2 \\ = (9\Sigma_0\sigma_u^6 + 9\Sigma_0\sigma_u^4\sigma_z^2 + 24\Sigma_0\sigma_u^4\sigma_\varepsilon^2 + 24\Sigma_0\sigma_u^2\sigma_z^2\sigma_\varepsilon^2 + 16\Sigma_0\sigma_u^2\sigma_\varepsilon^4 + 16\Sigma_0\sigma_z^2\sigma_\varepsilon^4).$$

Inserting equations (B4) and (B5) into the above equation, we can compute

$$\sigma_z^2 = \frac{\sigma_u^2 (\sigma_u^2 + 4\sigma_\varepsilon^2) (\sigma_u^4 - 16\sigma_\varepsilon^4 - 32\sigma_u^2\sigma_\varepsilon^2)}{(3\sigma_u^2 + 4\sigma_\varepsilon^2) (13\sigma_u^4 + 16\sigma_\varepsilon^4 + 40\sigma_u^2\sigma_\varepsilon^2)}, \quad (\text{B6})$$

which gives the expression of σ_z in Proposition 1.

In order for equation (B6) to indeed construct a mixed strategy equilibrium, we need

$$\sigma_z^2 = \frac{\sigma_u^2 (\sigma_u^2 + 4\sigma_\varepsilon^2) (\sigma_u^4 - 16\sigma_\varepsilon^4 - 32\sigma_u^2\sigma_\varepsilon^2)}{(3\sigma_u^2 + 4\sigma_\varepsilon^2) (13\sigma_u^4 + 16\sigma_\varepsilon^4 + 40\sigma_u^2\sigma_\varepsilon^2)} > 0 \Leftrightarrow \frac{\sigma_\varepsilon^2}{\sigma_u^2} < \frac{\sqrt{17}}{4} - 1.$$

Also, inserting equation (B6) into equation (B4), we see that (B4) is always positive. Finally, by equation (19) and $\lambda_2 = \lambda_1$, we know $\lambda_2 > 0$, i.e., the SOC is satisfied. Thus, when $\frac{\sigma_\varepsilon^2}{\sigma_u^2} < \frac{\sqrt{17}}{4} - 1$, the expression of σ_z^2 in equation (B6) indeed constructs a mixed strategy equilibrium.

Clearly, if $\frac{\sigma_\varepsilon^2}{\sigma_u^2} \geq \frac{\sqrt{17}}{4} - 1$, then the solved σ_z^2 would be non-positive in (B6), which implies the non-existence of a linear mixed strategy equilibrium.

B.3 Proof of Proposition 2

For a pure strategy equilibrium, we have $\sigma_z = 0$ and need to compute eight parameters, $\beta_{v,1}, \beta_{v,2}, \beta_{y_1}, \beta_{x_1}, \delta_s, \delta_{y_1}, \lambda_1$ and λ_2 . These parameters are determined by equations (14), (17), (18), (19), (20) and (23), together with two SOC's, (13) and (24). In particular, after setting $\sigma_z = 0$, we can simplify equations (17), (19) and (20) as follows:

$$\delta_s = \frac{\frac{\sigma_\varepsilon^{-2}}{(\beta_{v,1}^2 \Sigma_0)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}}{4 - \frac{\sigma_\varepsilon^{-2}}{(\beta_{v,1}^2 \Sigma_0)^{-1} + \sigma_\varepsilon^{-2} + \sigma_u^{-2}}} \lambda_2 \beta_{v,1}, \quad (\text{B7})$$

$$\lambda_1 = \frac{\beta_{v,1} \Sigma_0}{\beta_{v,1}^2 \Sigma_0 + \sigma_u^2}, \quad (\text{B8})$$

$$\lambda_2 = \frac{\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2} \beta_{v,1}\right) \frac{1}{\Sigma_0^{-1} + \beta_{v,1}^2 \sigma_u^{-2}}}{\left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2} \beta_{v,1}\right)^2 \frac{1}{\Sigma_0^{-1} + \beta_{v,1}^2 \sigma_u^{-2}} + \delta_s^2 \sigma_\varepsilon^2 + \sigma_u^2}. \quad (\text{B9})$$

Note that equation (B8) is the expression of λ_1 in Proposition 2.

The idea to compute the system characterizing a pure strategy equilibrium is to simplify it to a system in terms of $(\lambda_1, \lambda_2, \beta_{v,1}, \delta_s)$ and then characterize this simplified system as a single equation of $\beta_{v,1}$.

If we know $(\lambda_1, \lambda_2, \delta_s)$, then δ_{y_1} is known by equation (18), and $\beta_{y_1}, \beta_{v,2}$ and β_{x_1} are known by equation (14). Thus, the four unknowns $(\lambda_1, \lambda_2, \beta_{v,1}, \delta_s)$ are determined by the remaining four equations, (23) and (B7)–(B9), and the two SOC's, (13) and (24).

Now, we simplify this four-equation system as a single equation of $\beta_{v,1}$. The idea is to express $\lambda_1, \lambda_2 \delta_s$ and λ_2 as functions of $\beta_{v,1}$, and then insert these expressions into equation (23). By (B7),

$$\lambda_2 \delta_s = \frac{\beta_{v,1} \sigma_u^2 \Sigma_0}{4\sigma_u^2 \sigma_\varepsilon^2 + 3\beta_{v,1}^2 \Sigma_0 \sigma_u^2 + 4\beta_{v,1}^2 \Sigma_0 \sigma_\varepsilon^2}. \quad (\text{B10})$$

By (B9),

$$\lambda_2 = \sigma_u^{-1} \sqrt{\left(\frac{1}{2} + \frac{\lambda_2 \delta_s}{2} \beta_{v,1}\right) \frac{1}{\Sigma_0^{-1} + \beta_{v,1}^2 \sigma_u^{-2}} - \left(\frac{1}{2} + \frac{\lambda_2 \delta_s}{2} \beta_{v,1}\right)^2 \frac{1}{\Sigma_0^{-1} + \beta_{v,1}^2 \sigma_u^{-2}} - (\lambda_2 \delta_s)^2 \sigma_\varepsilon^2}.$$

Inserting (B10) into the above expression, we obtain

$$\lambda_2^2 = \Sigma_0 \frac{(2\sigma_u^4 + 4\sigma_\varepsilon^4 + 5\sigma_u^2 \sigma_\varepsilon^2) \Sigma_0^2 \beta_{v,1}^4 + (8\sigma_u^2 \sigma_\varepsilon^4 + 5\sigma_u^4 \sigma_\varepsilon^2) \Sigma_0 \beta_{v,1}^2 + 4\sigma_u^4 \sigma_\varepsilon^4}{(\beta_{v,1}^2 \Sigma_0 + \sigma_u^2) (3\sigma_u^2 \Sigma_0 \beta_{v,1}^2 + 4\sigma_\varepsilon^2 \Sigma_0 \beta_{v,1}^2 + 4\sigma_u^2 \sigma_\varepsilon^2)^2}, \quad (\text{B11})$$

which gives the expression of λ_2 in Proposition 2.

We can rewrite equation (23) as

$$2\lambda_2(2\beta_{v,1}\lambda_1 - 1) = \left[\beta_{v,1} \left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right) - 1 \right] \times \left[\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right]. \quad (\text{B12})$$

We then want to take square on both sides of (B12) in order to use (B11) to substitute λ_2^2 . Doing this requires that the terms $2\beta_{v,1}\lambda_1 - 1$ and $\beta_{v,1} \left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right) - 1$ have the same sign, that is,

$$(2\beta_{v,1}\lambda_1 - 1) \left[\beta_{v,1} \left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right) - 1 \right] \geq 0.$$

Inserting the expression of λ_1 and $\lambda_2\delta_s$ in (B8) and (B10) into the above condition, we find that the above inequality is equivalent to requiring

$$\beta_{v,1} \leq \frac{\sigma_u}{\sqrt{\Sigma_0}}.$$

Thus, given $\beta_{v,1} \leq \frac{\sigma_u}{\sqrt{\Sigma_0}}$, we can take square of (B12), and set

$$4\lambda_2^2(2\beta_{v,1}\lambda_1 - 1)^2 - \left[\beta_{v,1} \left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right) - 1 \right]^2 \times \left[\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right]^2 = 0.$$

Inserting the expression of λ_1 , $\lambda_2\delta_s$ and λ_2^2 in (B8), (B10) and (B11) into the above equation, we have the 7th order polynomial of $\beta_{v,1}^2$ as follows:

$$f(\beta_{v,1}^2) = A_7\beta_{v,1}^{14} + A_6\beta_{v,1}^{12} + A_5\beta_{v,1}^{10} + A_4\beta_{v,1}^8 + A_3\beta_{v,1}^6 + A_2\beta_{v,1}^4 + A_1\beta_{v,1}^2 + A_0 = 0, \quad (\text{B13})$$

where

$$A_7 = \Sigma_0^7 (2\sigma_u^4 + 4\sigma_\varepsilon^4 + 5\sigma_u^2\sigma_\varepsilon^2) (3\sigma_u^2 + 4\sigma_\varepsilon^2)^2, \quad (\text{B14})$$

$$A_6 = 2\Sigma_0^6\sigma_u^2 (4\sigma_\varepsilon^2 - \sigma_u^2) (3\sigma_u^2 + 4\sigma_\varepsilon^2) (3\sigma_u^4 + 6\sigma_\varepsilon^4 + 8\sigma_u^2\sigma_\varepsilon^2), \quad (\text{B15})$$

$$A_5 = -\Sigma_0^5\sigma_u^6 (27\sigma_u^6 + 336\sigma_\varepsilon^6 + 524\sigma_u^2\sigma_\varepsilon^4 + 246\sigma_u^4\sigma_\varepsilon^2), \quad (\text{B16})$$

$$A_4 = 4\Sigma_0^4\sigma_u^6 (3\sigma_u^8 - 144\sigma_\varepsilon^8 - 304\sigma_u^2\sigma_\varepsilon^6 - 182\sigma_u^4\sigma_\varepsilon^4 - 23\sigma_u^6\sigma_\varepsilon^2), \quad (\text{B17})$$

$$A_3 = -\Sigma_0^3\sigma_u^8 (\sigma_u^8 + 704\sigma_\varepsilon^8 + 752\sigma_u^2\sigma_\varepsilon^6 + 76\sigma_u^4\sigma_\varepsilon^4 - 57\sigma_u^6\sigma_\varepsilon^2), \quad (\text{B18})$$

$$A_2 = -4\Sigma_0^2\sigma_u^{10}\sigma_\varepsilon^2 (\sigma_u^6 + 48\sigma_\varepsilon^6 - 24\sigma_u^2\sigma_\varepsilon^4 - 31\sigma_u^4\sigma_\varepsilon^2), \quad (\text{B19})$$

$$A_1 = -4\Sigma_0\sigma_u^{12}\sigma_\varepsilon^4 (\sigma_u^4 - 32\sigma_\varepsilon^4 - 36\sigma_u^2\sigma_\varepsilon^2), \quad (\text{B20})$$

$$A_0 = 64\sigma_u^{14}\sigma_\varepsilon^8. \quad (\text{B21})$$

The final requirement is to ensure that a root to the polynomial also satisfies the two SOC's, (13) and (24). Given the expression of λ_2 in Proposition 2, (13) is redundant. Also, (24) implies $\beta_{v,1} > 0$, because (24) implies $\lambda_1 > 0$, which by (B8), in turn implies $\beta_{v,1} > 0$. So, the final constraint on $\beta_{v,1}$ is $0 < \beta_{v,1} \leq \frac{\sigma_u}{\sqrt{\Sigma_0}}$ and condition (24).

B.4 Proof of Proposition 3

When σ_ε is small: By Proposition 1, when σ_ε is small, there is a mixed strategy equilibrium. The task is to show that there is no pure strategy equilibrium. By (23) and the fact $\beta_{v,1} > 0$ in a pure strategy equilibrium, we have

$$\beta_{v,1} = \frac{1 - \frac{\lambda_1 + \lambda_2 \delta_s - \lambda_2 \delta_{y_1}}{2\lambda_2}}{2 \left[\lambda_1 - \frac{(\lambda_1 + \lambda_2 \delta_s - \lambda_2 \delta_{y_1})^2}{4\lambda_2} \right]} > 0. \quad (\text{B22})$$

Note that the denominator is the SOC in (24), which is positive. So, we must have

$$1 - \frac{\lambda_1 + \lambda_2 \delta_s - \lambda_2 \delta_{y_1}}{2\lambda_2} > 0 \Rightarrow 4\lambda_2^2 - (\lambda_1 + \lambda_2 \delta_s - \lambda_2 \delta_{y_1})^2 > 0.$$

Using (18) we can rewrite the above inequality as follows:

$$4\lambda_2^2 - \left(\frac{2}{3}\lambda_1 + \lambda_2 \delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right)^2 > 0. \quad (\text{B23})$$

Plugging the expression of λ_1 , $\lambda_2 \delta_s$ and λ_2^2 in (B8), (B10) and (B11) into the left-hand-side (LHS) of (B23), we find that (B23) is equivalent to

$$\begin{aligned} & (16\beta_{v,1}^4 \Sigma_0^2 + 32\beta_{v,1}^2 \Sigma_0 \sigma_u^2 + 16\sigma_u^4) \sigma_\varepsilon^4 + (8\beta_{v,1}^4 \Sigma_0^2 \sigma_u^2 - 4\beta_{v,1}^6 \Sigma_0^3 + 12\beta_{v,1}^2 \Sigma_0 \sigma_u^4) \sigma_\varepsilon^2 \\ & - \beta_{v,1}^2 \Sigma_0 \sigma_u^2 (\beta_{v,1}^2 \Sigma_0 - \sigma_u^2)^2 > 0. \end{aligned} \quad (\text{B24})$$

We prove that the above condition is not satisfied in a pure strategy equilibrium, as $\sigma_\varepsilon \rightarrow 0$. Proposition 2 implies that in a pure strategy equilibrium, $\beta_{v,1}^2 \in \left(0, \frac{\sigma_u^2}{\Sigma_0} \right]$. So, as $\sigma_\varepsilon \rightarrow 0$, the first two terms of the LHS of (B24) go to 0. Thus, if as $\sigma_\varepsilon \rightarrow 0$, $\beta_{v,1}^2$ does not go to 0 or $\frac{\sigma_u^2}{\Sigma_0}$ in a pure strategy equilibrium, then the third term of the LHS of (B24) is strictly negative, which proves our statement.

Now we consider the two cases that $\beta_{v,1}^2$ converges to 0 or to $\frac{\sigma_u^2}{\Sigma_0}$ as $\sigma_\varepsilon \rightarrow 0$. We will show that both lead to contradictions to a pure strategy equilibrium.

Note that if $\sigma_\varepsilon = 0$, the polynomial (B13) is negative at $\frac{\sigma_u^2}{\Sigma_0}$; that is, $f\left(\frac{\sigma_u^2}{\Sigma_0}\right) = -16\sigma_u^{22} < 0$ if $\sigma_\varepsilon = 0$. Thus, if for any sequence of $\sigma_\varepsilon \rightarrow 0$, we have $\beta_{v,1}^2 \rightarrow \frac{\sigma_u^2}{\Sigma_0}$ in a pure strategy equilibrium, then we must have $f(\beta_{v,1}^2) \rightarrow -16\sigma_u^{22} < 0$, which contradicts with Proposition 2 which says that $f(\beta_{v,1}^2) \equiv 0$ in a pure strategy equilibrium. Thus, $\beta_{v,1}^2 \not\rightarrow \frac{\sigma_u^2}{\Sigma_0}$ as $\sigma_\varepsilon \rightarrow 0$.

Suppose $\beta_{v,1}^2 \rightarrow 0$ in a pure strategy equilibrium for some sequence of $\sigma_\varepsilon^2 \rightarrow 0$. By (23), we have

$$\left(\frac{2}{3}\lambda_1 + \lambda_2 \delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right)^2 > \left(1 - 2 \frac{\beta_1 \Sigma_0}{\beta_1^2 \Sigma_0 + \sigma_u^2} \beta_1 \right)^2 4\lambda_2^2.$$

Combining the above condition with condition (B23), we know $\left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2$ has the same order as λ_2^2 :

$$O\left(\left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2\right) = O(\lambda_2^2).$$

Substituting into the above equation the expression of λ_1 , $\lambda_2\delta_s$ and λ_2^2 from (B8), (B10) and (B11) and matching the highest-order terms, we can show that $\beta_{v,1}^2$ has the same order as σ_ε^4 . As a result, by (B8), $\lambda_1 \rightarrow 0$; by (B10), $\lambda_2\delta_s$ goes to a positive finite number; and by (B11), λ_2 goes to a positive finite number. This in turn implies the SOC (24) is violated. Specifically, by (18), the SOC is equivalent to

$$\lambda_1 - \frac{\left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2}{4\lambda_2} > 0. \quad (\text{B25})$$

However, as $\sigma_\varepsilon^2 \rightarrow 0$, we have $\lambda_1 - \frac{\left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2}{4\lambda_2} \rightarrow -\frac{(\lambda_2\delta_s)^2}{4\lambda_2} < 0$, a contradiction.

When σ_ε is large: By Proposition 1, when σ_ε is sufficiently large, there is no mixed strategy equilibrium. The task is to show that a linear pure strategy equilibrium exists and is unique.

By equations (B14)–(B21), we have $A_7 > 0$, $A_6 > 0$, $A_5 < 0$, $A_4 < 0$, $A_3 < 0$, $A_2 < 0$, $A_1 > 0$ and $A_0 > 0$, when σ_ε^2 is sufficiently large. Thus, by Descartes' Rule of Signs, there are at most two positive roots of $\beta_{v,1}^2$.

By equation (B13), we have

$$\begin{aligned} f(0) &= 64\sigma_u^{14}\sigma_\varepsilon^8 > 0, \\ \lim_{\beta_{v,1}^2 \rightarrow \infty} f(\beta_{v,1}^2) &\propto \Sigma_0^7 (2\sigma_u^4 + 4\sigma_\varepsilon^4 + 5\sigma_u^2\sigma_\varepsilon^2) (3\sigma_u^2 + 4\sigma_\varepsilon^2)^2 \times \infty > 0. \end{aligned}$$

In addition, as $\sigma_\varepsilon^2 \rightarrow \infty$, $f\left(\frac{\sigma_u^2}{\Sigma_0}\right) \propto -1024\sigma_u^{14}\sigma_\varepsilon^8 < 0$. So, there is exactly one root of $\beta_{v,1}^2$ in the range of $\left(0, \frac{\sigma_u^2}{\Sigma_0}\right)$ and one root in the range of $\left(\frac{\sigma_u^2}{\Sigma_0}, \infty\right)$. Given that in a pure strategy equilibrium, we require $0 < \beta_{v,1}^2 \leq \frac{\sigma_u}{\Sigma_0}$ by Proposition 2, only the small root is a possible equilibrium candidate (which is indeed an equilibrium if the SOC is also satisfied).

Finally, we can show that the small root of $\beta_{v,1}^2 \in \left(0, \frac{\sigma_u^2}{\Sigma_0}\right)$ satisfies the SOC as $\sigma_\varepsilon^2 \rightarrow \infty$. Specifically, by (B25), the SOC is

$$\lambda_1 - \frac{\left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2}\right)\right)^2}{4\lambda_2} > 0 \Leftrightarrow$$

$$16\lambda_2^2\lambda_1^2 - \left(\frac{2}{3}\lambda_1 + \lambda_2\delta_s \left(1 + \frac{4\sigma_\varepsilon^2}{3\sigma_u^2} \right) \right)^4 > 0.$$

Plugging the expression of λ_1 , $\lambda_2\delta_s$ and λ_2^2 in (B8), (B10) and (B11) into the LHS of the above condition, we can show that the above condition holds if and only if

$$B_4\sigma_\varepsilon^8 + B_3\sigma_\varepsilon^6 + B_2\sigma_\varepsilon^4 + B_1\sigma_\varepsilon^2 + B_0 > 0, \quad (\text{B26})$$

where,

$$\begin{aligned} B_4 &= 768\beta_{v,1}^{10}\Sigma_0^5 + 4096\beta_{v,1}^8\Sigma_0^4\sigma_u^2 + 8704\beta_{v,1}^6\Sigma_0^3\sigma_u^4 + 9216\beta_{v,1}^4\Sigma_0^2\sigma_u^6 + 4864\beta_{v,1}^2\Sigma_0\sigma_u^8 + 1024\sigma_u^{10}, \\ B_3 &= 2048\beta_{v,1}^{10}\Sigma_0^5\sigma_u^2 + 8704\beta_{v,1}^8\Sigma_0^4\sigma_u^4 + 13824\beta_{v,1}^6\Sigma_0^3\sigma_u^6 + 9728\beta_{v,1}^4\Sigma_0^2\sigma_u^8 + 2560\beta_{v,1}^2\Sigma_0\sigma_u^{10}, \\ B_2 &= 2144\beta_{v,1}^{10}\Sigma_0^5\sigma_u^4 + 6720\beta_{v,1}^8\Sigma_0^4\sigma_u^6 + 6912\beta_{v,1}^6\Sigma_0^3\sigma_u^8 + 2240\beta_{v,1}^4\Sigma_0^2\sigma_u^{10} - 96\beta_{v,1}^2\Sigma_0\sigma_u^{12}, \\ B_1 &= 1056\beta_{v,1}^{10}\Sigma_0^5\sigma_u^6 + 2112\beta_{v,1}^8\Sigma_0^4\sigma_u^8 + 912\beta_{v,1}^6\Sigma_0^3\sigma_u^{10} - 160\beta_{v,1}^4\Sigma_0^2\sigma_u^{12} - 16\beta_{v,1}^2\Sigma_0\sigma_u^{14}, \\ B_0 &= 207\beta_{v,1}^{10}\Sigma_0^5\sigma_u^8 + 180\beta_{v,1}^8\Sigma_0^4\sigma_u^{10} - 54\beta_{v,1}^6\Sigma_0^3\sigma_u^{12} - 12\beta_{v,1}^4\Sigma_0^2\sigma_u^{14} - \beta_{v,1}^2\Sigma_0\sigma_u^{16}. \end{aligned}$$

Given that $\beta_{v,1}$ is bounded, we have that as σ_ε^2 is large, the LHS of condition (B26) is determined by $B_4\sigma_\varepsilon^8$, which is always positive: $B_4\sigma_\varepsilon^8 > 1024\sigma_u^{10}\sigma_\varepsilon^8 > 0$.

B.5 Proof of Corollary 1

Now suppose $\sigma_\varepsilon \rightarrow \infty$. By Proposition 3, as σ_ε is large, there is a unique linear equilibrium, which is a pure strategy equilibrium. In a pure strategy equilibrium, we always have $f(\beta_{v,1}^2) = 0$. If we rewrite the polynomial f as a polynomial in terms of σ_ε , we must have that as $\sigma_\varepsilon \rightarrow \infty$, the coefficients on the highest order of σ_ε goes to 0. This exercise yields the following condition that as $\sigma_\varepsilon \rightarrow \infty$, we have

$$64\Sigma_0^7\beta_{v,1}^{14} + 192\Sigma_0^6\sigma_u^2\beta_{v,1}^{12} - 576\Sigma_0^4\sigma_u^6\beta_{v,1}^8 - 704\Sigma_0^3\sigma_u^8\beta_{v,1}^6 - 192\Sigma_0^2\sigma_u^{10}\beta_{v,1}^4 + 128\Sigma_0\sigma_u^{12}\beta_{v,1}^2 + 64\sigma_u^{14} \rightarrow 0. \quad (\text{B27})$$

Define $x \equiv \beta_{v,1}^2 \frac{\Sigma_0}{\sigma_u^2} \in [0, 1]$ in a pure strategy equilibrium. Condition (B27) becomes

$$-2x - x^2 + x^3 + 1 \rightarrow 0, \text{ as } \sigma_\varepsilon \rightarrow \infty. \quad (\text{B28})$$

That is, as $\sigma_\varepsilon \rightarrow \infty$, we must have that (B28) holds.

In a standard Kyle setting, the unique equilibrium is defined by

$$-2x^* - x^{*2} + x^{*3} + 1 = 0. \quad (\text{B29})$$

Specifically, Proposition 1 of Huddart, Hughes, and Levine (2001) characterizes the equilibrium in a two-period Kyle model by a cubic in terms of K ,

$$8K^3 - 4K^2 - 4K + 1 = 0. \quad (\text{B30})$$

By the expressions of $\beta_1 = \frac{2K-1}{4K-1} \frac{1}{\lambda_1}$ and $\lambda_1 = \frac{\sqrt{2K(2K-1)} \sqrt{\Sigma_0}}{4K-1} \frac{1}{\sigma_u}$ in Proposition 1 of Huddart, Hughes, and Levine (2001), we have $K = \frac{1}{2(1-x^*)}$, where $x^* = \beta_1^2 \frac{\Sigma_0}{\sigma_u^2}$. Then, equation (B30) is equivalent to equation (B29). Given that $-2x - x^2 + x^3 + 1$ is monotone and continuous in the range of $[0, 1]$, we have $x \rightarrow x^*$ as $\sigma_\varepsilon \rightarrow \infty$, by conditions (B28) and (B29).

B.6 Proof of Proposition 4

We here give the expression of the variables in the proposition. The comparison follows from setting $\sigma_\varepsilon = 0$ and $\sigma_\varepsilon = \infty$ in these expressions and from straightforward computations.

Setting $\sigma_\varepsilon = 0$ in Proposition 1 yields $\sigma_z = \sqrt{\frac{1}{39}} \sigma_u$ and $\beta_{v,1} = \frac{2}{\sqrt{13}} \frac{\sigma_u}{\sqrt{\Sigma_0}}$. Plugging these two expressions of σ_z and $\beta_{v,1}$ into the expressions of λ_1 and λ_2 in Proposition 1 gives λ_1^0 and λ_2^0 . In a pure strategy equilibrium, λ_1 and λ_2 are given by Proposition 2. Setting $\sigma_\varepsilon = \infty$ yields the expressions of λ_1^{Kyle} and λ_2^{Kyle} .

Direct computation shows that in a mixed strategy equilibrium, the price discovery variables are given by

$$\begin{aligned} \Sigma_1^{mixed} &= \frac{(\sigma_z^2 + \sigma_u^2) \Sigma_0}{\beta_{v,1}^2 \Sigma_0 + \sigma_z^2 + \sigma_u^2}, \\ \Sigma_2^{mixed} &= \frac{(4\sigma_u^4 + 4\sigma_u^2 \sigma_z^2 + \sigma_u^2 \delta_s^2 \sigma_z^2 + 4\sigma_u^2 \delta_s^2 \sigma_\varepsilon^2 + 4\delta_s^2 \sigma_z^2 \sigma_\varepsilon^2) \lambda_2^2 \Sigma_0}{\left((\Sigma_0 \lambda_2^2 \sigma_u^2 \delta_s^2 + 4\Sigma_0 \lambda_2^2 \sigma_u^2 + 4\Sigma_0 \lambda_2^2 \delta_s^2 \sigma_\varepsilon^2) \beta_{v,1}^2 + 2\lambda_2 \Sigma_0 \sigma_u^2 \delta_s \beta_{v,1} \right. \\ &\quad \left. + \lambda_2^2 (4\sigma_u^4 + 4\sigma_u^2 \sigma_z^2 + \sigma_u^2 \delta_s^2 \sigma_z^2 + 4\sigma_u^2 \delta_s^2 \sigma_\varepsilon^2 + 4\delta_s^2 \sigma_z^2 \sigma_\varepsilon^2) + \Sigma_0 (\sigma_u^2 + \sigma_z^2) \right)}. \end{aligned}$$

Plugging $\sigma_\varepsilon = 0$, $\sigma_z = \sqrt{\frac{1}{39}} \sigma_u$ and $\beta_{v,1} = \frac{2}{\sqrt{13}} \frac{\sigma_u}{\sqrt{\Sigma_0}}$ into the above expressions yields Σ_1^0 and Σ_2^0 . In a pure strategy equilibrium, we can compute

$$\begin{aligned} \Sigma_1^{pure} &= \frac{\sigma_u^2 \Sigma_0}{\sigma_u^2 + \beta_{v,1}^2 \Sigma_0}, \\ \Sigma_2^{pure} &= \frac{1}{\Sigma_0^{-1} + \beta_{v,1}^2 \sigma_u^{-2} + \left(\frac{1}{2\lambda_2} + \frac{\delta_s}{2} \beta_{v,1} \right)^2 (\delta_s^2 \sigma_\varepsilon^2 + \sigma_u^2)^{-1}}. \end{aligned}$$

Setting $\sigma_\varepsilon = \infty$ in Proposition 2, computing $\beta_{v,1}$, λ_2 and δ_s , and inserting these expressions into the above expressions, we have the expressions of Σ_1^{Kyle} and Σ_2^{Kyle} .

Finally, we present the profit expressions. By (15), the fundamental investor's *ex-ante* expected period-2 profit is

$$E(\pi_2) = \frac{E[v - p_1 - \lambda_2(-\delta_y y_1 + \delta_s x_1)]^2}{4\lambda_2}.$$

Using equations (5) and (8), we can show

$$E(\pi_2) = \frac{[1 - (\lambda_1 - \lambda_2\delta_y + \lambda_2\delta_s)\beta_{v,1}]^2 \Sigma_0 + (\lambda_1 - \lambda_2\delta_y)^2 \sigma_u^2 + (\lambda_1 - \lambda_2\delta_y + \lambda_2\delta_s)^2 \sigma_z^2}{4\lambda_2}.$$

For $E(\pi_2^0)$, we set $\sigma_\varepsilon = 0$, compute $\sigma_z, \beta_{v,1}, \lambda_1, \lambda_2, \delta_s$ and δ_y in Proposition 1, and insert these expressions into the above equation. For $E(\pi_2^{Kyle})$, we set $\sigma_\varepsilon = \infty$ in Proposition 2 and compute the relevant parameters accordingly.

To give the expression of $E(\pi_1)$, we first compute $E(\pi_1 + \pi_2)$, and then use the above computed $E(\pi_2)$ to compute $E(\pi_1) = E(\pi_1 + \pi_2) - E(\pi_2)$. Using equation (21), we can show that in a mixed strategy equilibrium,

$$E(\pi_1^{mixed} + \pi_2^{mixed}) = \frac{\Sigma_0 + \sigma_u^2 \lambda^2 (1 - \delta_y)^2}{4\lambda}$$

while in a pure strategy equilibrium

$$E(\pi_1^{pure} + \pi_2^{pure}) = \frac{\left(1 - \frac{\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_y}{2\lambda_2}\right)^2}{4\left(\lambda_1 - \frac{(\lambda_1 + \lambda_2\delta_s - \lambda_2\delta_y)^2}{4\lambda_2}\right)} \Sigma_0 + \frac{\Sigma_0 + \sigma_u^2 (\lambda_1 - \lambda_2\delta_y)^2}{4\lambda_2}.$$

Then, setting $\sigma_\varepsilon = 0$ and $\sigma_\varepsilon = \infty$ in the above two expressions gives $E(\pi_1^0 + \pi_2^0)$ and $E(\pi_1^{Kyle} + \pi_2^{Kyle})$, respectively.

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