Information Aggregation with Symmetry

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October 24, 2016

PRELIMINARY DRAFT: PLEASE DO NOT QUOTE OR CIRCULATE.

Abstract

Price taking behavior and fully revealing prices are two defining characteristics of rational expectations equilibrium. In a one-period model of strategic informed trading in a CARA-normal setting, we find that as private information is shared among more, smaller agents, either traders become price takers or prices become fully revealing, but not both. The key intuition comes from the opposite effects of endogenous competitiveness on prices and the quantity. More aggressive trading does not lead to more informative prices; more informative prices do not imply more aggressive trading. Our result is robust to correlated private values, endowment shocks, orthogonal convenience yields and exogenous noise trading.

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1 Introduction

If one million people have the same private information, will they effectively become price-takers? If all private information is shared among seven billion people, will the price be fully revealing? That is, as the market becomes more competitive, will a strategic equilibrium approach rational expectations equilibrium?

We find the answer to the above question to be no in a setting we describe below. As the number of competitors for a given private information grows, either agents become price-takers or the price becomes fully revealing but not both. Price-taking behaviors and fully revealing prices, which are defining characteristics of rational expectations equilibrium, cannot be achieved together at the same time.

We study a one-period model of strategic informed trading in a CARA-normal setting. All strategic traders have the same risk aversion, and their private information are of the same quality. This symmetry assumption allows the model to nest correlated private values, endowment shocks, orthogonal convenience yields, overconfidence and underconfidence and exogenous noise trading in one analytically tractable setting. To understand the effect of competitiveness, we let each private information be shared among a group of traders. We vary the number of competitors within each group while keeping the market size fixed.

Our first result is that price informativeness, the extent to which a trader can learn other information from the price, is independent of the number of competitors as long as there is no exogenous noise trading. As more traders share the same information, they trade more aggressively and do become price takers as the number of competitors approaches infinity. This, however, does not make the price more informative. While trades trade more aggressively on their private information, they also trade more aggressively on their private values, endowment shocks and/or convenience yields - which we refer to as endogenous noise - so that the price informativeness remains exactly the same. Traders become price takers, but the price does not become fully revealing.

The existence of an equilibrium requires sufficient amount of endogenous noise to overcome adverse selection. We propose that we can always find a well-defined equilibrium by adding vanishing exogenous noise trading in spirit of a trembling hand perfect equilibrium. In this equilibrium the price becomes more informative and does become fully revealing as the number of competitors approaches infinity. However, the market is infinitely noncompetitive and traders do not trade even when initial allocations are very inefficient. The price becomes fully revealing, but traders do not become price takers.

To understand the economic intuition we measure the competitiveness of equilibrium outcome by the ratio between the optimal demand of a strategic trader and that of a price taker. Importantly this measure of competition has opposite implications for the quantity and the price. As the market becomes more competitive, and therefore each strategic agent trades more, a trader incorporates less of his private information into the price. When the market becomes infinitely noncompetitive, a trader incorporates the largest fraction of his private information into the price, which is half.

On one hand this is quite obvious. A trader behaves as a price taker exactly because he does not have price impact. Traders are simply trading off the quantity and the price. On the other hand, this is counter to common intuition. For example, when traders shade their bid more, some infer that these traders hide their information more. However, the trader who shades their bid the most, almost not trading at all, incorporates the most fraction of his information into the price. Also, when traders do not leave money on the table, therefore, they reach their target inventory, some infer that they should incorporate their information the most. However, the trader who reaches their target inventory, does not incorporate his information into the price at all.

Therefore, price is not fully revealing even when there is very little noise. With vanishing noise trading the market is infinitely noncompetitive and each trader incorporates half of their private information into the price. If all traders with the same information collude, the price information will be also half. As the number of competitors increases, the price informativeness increases to approximately 2/3, 3/4, ... and becomes fully revealing at the limit. Interestingly, the equilibrium price informativeness resembles the total quantity produced in a Cournot equilibrium when the number of competitors for private information is replaced by the number of producers for given good. The price is partially revealing unless the number of competitors is infinity.

This gives an insight to understand what is going on in Grossman and Stiglitz (1980) (also Hellwig (1980)). They assume price taking behavior and make exogenous noise trading disappear. However, in a strategic model, the market becomes more noncompetitive as noise disappears. At the limit traders are not price takers even if there are infinite number of competitors. In the context of our model, the Grossman-Stiglitz paradox is not a paradox. Price is going to be fully revealing only if there are infinite number of competitors, which will never happen if information is costly.

Later we allow traders to have heterogeneous beliefs on the precision of private information. There is agreeing to disagreement. With overconfident traders, overconfidence can make price more informative or make people trade more aggressively but not both at the same time. With underconfidence and sufficiently small noise, there is an upside down equilibrium where people buy when their private information is negative and the price is high. The intuition is that traders think that the price incorporates too much of their own information and too little of everybody else's. In this equilibrium price impact is negative and traders trade more aggressively than price takers. Bergemann, Heumann and Morris (2015) also show that an equilibrium price impact (and thus χ) can be negative. In their paper the negative price impact comes from the information's relative weight on the common versus private values.

Developing microfoundations for competitive rational expectations equilibrium has been the topic of many papers. Kyle (1989) shows that the limit of strategic trading model as the noise trading (σ_Z^2) goes to infinity can be a competitive REE. In a setting where traders have interdependent private values, Reny and Perry (2006) and Vives (2011*a*,*b*) show that as the number of strategic traders goes to infinity, price becomes almost fully revealing and traders become almost price-takers. Rostek and Weretka (2012, 2015) also achieve this result provided that pairwise correlations among traders' values are homogeneous, which is the case in our paper. In contrast to our model with exponetial utility, Vives (2011*a*,*b*), Rostek and Weretka (2012, 2015), Bergemann, Heumann and Morris (2015) use quadratic utility.

Our paper is different from the papers above in that we focus on the effect of competitiveness while keeping the size of the market constant. This is achieved by allowing the same private information to be shared by M traders whose risk bearing capacity is equally divided among them. Fixing the size of market is important because more traders with new signals entering the market changes the quality of signals of each traders and thus price informativeness.

The order of limits is crucial in the results. While any order of limits would make a mathematical sense, it may not make an economic sense. In this paper we take a stand on the order of limits to study the perfect competition limit. We believe the market with one billion traders should be viewed as perfectly competitive. In other words, any limits should be taken for large but finite number of traders before taking the number of traders to infinity. Saying it differently, if a strategic trading equilibrium were to converge to competitive REE, a model where one billion traders have the exactly same information should look almost like a competitive REE, which is not the case. There are finite number of humans in the real world.

Our symmetric model is generally simpler than asymmetric models in which traders differ in risk aversion or quality of information. It allows us to nest many trading motives that have been studied separately in the existing literature such as endowment shock (as in Diamond and Verrecchia (1981)), private values (as in Vives (2011*b*)), overconfidence (as in Kyle, Obizhaeva and Wang (2016)). Incorporating asymmetry in a tractable manner is left for future work.

The plan for this paper is as follows. Section 2 provides a simple version of the model to motivate the discussions. Section 3 describes the set-up of the full model. Section 4 characterizes an equilibrium with key endogenous variables: price informativeness φ and scaled illiquidity χ. Section 5 provides necessary and sufficient conditions for the existence and uniqueness of an equilibrium and proposes adding vanishing noise trading to remedy the nonexistence of an equilibrium. Section 6 allows the possibility of heterogeneous beliefs to the main model and analyzes its equilibrium implications. Section 7 concludes.

2 Motivating Example

The price informativeness is independent of the competitiveness of the economy. To illustrate this important result, consider the following simplified version of the model with fewer ingredients. A more general version of this result is proved in later sections.

Setup. In a one-period model of informed trading, traders exchange a single risky asset against a safe asset whose return is normalized to one. The exogenous liquidation value of the risky asset v is distributed normally with mean zero and variance $\sigma_V^2 > 0$. There are N

symmetric groups of traders. Each group consists of M identical informed speculators. There are MN traders indexed (m, n). Each informed trader has exponential utility with risk aversion parameter $M\rho$; thus, $\rho > 0$ measures the aggregate risk aversion of each group.

For now, assume $\rho = \sigma_V = 1$. As discussed later, dimensional analysis implies that this normalization is without loss of generality.

Before trading, each of the M traders in group n observes two signals. First, trader (m, n) observes a noisy observation of the liquidation value v given by

$$i_n := \sqrt{\tau_I} \cdot v + e_n, \qquad \text{where} \qquad e_n \sim N(0, 1). \tag{1}$$

The error terms $e_1, \ldots, e_N$ are distributed independently across groups.

Second, trader (m, n) observes the realization of group n's aggregate endowment shock in the risky asset in shares, denoted $\mathcal{S}_n$. The quantity $\mathcal{S}_n$ is distributed independently across groups with

$$\mathcal{S}_n \sim N\left(\bar{\mathcal{S}}_n, \sigma_S^2\right), \qquad \text{where} \qquad \sum_{n=1}^{N} \bar{\mathcal{S}}_n = 0. \tag{2}$$

The endowment of trader (m, n) is the trader's pro rata share $\mathcal{S}_{m,n} := \frac{1}{M}\mathcal{S}_n$.

Within group n, each trader observes the same signal and has the same endowment. The signals and endowments are private information within groups.

The liquidation value v is distributed independently from $e_1, \ldots, e_N$ and $\mathcal{S}_1, \ldots, \mathcal{S}_N$.

These assumptions imply that the group's aggregate risk aversion (ρ), aggregate endowment shock (σ_S^2), and quality of private information (τ_I) do not change when the number of traders within each group (M) changes. Intuitively, varying the parameter M isolates changes in the competitiveness with which trading on private information about payoffs and endowments takes place, as the same information and risk are shared by more and more traders without changing other aspects of the economy.

Except for different mean endowments $\bar{\mathcal{S}}_n$, the model is symmetric in that it looks the same from the perspective of every trader. Before the realization of private information and endowment shocks, all traders have identical risk aversion, identical beliefs about the signal precision, and identical beliefs about the distribution of endowment shocks about their means. Furthermore, it is common knowledge that the mean aggregate endowment is zero ($\sum_{n=1}^{N} \bar{\mathcal{S}}_n = 0$).

Equilibrium. Each trader (m, n) submits a demand schedule $X_{m,n}(P \mid \mathcal{S}_n, i_n)$ which specifies, as a function of the group's endowment and information, the number of shares the trader is willing to buy or sell at any price P. An auctioneer aggregates all MN demand schedules $X := \{X_{m,n} : m = 1, \ldots, M; n = 1, \ldots, N\}$ to calculate a market clearing price $P(X)$. If there are many prices that clear the market, the auctioneer chooses the price that minimizes the amount of trading. An equilibrium is a Nash equilibrium in demand schedules in which each trader chooses his own demand schedule to maximize his own expected utility, taking the demand schedules of other traders as given. We consider only *symmetric linear equilibria*, defined as equilibria in which all traders choose the same linear demand schedule having the form

$$X_{m,n}(P \mid \mathcal{S}_n, i_n) = \pi_{\text{const}} - \pi_S \mathcal{S}_n + \pi_I i_n - \pi_P P. \tag{3}$$

Trader (m, n), conjecturing that the other $MN - 1$ traders use linear demand schedules above, faces a linear residual supply function of the form

$$P = p_{m,n} + \lambda X_{m,n}, \tag{4}$$

where $p_{m,n}$ measures the price if trader (m, n) does not trade and λ measures the price impact of each share traded (in dollars-per-share-squared).

Now consider the relationship between the informativeness of prices and the competitiveness of the equilibrium.

To measure the informativeness of prices, define φ implicitly as the unique solution to

$$\mathrm{var}^{m,n}\left\{v \mid P, i_n, S_n\right\}^{-1} = 1 + \tau_I + (N-1)\,\tau_I \varphi, \qquad \forall\,(m, n)\,. \tag{5}$$

In this equation, $\mathrm{var}^{m,n}\left\{v \mid P, i_n, S_n\right\}^{-1}$ is the ratio of the prior variance to the posterior variance, and φ intuitively measures the informativeness of prices as the fraction of other traders aggregate precision ($(N-1)\,\tau_I$) revealed by prices to trader (m, n). If $\varphi = 0$, a trader does not learn anything from price beyond the private information he already has. If $\varphi = 1$, a trader learns all relevant information that is available to the economy.

To describe the competitiveness of the equilibrium, compare the quantity traded by a trader with the quantity the trader would trade if he were a perfect competitor. Perfect competition is usually defined as a trader "taking the market price as given." In a finance context, this captures the intuition that a trader ignores price impact in making a decision to buy or sell. Intuitively, perfect competition means that a strategic trader chooses to trade the same quantity that he would trade in the absence of price impact.

Define trader (m, n)'s "target inventory" as the quantity the trader would trade as a perfect competitor given the information revealed by the price in the imperfectly competitive equilibrium. In a CARA-normal setting, this quantity is given by

$$\mathcal{S}^{TI}_{m,n} := \frac{\mathrm{E}^{m,n}\left\{v_n \mid P, i_n, \mathcal{S}_n\right\} - p_{m,n}}{M/\left(1 + \tau_I + (N-1)\,\tau_I \varphi\right)}. \tag{6}$$

Define "scaled illiquidity" χ by scaling price impact λ as

$$\chi := \frac{\lambda}{M}\left(1 + \tau_I + (N-1)\,\tau_I\varphi\right). \tag{7}$$

The endogenous parameter χ turns out to be a convenient measure of the competitiveness of the economy as a result of the following theorem. All proofs are in the Appendix.

Theorem 1 (Competition and Price Informativeness)**.** *There exists a symmetric linear equilibrium if and only if $\sigma_S^2 > \left(\frac{N}{MN-2}\right)\tau_I$. A symmetric linear equilibrium is unique if it exists. In an equilibrium, price informativeness φ and scaled illiquidity χ are given by*

$$\varphi = \frac{\tau_I}{\tau_I + \sigma_S^2}, \tag{8}$$

$$\chi = \frac{1 + N\tau_I/\sigma_S^2}{MN - 2 - N\tau_I/\sigma_S^2}. \tag{9}$$

The equilibrium quantity traded by trader $(m,\, n)$ can be written

$$x_{m,n} = \frac{1}{2\chi + 1}\left(\mathcal{S}_{m,n}^{TI} - \mathcal{S}_{m,n}\right). \tag{10}$$

Under perfect competition, $\mathcal{S}_{m,n}^{TI} - \mathcal{S}_{m,n}$ would be the optimal demand. If a trader initially holds the target inventory $\mathcal{S}_{m,n}^{TI}$, then the equilibrium quantity traded is zero. The competitiveness parameter χ scales price impact λ so that the equilibrium quantity traded is a fraction $1/(2\chi + 1)$ of its hypothetical counterpart under perfect competition. A perfectly competitive equilibrium implies $\chi = 0$, with all traders reaching their target inventories after the one round of trading. Under imperfect competition, price impact induces traders to trade less than the quantity which reaches their target inventory, implying $\chi > 0$.

Competition and Price Informativeness. Theorem 1 provides a transparent result describing how the number of traders (M) affects the informativeness of prices (φ) and competitiveness (χ) of the equilibrium outcome.

Equation (8) shows that the value of φ does not depend on the number of traders M per group. Price informativeness is determined solely by the ratio between the quality of private information and the size of endowment shock (τ_I/σ_S^2). It does not depend on how many traders share the same private information. This is counter to the conventional wisdom that more competition makes prices more informative by inducing traders to shade their bids less and thus trade more aggressively.

As M increases, equation (9) shows that traders do indeed trade more aggressively. This more aggressive trading does not lead to more informative price because traders also shade their trading resulting from endowment shocks. Indeed, the proportion by which traders shade their trading, as a function of M, is the same for private information and endowment shocks because optimal exercise of monopoly power is governed by the same incentives for both private information and endowment shocks. Therefore, in a model with strategic traders, price informativeness remains constant, regardless of the competitiveness of the market, as long as the relative quality of their information (τ_I/σ_S^2) is the same.

3 Setup

Now consider the following more general model which nests the motivating example in section 2 as a special case. There is one round of trading in which traders exchange a single risky asset against a safe asset whose return is normalized to one. There are N symmetric groups, each with M identical informed speculators. Each informed trader has exponential utility with risk aversion coefficient $M\rho$, implying that ρ measures the aggregate risk aversion of one group. Before trading, each trader (m, n) in group n obtains two identical signals: private information i_n and risky endowments $\mathcal{S}_{m,n} = \frac{1}{M}\mathcal{S}_n$. The quantity $\mathcal{S}_n$ is distributed independently across groups as (2).

Value of the Risky Asset. For traders in group n, the value of the risky asset is the sum of two random variables, v_n and o_n. The random variables v_n, which we call group n's fundamental value, are jointly normally distributed with

$$v_n \sim N\left(v_0, \sigma_V^2\right) \qquad \text{and} \qquad \mathrm{Cov}\left(v_m, v_n\right) = \zeta\sigma_V^2, \qquad \forall m \neq n. \tag{11}$$

The parameter $1/\sigma_V^2$ measures the prior precision of trader n's beliefs about the value v_n, and the parameter ζ, which must satisfy $\zeta \in \left[-\frac{1}{N-1}, 1\right]$, measures the correlation in traders' valuations across groups.

The random variables o_n, which we call a "convenience yield," has the normal distribution

$$o_n \sim N\left(0, \sigma_O^2\right) \tag{12}$$

A common values model is obtaining by assuming $\zeta = 1$ and $\sigma_O^2 = 0$ because these assumptions imply $v_n = v_{n'}$, $\forall n, n'$ and $o_n = 0$. Assuming $\zeta < 1$ or $\sigma_O^2 > 0$ implies traders have private values.

Traders in group n have identical private information about v_n given by

$$i_n = \tau_I^{1/2}\left(\frac{v_n}{\sigma_V}\right) + e_n, \qquad \text{where} \qquad e_n \sim N\left(0, 1\right). \tag{13}$$

Thus, the parameter ζ measures the importance of private values concerning which there is also private information while the parameter σ_O measures the importance of private values for which there is no private information.

Exogenous Noise Traders Exogenous noise traders demand a random quantity z which is distributed $N\left(0, \sigma_z^2\right)$. Noise traders do not optimize anything; their trading is exogenous. Since the model does not require exogenous noise traders, the model allows $\sigma_Z^2 = 0$.

Assume all random variables are jointly normally distributed, with $v_1, \ldots, v_N$ distributed independently from $e_1, \ldots, e_N$; $o_1, \ldots, o_N$; $\mathcal{S}_1, \ldots, \mathcal{S}_N$; and z.

Trading. There is one round of trading. After observing their own endowment shocks $\mathcal{S}_n$, private value o_n, and private signal i_n, each trader (m, n) submits a demand schedule $X_{m,n}(P \mid \mathcal{S}_n, o_n, i_n)$. This notation means that $X_{m,n}$ as a function of the price P, and the function is measurable with respect to $\mathcal{S}_n$, o_n, and i_n.

Let X denote the $M \times N$ matrix of submitted demand functions whose (m, n)th element corresponds to $X_{m,n}$. An auctioneer aggregates all MN functions to calculate a market clearing price, denoted $P(X)$, which satisfies the market clearing condition

$$\sum_{n=1}^{N} \sum_{m=1}^{M} X_{m,n}\left(P(X)\right) + z = 0. \tag{14}$$

If there is no market clearing price, then there is no trade ($x_{m,n} = 0, \forall (m, n)$. If there are many market clearing prices, then the auctioneer choose the smallest price which minimizes trading volume. Given the matrix of submitted demand schedules, trader (m, n) realizes wealth

$$W_{m,n}(X) := (v_n + o_n) \frac{\mathcal{S}_n}{M} + \left(v_n + o_n - P(X)\right) X_{m,n}\left(P(X)\right) \tag{15}$$

and achieves expected utility $u_{m,n}(X)$ given by[1]

$$u_{m,n}(X) := E^{m,n}\{-\exp\left(-M\rho W_{m,n}(X)\right)\}. \tag{16}$$

The model is described by nine exogenous parameters: M, N, ρ, τ_I, ζ, σ_V^2, σ_Z^2, σ_S^2, and σ_O^2. There are two dimensions of measurement: dollars and shares. The parameters σ_z and σ_S have dimensions of shares, the parameters σ_O and σ_V have dimensions of dollars-per-share, and ρ has dimensions of per-dollar. In what follows, all endogenous parameters are normalized to be dimensionless by scaling by ρ^{-1} (dollars) and $(\rho\sigma_V)^{-1}$ (shares). Changing units of measurement has no real effect on equilibrium.

A description of exogenous parameters is provided in Table 1. A description of endogenous parameters is provided in Table 2. A description of random variables is provided in Table 3.

4 Characterization of Equilibrium

This section characterizes symmetric linear equilibrium and provides necessary and sufficient conditions for the existence and uniqueness of a symmetric linear equilibrium.

Definition. An *equilibrium* is a matrix of demand schedules X such that (1) a market clearing price $P(X)$ is always well-defined and (2) for all $m = 1, \ldots, M$ and $n = 1, \ldots, N$, trader (m, n) chooses his demand schedule $X_{m,n}$ to maximize his expected utility $u_{m,n}(X)$ in the following sense: If X' denotes the matrix of demand schedules obtained by changing the demand-

[1] The superscripts m, n in the expectation operator $E^{m,n}$ in equation (16) indicate that trader (m, n) uses his own beliefs about the precisions of other traders signals to calculate the probability distribution of his terminal wealth $W_{m,n}$. This notation is unnecessary in the current context where traders share a common prior. The notation has meaning in section 6, which generalizes the model to one with heterogenous beliefs.

schedule element (m, n) from $X_{m,n}$ to any other demand schedule $X'_{m,n}$, then

$$u_{m,n}(X) \geq u_{m,n}(X'). \tag{17}$$

This definition implies that traders understand how changing their demand schedules affects their own utility by affecting both the quantity they trade and the market clearing price. Traders not only understand how their trading affects prices, but they also learn from prices in the sense they can extract information about the value of the asset embedded in the residual demand schedule they face.

Define a *symmetric linear equilibrium* as an equilibrium in which all traders choose the same linear demand schedule

$$\rho\sigma_V X_{m,n}(P|\mathcal{S}_n, o_n, i_n) = \pi_{\text{const}} - \pi_S \rho\sigma_V \mathcal{S}_n + \pi_O \frac{o_n}{\sigma_V} + \pi_I i_n - \pi_P \frac{P}{\sigma_V}, \tag{18}$$

where the five dimensionless endogenous parameters π_{const}, π_S, π_O, π_I, and π_P define the same linear function $X_{m,n}$.

If $MN\pi_P = 0$, then every trader submits a totally inelastic demand schedule and the resulting aggregate demand is either identically zero or some random quantity which is non-zero with probability one. Market clearing requires this aggregate demand to be identically zero; this further requires noise trading to be zero with probability one ($\sigma_z^2 = 0$) and each trader's demand to be identically zero ($X_{m,n} \equiv 0, \forall m, n$). In such a no-trade equilibrium, the market clearing price is not uniquely determined, as any price can support the allocations. We call this a *trivial no-trade equilibrium.* A trivial no-trade equilibrium exists whenever $\sigma_z^2 = 0$ and traders submit flat demand schedules ($\pi_P = 0$).

This paper focuses on other types of symmetric linear equilibria, in which the price is always well-defined because traders submit either downward-sloping or upward-sloping schedules. Such equilibria may involve no trade. Depending on the values of the exogenous parameters, equilibrium demand schedules may be either upward sloping (the usual case) or downward sloping ("upside-down equilibrium" discussed in section 6.2). Discussing equilibria with non-linear strategies takes us beyond the scope of this paper.

4.1 Linear Filtering

If a symmetric linear equilibrium has a well-defined price ($\pi_P \neq 0$), then equation (18) implies trader (m, n) always has a well-defined residual supply schedule given by

$$\frac{P}{\sigma_V} = p_{m,n} + \lambda \cdot \rho\sigma_V \cdot X_{m,n}. \tag{19}$$

In equation (19), consistent with the convention of scaling endogenous quantities to be dimensionless, the intercept of the residual supply schedule $p_{m,n}$ and the market impact parameter λ are both dimensionless. This is done by scaling P (dollars-per-share) by σ_V to make P/σ_V dimensionless on the left side and by scaling $X_{m,n}$ (shares) by $\rho\sigma_V$ to make it dimensionless on the right side.

In such a linear equilibrium, the price also has the linear form

$$\frac{P}{\sigma_V} = \alpha + \beta \cdot \frac{\sum_{n=1}^{N} i_n}{N} + \gamma\rho\sigma_V \cdot z - \delta\rho\sigma_V \cdot \frac{\sum_{n=1}^{N} S_n}{N} + \eta \cdot \frac{\sum_{n=1}^{N} o_n}{N\sigma_V}. \tag{20}$$

Linear demand schedules (18) and the market clearing condition (14) imply that these six dimensionless endogenous parameters (α, β, γ, δ, η, and λ) are given by

$$\alpha = \frac{\pi_{\text{const}}}{\pi_P}, \quad \beta = \frac{\pi_I}{\pi_P}, \quad \gamma = \frac{1}{MN\pi_P}, \quad \delta = \frac{\pi_S}{\pi_P}, \quad \eta = \frac{\pi_O}{\pi_P} \quad \text{and} \quad \lambda = \frac{1}{(MN-1)\,\pi_P}. \tag{21}$$

Traders learn about v_n from the equilibrium price. Since random variables are jointly normally distributed and the price function (20) is linear, trader (m, n)'s conditional expectation of v_n is a linear function of $p_{m,n}$, i_n, $\mathcal{S}_n$, and o_n.

Let τ^* denote the dimensionless ratio of the prior variance to the posterior variance given by

$$\tau^* = \frac{\sigma_V^2}{\text{var}^{m,n}\left\{v_n \mid P,\, i_n,\, S_n,\, o_n\right\}}. \tag{22}$$

Since the posterior variance is at least as accurate as the prior variance, the inequality $\tau^* \geq 1$ holds by definition. Trader (m, n)'s learning from price is described by the following lemma.

Lemma 1 (Learning From Prices.)**.** *Assume $\beta \neq 0$ and $(N-1)\,\zeta^2\tau_I \neq 0$. Then φ solves*

$$\frac{1-\varphi}{\varphi} = \tau_I\,(1-\zeta)\,(1+(N-1)\,\zeta) + \left[\left(\frac{N^2}{N-1}\right)\left(\frac{\gamma\rho\sigma_V\sigma_Z}{\beta}\right)^2 + \left(\frac{\delta\rho\sigma_V\sigma_S}{\beta}\right)^2 + \left(\frac{\eta\sigma_O}{\beta\sigma_V}\right)^2\right], \tag{23}$$

and τ^* is given by

$$\tau^* = 1 + \tau_I + (N-1)\,\tau_I\zeta^2\varphi. \tag{24}$$

Discussion. The dimensionless endogenous parameter φ is both intuitively and analytically important in this paper. Intuitively, the parameter φ measures how much information about v_n trader (m, n) extracts from prices. If $\varphi = 0$, then no information is extracted. If $\varphi = 1$ and $\zeta = 1$, then the maximum amount of information is extracted. When traders are identical ($N = 1$), private values are uncorrelated ($\zeta = 0$), traders have no private information ($\tau_I = 0$), or traders do not trade on the private information they have ($\beta = 0$), then there is no learning from prices. We set $\varphi = 0$ in these cases.

4.2 Characterization of Equilibrium

Equation (23) of φ has endogenous variables in the price function. The analytical strategy in this paper is to derive a polynomial expression of φ as a function of exogenous parameters only. This requires developing solutions for the five endogenous parameters (α, β, γ, δ, and η) defining the linear price function (20) in terms of exogenous parameters and φ. In pursuing this analytical strategy, it turns out that the five parameters (π_{const}, π_S, π_O, π_I, and π_P) defining the linear demand function $X_{m,n}$ in equation (18) are not important; instead, they are obtained as an afterthought at the end of our analysis.

The following theorem provides closed-form solutions for all of the other endogenous parameters in terms of exogenous parameters and φ.

Theorem 2 (Characterization of Symmetric Linear Equilibrium)**.** *Suppose $\rho > 0$, $\sigma_V > 0$, and $MN > 2$. If $(N-1)\,\zeta^2\tau_I^2 > 0$, then the set of symmetric linear equilibria, excluding trivial no-trade equilibria, is characterized by the set of all endogenous parameters φ such that (1) φ solves*

$$\begin{aligned}\frac{1-\varphi}{\varphi} &= \tau_I\,(1-\zeta)\,(1+(N-1)\,\zeta) + \frac{1}{\tau_I}\left[\left(1+\tau_I+(N-1)\,\tau_I\zeta^2\varphi\right)\frac{\sigma_O}{\sigma_V}\right]^2 \\ &\quad + \frac{(\rho\sigma_V\sigma_S)^2}{\tau_I} + \frac{(\rho\sigma_V\sigma_Z)^2/(N-1)}{\tau_I\left[\frac{MN-2}{MN-1} - \left(\frac{MN+N-2}{MN-1}\right)\zeta\varphi\right]^2},\end{aligned} \tag{25}$$

and (2) φ satisfies the second order condition

$$\varphi < \left(\frac{MN-2}{MN+N-2}\right)\frac{1}{\zeta}. \tag{26}$$

In such an equilibrium, with τ^* given by (24), the demand schedule of trader (m, n) is given by

$$\rho\sigma_V x_{m,n} = \frac{1}{M}\left[\frac{MN-2}{MN-1} - \left(\frac{MN+N-2}{MN-1}\right)\zeta\varphi\right]\times \\ \left[\left(\frac{1-\tau_I(N-1)\zeta(1-\zeta)\varphi}{1+(N-1)\zeta\varphi}\right)\frac{v_0}{\sigma_V} - \frac{\tau^* P/\sigma_V}{1+(N-1)\zeta\varphi} + \frac{\tau^* o_n}{\sigma_V} + \sqrt{\tau_I}\, i_n - \rho\sigma_V S_n\right], \tag{27}$$

and the market clearing price is given by

$$\frac{P}{\sigma_V} = \left[\frac{1-\tau_U(N-1)\zeta(1-\zeta)\varphi}{\tau^*}\right]\frac{v_0}{\sigma_V} + \\ \left(\frac{1+(N-1)\zeta\varphi}{N\tau^*}\right)\left(\sqrt{\tau_I}\sum_{i=1}^{N} i_n + \tau^*\sum_{n=1}^{N}\frac{o_n}{\sigma_V} - \sum_{n=1}^{N}\rho\sigma_V S_n + \frac{\rho\sigma_V z}{\frac{MN-2}{MN-1} - \left(\frac{MN+N-2}{MN-1}\right)\zeta\varphi}\right). \tag{28}$$

If $(N-1)\zeta^2\tau_I^2 = 0$, an equilibrium is characterized with $\varphi = 0$, and the demand schedule and market clearing price are given by (27) and (28) respectively.

The following corollary is an immediate implication of Theorem 2, equation (25):

Corollary 1. *If a symmetric linear equilibrium exists, then φ does not depend on M when $\sigma_Z^2 = 0$, and φ is increasing in M when $\sigma_Z^2 > 0$.*

Competition and Price Informativeness. Equation (25) is the most important equation in this paper. It represents the intuition that, in order to understand the equilibrium, it is first necessary to understand how much traders learn from prices. It also shows our first result that price informativeness is independent of competition applies in a more general setting.

On the right hand side of (25) there are different types of "noise" including correlated private values, endowment shocks, convenience yields, and exogenous noise trading. Without exogenous noise trading ($\sigma_Z^2 = 0$), the competitiveness of the market (number of traders per group M) does not affect the informativeness of price. Algebraically, this occurs because the parameter M only appears in equation (25) in the term involving exogenous noise trading σ_z^2.

The economic intuition for why more aggressive trading in a competitive market does not lead to more informative price remains the same as in Section 2. Strategic traders' incentives to trade more or less aggressively govern private information, endowment shocks, private value and convenience yields in the same manner. This does not apply to exogenous noise trading because it is exogenous.

Equation (25) describes economic tradeoffs involving different types of noise. For example, a change of one unit in the normalized endowment shock ($\rho\sigma_V\sigma_S$) has the same effect on the informativeness of price as a change of τ^* unit in convenience yields (σ_O/σ_V). The amount of "noise" associated with private values is given by $\tau_I\,(1-\zeta)\,(1+(N-1)\,\zeta)$; there is not private-values noise when $\zeta = 1$. From the perspective of traders in group n, the uncorrelated component of other groups' fundamental values (v_m, $\forall m \neq n$) are simply noise. This noise mitigates the adverse selection problem but makes prices less informative. Whether the price will be more informative as the number of groups (N) increases depends on how $(1-\zeta)\,(1+(N-1)\,\zeta)$ changes in N.

4.3 Scaled Illiquidity.

Consistent with the motivating example of section 2 (equation (7)), define scaled illiquidity χ by

$$\chi := \frac{\lambda\tau^*}{M}. \tag{29}$$

Since χ determines the equilibrium quantity demanded as a fraction of its counterpart under price taking, χ measures the competitiveness of the market. To interpret χ further, define target inventory $\mathcal{S}^{TI}_{m,n}$ as the level of inventory at which the trader would not want to trade any more

$$\rho\sigma_V\mathcal{S}^{TI}_{m,n} = \frac{\mathrm{E}^{m,n}\left\{\frac{v_n}{\sigma_V} \mid p_{m,n},\, i_n,\, \mathcal{S}_n,\, o_n\right\} + \frac{o_n}{\sigma_V} - p_{m,n}}{M/\tau^*}. \tag{30}$$

Now we present the results.

Theorem 3. *For all* $(m,\ n)$*, scaled illiquidity* χ*, defined by* (29)*, satisfies*

$$\chi = \frac{\lambda\rho\sigma_V^2\mathcal{S}^{TI}_{m,n}}{E^{m,n}\left\{v_n + o_n - \sigma_V p_{m,n} \mid p_{m,n},\, i_n,\, S_n,\, o_n\right\}}. \tag{31}$$

The optimal quantity traded $x_{m,n}$ *and market clearing price* P *can be expressed as*

$$x_{m,n} = \frac{1}{1+2\chi}\left(\mathcal{S}^{TI}_{m,n} - \mathcal{S}_{m,n}\right), \tag{32}$$

and

$$P - \sigma_V p_{m,n} = \frac{1}{2}\left(1 - \frac{1}{1+2\chi}\right) E^{m,n}\left\{v_n + o_n - \sigma_V p_{m,n} \mid p_{m,n},\, i_n,\, S_n,\, o_n\right\}. \tag{33}$$

Comments. Equation (31) provides a practical interpretation of χ as a trader's cost of reaching the target inventory as a fraction of a paper trading profit (which is the alpha achieved in an absence of price impact). In (32), $1/(1+2\chi)$ is a trader's equilibrium demand as a fraction of his target demand. This makes χ a measure of the competitiveness of equilibrium trading. If $\chi = 0$, the trader acts as a price taker and the outcome is perfectly competitive. As χ increases above zero, the economic outcome is less competitive. In equation (33), the quantity $\chi/(1+2\chi)$ measures the difference between the equilibrium price and the price if a trader does not trade as a fraction of the difference between a trader's valuation and the price if he does not trade. When $\chi = 0$, this fraction is zero; the price-taking trader has no effect of prices. Optimal exercise of monopoly power implies that a trader will never move the equilibrium price more than halfway between his own valuation and the price that would prevail if he did not trade. The limit of $\chi/(1+2\chi) \to 1/2$ is reached in the case of perfect monopoly as $\chi \to \infty$.

As an index of competitiveness, χ may be broadly applied outside our symmetric model to compare how much "illiquidity" affects the trading of different traders in different markets. Scaled illiquidity χ may be high either because price impact λ is high or risk aversion $M\rho$ is low. The reciprocal of risk aversion $1/M\rho$, which measures risk tolerance, intuitively corresponds to assets under management.[2] If an asset manager with large assets under management is compared to an asset manager with small assets under management, the former may be greatly constrained by market liquidity (large χ) while the latter is unconstrained (small χ).[3]

The value of χ can be expressed as a function of the informativeness of prices φ and exogenous parameters.

Theorem 4. *Scaled illiquidity χ is given by*

$$\chi = \frac{1 + (N-1)\zeta\varphi}{MN - 2 - (MN + N - 2)\zeta\varphi}. \tag{34}$$

[2]For a small mean μ and small variance σ^2, the competitive demand function for a log-utility investor with wealth W is approximately $W\mu/\sigma^2$. When this is compared to the CARA-normal competitive demand $\rho^{-1}\mu\sigma^2$, it is easy to see that $1/\rho$ maps directly into wealth W. If an asset manager is managing the wealth of log-utility investors, then W corresponds to assets under management.

[3]The price impact parameter in this paper is the same as the price impact parameter λ in Kyle (1985) and Kyle (1989). In the one-period model of Kyle (1985), the informed trader is a monopolist. This corresponds to $\chi \to \infty$, with the informed trader moving the price to a point half-way between his value and his expected price if he does not trade. In the one-period model of Kyle (1989), our parameter $\chi/(1+2\chi)$ is similar to the "information incidence" parameter ζ, which is always positive but less than one-half.

and can be written as a convergent power series

$$\chi = \frac{1}{MN-2} + \left(\frac{MN-1}{MN-2}\right) \sum_{k=1}^{\infty} \left(\frac{N}{MN-2}\right)^k \left(\frac{\zeta\varphi}{1-\zeta\varphi}\right)^k, \tag{35}$$

if and only if an equilibrium exists. The following three conditions are equivalent: (1) The power series (35) *converges; (2) an equilibrium exists; and (3) the second order condition* (26) *holds.*

Comments. When there are more traders within each group, traders reach closer toward their target inventory by trading more aggressively. This is easy to see from equation (35). Theorem 2 shows that price informativeness φ is independent of M without exogenous noise trading and is increasing in M with exogenous noise trading. Thus, regardless of exogenous noise trading, χ is decreasing in M. As M approaches infinity, χ approaches zero and all traders "take price as given" and reach their target inventory after one round of trading. This is why, thus far, we have used M and χ as measures of competitiveness interchangeably.

Equation (35) decomposes χ into two parts. The first term ($1/(MN-2)$) captures the pure effect of competition, which disappears when there are many traders. The second term captures the effect of information on competitiveness of the economy. It disappears when there is no private information ($\tau_I = 0$) or the asset values are independent across traders ($\zeta = 0$). As discussed above, this effect disappears as M increases, ceteris paribus.

The effect of the number of groups N on competitiveness χ is less clear. If price informativeness φ is held constant, increasing N does not make the second factor in the power series disappear. The economic intuition is that even when there are many groups, each group's private information remains unique. As long as there are a finite number of traders within each group, they maintain incentives to trade less aggressively. This is why the model is set up with MN traders, distinguishing the number of traders per group M from the number of groups N.

Allocational and Informational Efficiency. The parameter χ can also be interpreted as a measure of allocational efficiency. An equilibrium is allocationally efficient when traders trade all the way to their target inventories based on all information available to them $(i_n, \mathcal{S}_n, o_n, p_{m,n})$. This occurs if and only if $\chi = 0$, implying traders are price takers. Allocational efficiency, when defined as the extent to which traders trade towards their target inventory, is decreasing in χ.

Allocational efficiency ($\chi \to 0$) does not lead to informational efficiency ($\varphi \to 1$). Equation (35) shows that for given M and N, χ is increasing in φ. In other words, informational efficiency is inversely related to allocational efficiency. The economic intuition behind this result is the following. An economy is more informationally efficient when traders have better information (higher information-to-noise ratio, e.g., τ_I/σ_S^2). Holding the number of groups and traders constant, traders with better information are effectively "larger" and make the economy less competitive, when measured by χ. The less competitive economy is less allocationally efficient. To put it differently, it is not that price informativeness depends on the competitiveness but that competitiveness depends on price informativeness. We discuss this point in more detail in the next section.

4.4 Dimensional Analysis.

In the characterization of equilibrium, there are three dimensionless products of dimensional quantities: $\rho\sigma_V\sigma_z$, $\rho\sigma_V\sigma_S$, and σ_O/σ_V. Changing units of measurement has no real effect on equations (25), (27) and (28). This implies we can assume $\rho = \sigma_V = 1$ without loss of generality. Furthermore, if the five dimensional variables change in such a way that the three dimensionless products do not change, then φ does not change, and, in many respects, the properties of the equilibrium do not change.

This property is shared by many finance models. Fundamental model properties depend on the ratio of the risks to be borne—measured by $\sigma_V\sigma_z$ and $\sigma_V\sigma_S$—to the dollar risk-bearing capacity ρ^{-1}. For example, $\rho\sigma_V\sigma_Z$ or $\rho\sigma_V\sigma_S$ can become small either because risk bearing capacity increases (ρ becomes small) or because the risks to be borne $\sigma_V\sigma_Z$ and $\sigma_V\sigma_S$ become small. Either way, the effect on equilibrium is similar.

This provides useful insights. This allows us to answer the following seemingly complex questions in one second. Suppose $\sigma_O = 0$. Would the market become more or less competitive when the prior uncertainty about the risky asset decreases ceteris paribus? Would traders trade more or less aggressively when everyone becomes more risk averse and all the other exogenous variables remain the same? are (1) less competitive and (2) more aggressively.

At first glance, the first question might look tricky. Perhaps one would suspect that traders would be less afraid of adverse selection and behave more like price takers because there is less uncertainty about the fundamental value of the risky asset to begin with. From dimensional analysis, we know that lower σ_V is equivalent to lower σ_S, which makes the price more informative. As long as an equilibrium exists, theorem 4 shows that the market becomes less competitive. The same logic can apply to the second question. One might guess that more risk averse traders might trade less aggressively because they are more concerned about uncertainty. Again, from dimensional analysis, the equilibrium effect of larger risk aversion ρ is equivalent to higher σ_S, which makes the price less informative and the market more competitive.

5 Existence of an Equilibrium

The following theorem characterizes existence and uniqueness of equilibrium using only exogenous parameters, not the endogenous parameter φ.

Theorem 5 (Existence and Uniqueness)**.** *Assume $\rho > 0$ and $\sigma_V > 0$. Then there exists a symmetric linear equilibrium, excluding trivial no-trade equilibria, if and only if $MN > 2$ and at least one of the following three conditions holds:*

$$(N-1)\,\zeta^2\tau_I = 0, \tag{36}$$

$$\sigma_Z^2 > 0, \tag{37}$$

$$\begin{aligned}\left[\left(1+\frac{N}{MN-2}\right)\zeta - 1\right]\tau_I < \tau_I^2\,(1-\zeta)\,(1+(N-1)\,\zeta) + (\rho\sigma_V\sigma_S)^2 \\ + \left[1+\tau_I+(N-1)\,\tau_I\zeta\left(\frac{MN-2}{MN+N-2}\right)\right]^2\left(\frac{\sigma_O}{\sigma_V}\right)^2.\end{aligned} \tag{38}$$

If an equilibrium exists, it is unique.

Comments. An equilibrium exists when either (1) there is no adverse selection (equation (36)); or (2) there is a small amount exogenous noise trading (equation (37)); or (3) there is sufficient amount of endogenous "noise" such as private values, endowment shocks and convenience yields (equation (38)). Equation (38) is equivalent to the second order condition (26) in Theorem 2 and implies that for an equilibrium to exist, the price cannot be too informative.

Equation (38) can be interpreted to imply that there is no equilibrium without a sufficient amount of "noise" because the severity of adverse selection makes the market too illiquid. While this intuition is correct, another less intuitive explanation is also possible. Suppose (38) does not hold, or equivalently,

$$\varphi = \left(\frac{MN-2}{MN+N-2}\right)\frac{1}{\zeta} \qquad \text{or} \qquad \varphi > \left(\frac{MN-2}{MN+N-2}\right)\frac{1}{\zeta}. \tag{39}$$

Then equation (34) in Theorem 4 shows that there are two separate reasons that an equilibrium may not exist.

First, when (39) holds with equality, χ is indeterminate. In this case the equilibrium quantities traded are $x_{m,n} \equiv 0$ for all traders, and the only possible equilibrium is a trivial no-trade equilibrium.

Second, when (39) holds with strict inequality, the algebraic solution implies that χ is negative (and less than $-\frac{1}{2}$). Traders trade more aggressively than price-takers because negative trading costs induce them to trade beyond their target inventories. In this case, the trader's objective function becomes convex and traders can achieve infinite expected utility by buying or selling arbitrarily large quantities to profit from negative price impact. Therefore, an equilibrium does not exist because the market is *too illiquid* when (39) holds with equality; and an equilibrium does not exist because market is *too liquid* when (39) holds with strict inequality. While this logic may seem suspicious because it is based on algebra and not economic intuition, we show in section 6 that equilibria with negative χ can indeed exist under heterogeneous beliefs.

When an equilibrium exists, it is a unique equilibrium. This is easily inferred from equation (25). Its left hand side is strictly decreasing in φ and the right hand side is increasing in φ as long as an equilibrium exists. By focusing on existence and uniqueness using the endogenous parameter φ, our equilibrium characterization allows a transparent analysis of both first-order conditions and second-order conditions at the same time.

Theorem 5 contrasts the effect of exogenous noise trading (σ_Z^2) and various types of endogenous noise. An arbitrarily small amount of exogenous noise trading guarantees existence of equilibrium. Since the amount traded by exogenous noise traders is not optimized, exogenous noise traders do no not reduce their trading when trading costs are high. Unlike optimizing traders who receive endowment shocks (endogenous noise), noise traders are willing to incur whatever trading costs are necessary to sustain an equilibrium with trade.

5.1 Vanishing Exogenous Noise Trading

Nonexistence of equilibrium represents a failure of the model. We propose adding a vanishingly small amount of exogenous noise trading ($\sigma_Z^2 \to 0$) as a possible remedy when an equilibrium does not otherwise exist. Any arbitrarily small amount of exogenous noise trading allows an equilibrium to exist. While this solution is in the spirit of trembling hand perfect equilibrium, it is somewhat different because trembling hand perfect equilibrium involves perturbations of the players' actions rather than exogenous perturbations. The next theorem formalizes our result.

Theorem 6. *Let $\mathcal{E}$ denote an economy described by exogenous parameters $\{M, N, \rho, \tau_I, \zeta, \sigma_V^2, \sigma_Z^2, \sigma_S^2, \sigma_O^2\}$ with $MN > 2$ that do not satisfy (36), (37) or (38). Consider any sequence of economies $(\mathcal{E}_l)_{l=0}^{\infty} = (\mathcal{E}_0, \mathcal{E}_1, \dots)$ such that $\mathcal{E}_l, \forall l$ has exactly the same exogenous parameters as $\mathcal{E}$ except for exogenous noise trading $\sigma_Z^2(l) > 0$ and $\sigma_Z^2(l)$ approaches zero as l approaches infinity. Then there exists a unique equilibrium in $\mathcal{E}_l, \forall l$. Moreover, φ_l in economy $\mathcal{E}_l$ satisfies*

$$\varphi_l \to \left(\frac{MN-2}{MN+N-2}\right)\frac{1}{\zeta} \qquad \text{as} \qquad l \to \infty, \tag{40}$$

and χ_l, given by (35), satisfies

$$\chi_l \to \infty \qquad \text{as} \qquad l \to \infty. \tag{41}$$

Comments. To remedy the non-existence problem, we consider this limit to define the unique equilibrium when the existence conditions in Theorem 5 fail. In the limit as noise trading vanishes, prices remain noisy because $\varphi < 1$, the noisy price is well-defined in the limit because φ converges to a well-defined limit with $\varphi < 1$, and there is no trade in the limit because $\chi \to \infty$.[4]

The limit φ in (40) is the upper bound of the second order condition (26). Therefore, price cannot be more informative than this limit. It is puzzling that the price remains noisy even when all noise vanishes.

To consider this further, shut down endowment shocks in the simple motivating example of section 2 by considering the simple case $\zeta = 1$, $\sigma_S^2 = 0$ and $\sigma_O^2 = 0$. These assumptions correspond to shutting down all quantity noise so that the only motive for trading is private information. With vanishing exogenous noise trading, the equilibrium price remains noisy. The economic intuition for the noisy price when all quantity noise vanishes is as follows. With so little noise, the market becomes infinitely noncompetitive (since $\chi \to \infty$ from equation (41)). This implies that each trader incorporates at most half of his private information into prices (since $\varphi \approx 1/2$ in equation (33) in Theorem 3). If all traders with the same private information collude (equivalent to $M = 1$), this implies that price informativeness φ satisfies $\varphi \approx 1/2$ for large enough N (or, more precisely, $\frac{1}{2}\left(1 - \frac{1}{N-1}\right)$). This corresponds to the monopolistic competition limit in Kyle (1989) as the number of different informed traders (N) approaches infinity.

As the number of non-colluding, oligopolistic competitors within each group increases to $M = 2, 3, \ldots$, the informativeness of prices φ increases to approximately $2/3, 3/4, 4/5, \ldots$ for large enough N and becomes fully revealing at the limit as $M \to \infty$. While each trader still incorporates half of his marginal private information into prices, the price becomes more informative when the overlapping private information is shared among many traders M. The outcome of this oligopolistic competition resembles quantity competition in a Cournot equi-

[4] Although the limit as noise trading vanishes has no trade, this limit is not a trivial no-trade equilibrium because the price is well-defined in the limit.

librium. Each firm tries to maximize its profit by supplying only the half of its residual demand. But as the number of firms increases, each firm becomes smaller and the total quantity produced increases to $2/3,\ 3/4,\ \ldots$ of the quantity with perfect competition. Of course, the important distinction is that even when traders become "small" in terms of their private information, they remain "large" in the market because their trading costs are constrained by vanishingly low market liquidity, not by risk aversion.

Comparison with Milgrom and Stokey (1982). Our results in Theorem 6 are related to but different from the no-trade theorem of Milgrom and Stokey (1982).[5] In our model, there will be no trade even when initial allocations to be Pareto optimal. In the absence of quantity noise of any kind, traders may nevertheless have different, arbitrarily-large, deterministic inventories $\bar{S}_n$. Even though there may be large potential gains from trading to equalize inventories across traders, traders will not participate in any trade at all. Moreover, fully revealing prices are not supported for any finite M. This is clear from the upper bound of φ in equation (40), which increases in M and only approaches one as M approaches infinity.

The equilibrium in demand schedules is attractive because all traders are treated symmetrically and limit orders are protected. These are properties of well-functioning markets which organized exchanges and their regulators strive to implement. From the perspective of welfare economics, the main weakness of the equilibrium in demand schedules studied here is that modest adverse selection can make trade break down even when there are large gains from trade due to large non-stochastic initial endowments. Whether there are better trading mechanisms for internalizing gains from trade is an interesting issue. Liu and Wang (2016) examine a dealer-market model in which dealers make profits by buying at the bid and selling at the offer while customers are not allowed to trade with one another at the the same price (such as

[5] See also Dow, Madrigal and da Costa Werlang (1990).

the midpoint of the bid-ask spread). The monopolistic spread profits earned by dealers may allow trade to occur when it would not occur in our equilibrium in demand schedules. Duffie and Zhu (2015) study a workup process that allows traders to trade at fixed prices which do not necessarily clear the market. Glode and Opp (2016) study the welfare effects of trading with intermediation chains.

Comparison with Models of Noisy Competitive Rational Expectations Equilibrium. The rational expectations literature has a long history of studying whether prices are fully revealing. The noisy competitive rational expectations models of Grossman and Stiglitz (1980) and Hellwig (1980) have the two properties that, in the limit as noise trading vanishes, (1) traders remain price takers and (2) prices become fully revealing. These limiting results might be informally justified with the following questionable logic: Competitive rational expectations models are an approximation to what would happen in an oligopolistc model with an arbitrarily large number of traders. As noise trading vanishes, the noise from prices will disappear. Thus, it is reasonable to believe that price noise will disappear and price taking behavior will apply in a non-competitive model with a large number of traders observing the same private information.

Theorem 6 provides a critique of this logic. It shows that these two properties are mutually exclusive in an otherwise similar model with strategic trading. Fully revealing prices ($\varphi = 1$) are achieved only in the limit (as in equation (40)) when there is vanishingly small noise trading, no private values ($\zeta = 1$), and an infinite number of oligopolistic traders trading with the same private information ($M \to \infty$). In contrast to the competitive rational expectations models, under these assumptions, the market becomes infinitely non-competitive because χ diverges to infinity regardless of the number of traders within each group M. Traders understand their infinitely large price impact and refuse to trade. They are not price takers, no matter how large the number of traders sharing the same information. Therefore, either price is fully revealing or traders are price-takers, but these two cannot hold at the same time.

In general our symmetric model is simpler than asymmetric models in which traders differ in quality of information. For example, Grossman and Stiglitz (1980) is an asymmetric model whose equilibrium is made more complicated by asymmetry. Our model with $N = 2$, $M \to \infty$, $\sigma_S^2 = \sigma_O^2 = 0$, $\zeta = 1$ and $\sigma_Z^2 > 0$ resembles theirs in which the informed and uninformed are replaced by two groups of symmetrically informed traders with independent private signal.

As noise trading vanishes, however, our model is different from the Grossman and Stiglitz model. They assume price-taking behavior ($\chi = 0$) regardless of noise trading. However, as discussed above, as noise trading is vanishing, the market becomes less competitive and χ goes to infinity. Price-taking assumption cannot be made independent of price informativeness. If price is fully revealing, no one is small.

6 Heterogeneous Beliefs

In this section, we introduce the possibility that informed traders disagree about the informativeness of the private signals of different groups. M traders in group n now observe private information i_n given by

$$i_n = \tau_n^{1/2} \left(\frac{v_n}{\sigma_V} \right) + e_n, \qquad \text{where} \qquad e_n \sim N(0, 1). \tag{42}$$

All traders believe that the precision of their own group's private signal is τ_I, but the precision of the other groups' private signal is τ_U. We assume that these beliefs are common knowledge.

To parameterize disagreement conveniently, define

$$C :\equiv \sqrt{\frac{\tau_I}{\tau_U}}. \tag{43}$$

Then $C > 1$ corresponds to overconfidence, $C < 1$ corresponds to underconfidence, and $C = 1$ corresponds to a common prior. If $C \neq 1$, there is agreement to disagree.

The ratio between prior and posterior variance τ^* now satisfies

$$\tau^* = 1 + \tau_I + (N-1)\,\tau_I \left(\frac{\zeta}{C}\right)^2 \varphi, \tag{44}$$

as φ represents the informativeness of price as a fraction of what a trader believes to be all available information to the rest of the market.

Solving for an equilibrium in this case is similar to the previous case with $C = 1$. To avoid repetition we only present the result and leave details in Appendix.

Theorem 7 (Heterogeneous Beliefs)**.** *Suppose* $\rho > 0, \sigma_V > 0,$ *and* $MN > 2$. *If* $(N-1)\left(\frac{\zeta}{C}\right)^2 \tau_I^2 > 0$*, then the set of symmetric linear equilibria, excluding trivial no-trade equilibria, is characterized by the set of all endogenous parameters* φ *such that (1)* φ *solves*

$$\begin{aligned} \frac{1}{\varphi} - 1 = \tau_I (1-\zeta)(1+(N-1)\zeta) + \frac{1}{\tau_I}\left[\left(1 + \tau_I + (N-1)\,\tau_I \left(\frac{\zeta}{C}\right)^2 \varphi\right)\frac{\sigma_O}{\sigma_V}\right]^2 \\ + \frac{(\rho\sigma_V\sigma_S)^2}{\tau_I} + \frac{(\rho\sigma_V\sigma_Z)^2 / (N-1)}{\tau_I \left[\frac{MN-2}{MN-1} - \left(\frac{MN+N-2}{MN-1}\right)\frac{\zeta}{C}\varphi\right]^2}, \end{aligned} \tag{45}$$

and (2) φ *satisfies a second order condition*

$$\varphi < \left(\frac{MN-2}{MN+N-2}\right)\frac{C}{\zeta} \qquad or \qquad \varphi > \left(\frac{M}{M-1}\right)\frac{C}{\zeta}. \tag{46}$$

If $(N-1)\left(\frac{\zeta}{C}\right)^2 \tau_I^2 = 0$*, an equilibrium is characterized with* $\varphi = 0$*.*

χ *is given by*

$$\chi = \frac{1 + (N-1)\frac{\zeta}{C}\varphi}{MN - 2 - (MN + N - 2)\frac{\zeta}{C}\varphi}. \tag{47}$$

If φ *satisfies* (46)*,* χ *can be also written as*

$$\chi = \frac{1}{MN-2} + \left(\frac{MN-1}{MN-2}\right)\sum_{k=1}^{\infty}\left(\frac{N}{MN-2}\right)^k \left(\frac{\frac{\zeta}{C}\varphi}{1-\frac{\zeta}{C}\varphi}\right)^k. \tag{48}$$

Comments. Many equilibrium properties remain the same when allowing heterogeneous beliefs. Price informativeness φ is still independent of M when there is no exogenous noise trading. The economic intuition that the proportion by which strategic traders distort their trading is the same for private information and other types of noise remains unchanged.

With $\sigma_O^2 = \sigma_Z^2 = 0$, φ is also independent of C. Traders' over- or underconfidence affects their demand to the extent that it affects their learning, and thus τ^*. The change in τ^* affects trading on private information and endowment shock with the same proportion. Price informativeness remains unchanged. With independent convenience yields, C affects φ directly. Our interpretation of this result based on (25) is that τ^* units of convenience yields (σ_O/σ_V) correspond to the one unit of endowment shock $(\rho\sigma_V\sigma_S)$. Then as long as $\tau^* \cdot \sigma_O/\sigma_V$ is constant, φ does not depend on C.

The key difference from the common prior case is that there are two disjoint ranges of φ that satisfy the second order condition (46). The second inequality requires that traders are underconfident ($C < 1$) and there are at least two traders within each group ($M \geq 2$). We discuss the underconfidence case in detail below.

6.1 Overconfidence

If $C > 1$, traders are overconfident. They think their private information is of better quality (or more precise) than that of other traders. This disagreement being common knowledge implies that traders think their private information is better than other traders think it is. When there is no exogenous noise trading ($\sigma_Z^2 = 0$), the second order condition (46) requires that for an equilibrium to exist, the set of parameters that describe the economy satisfy

$$\left[\left(\frac{MN+N-2}{MN-2}\right)\frac{\zeta}{C}-1\right]\tau_I < \frac{\tau_I^2}{C^2}(1-\zeta)(1+(N-1)\zeta)+(\rho\sigma_V\sigma_S)^2 + \left[1+\tau_I+\tau_I\frac{(N-1)(MN-2)}{(MN+N-2)}\frac{\zeta}{C}\right]^2\left(\frac{\sigma_O}{\sigma_V}\right)^2. \tag{49}$$

If C is sufficiently high, fully revealing price can be supported in equilibrium with finitely many traders. Rewriting the first inequality in (46) with $\varphi = 1$ yields

$$C \geq \left(\frac{MN+N-2}{MN-2}\right)\zeta, \tag{50}$$

where the equality implies a limit equilibrium as noise trading is vanishing. This coincides with the existence condition in Kyle, Obizhaeva and Wang (2016) when $M = 1$.

With overconfidence, fully revealing price does not imply χ diverges to infinity except when (50) holds with equality. The economic intuition is that all traders are overconfident and think others' information is poorer than theirs. Traders underestimate others' information. This is why the market does not become infinitely illiquid ($\chi \to \infty$) even when the price becomes fully revealing.

If (50) does not satisfy, the upper bound on φ is given by

$$\varphi \leq \left(\frac{MN-2}{MN+N-2}\right)\frac{C}{\zeta} < 1, \tag{51}$$

and price is never fully revealing regardless of noise of any type.

6.2 Underconfidence

If $C < 1$, traders are underconfident. Traders think their private information is poorer (or less precise) than that of others. Again, this being common knowledge implies that all traders overestimate others' private information, and they think others overestimate their own information.

With underconfidence, the second order condition consists of two disjoint ranges. Without exogenous noise trading ($\sigma_Z^2 = 0$), the first inequality corresponds to (49) and an equilibrium is in many ways similar to an equilibrium with common prior ($C = 1$). The other range corresponds to

$$\begin{aligned}\left[\left(\frac{M-1}{M}\right)\frac{\zeta}{C}-1\right]\tau_I > \frac{\tau_I^2}{C^2}(1-\zeta)(1+(N-1)\zeta)+(\rho\sigma_V\sigma_S)^2 \\ +\left[1+\tau_I+(N-1)\tau_I\left(\frac{M}{M-1}\right)\frac{\zeta}{C}\right]^2\left(\frac{\sigma_O}{\sigma_V}\right)^2.\end{aligned} \tag{52}$$

Upside-Down Equilibrium When an economy is described by the set of parameters that satisfy (52), there is an equilibrium with many peculiar features. We call this equilibrium an upside-down equilibrium and describe the details below.

Theorem 8 (Upside-Down Equilibrium)**.** *If (52) holds, there exists $\bar{\sigma_Z}^2 > 0$ such that for all $\sigma_Z^2 \in [0, \bar{\sigma}_Z^2)$ there exists an upside-down equilibrium, in which every trader's demand sched-*

ule is upward-slopping; every trader trades against his private information, i.e., trader (m, n)'s demand schedule $\forall (m, n)$ is given by equation (18) with

$$\pi_P < 0 \qquad \text{and} \qquad \pi_I < 0. \tag{53}$$

Moreover, scaled illiquidity χ satisfies

$$-\frac{1}{2} < \chi < 0. \tag{54}$$

Comment. An upside-down equilibrium has everything upside-down. Demand schedules are sloped "wrong" way. The law of demand fails, and a risky asset resembles a Giffen good. Traders also trade against their private information. Bullish information prompts them to sell the asset; bearish information prompts them to buy the asset. Trading cost is negative. Traders can buy an asset at a low price and sell it at a high price.

The economic intuition is that underconfident traders believe that the equilibrium price incorporates their private information too much, while it incorporates others' information too little. Traders perceive others to have erroneous beliefs and optimally respond by trading against their private information and by having upward sloping demand schedule. This only happens when the equilibrium price is sufficiently informative. That is why the existence condition requires small or no exogenous noise trading. Also, this does not happen if $M = 1$ because then a trader has monopoly power on his private information and can prevent the price from incorporating his private information too much.

One might wonder if an upward-sloping demand schedule will result in infinite trades. This is not the case. As long as $\chi > -\frac{1}{2}$, the objective function is concave and thus, the optimal demand is well defined. Equation (32) in Theorem 3 shows that this second order condition implies that traders do not trade away from their target inventory ($1/(1+2\chi) > 1$).

Although ours is a one-period model, we believe the intuition of upside-down equilibrium is consistent with momentum trading (or trend chasing behavior) in a dynamic model. In a dynamic market, traders may buy when the price rises because they believe market price changes under-react to information of other traders which generates these price changes.

When there is exogenous noise trading ($\sigma_Z^2 > 0$), an upside-down equilibrium is always a part of multiple equilibria and there is always a unique "normal" equilibrium with familiar downward sloping demand schedule. We present the result below.

Theorem 9 (Multiple Equilibria)**.** *Provided that an upside-down equilibrium exists, there exist multiple equilibria excluding trivial no-trade equilibria if and only if $\sigma_Z^2 > 0$. Whenever there exist multiple equilibria, there is a unique normal equilibrium, whose variables have subscript N and there is one or more upside-down equilibria. When there are two or more upside-down equilibria, we pick an equilibrium with highest φ and denote its variables with subscript UD. Then*

$$\varphi_{UD} > \varphi_N \qquad \textit{and} \qquad \chi_{UD} < \chi_N. \tag{55}$$

However, it is ambiguous which equilibrium has the higher certainty equivalent.

When there are multiple equilibria, a criterion by which to select an equilibrium is called for. Given peculiar properties of an upside-down equilibrium one might suspect a normal equilibrium should be chosen. We compare price informativeness (φ) and scaled illiquidity (χ), and find that an upside-down equilibrium is more desirable than its normal counterpart. However, when comparing the certainty equivalent, it is ambiguous which equilibrium is more desirable. Therefore it is an open question which equilibrium should be selected.

7 Conclusion

Our general framework allows to investigate the robustness of implications of standard competitive rational expectations models. We also find the puzzling existence of an upside-down equilibrium in which demand schedule is sloped wrong way when there is underconfidence.

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A Proofs

Proof of Theorem 1. Theorem 1 is a special case of Theorem 7. See proof of Theorem 7 below.

Proof of Lemma 1. We prove a more general case of heterogeneous belief case, where traders believe the signal of their precision is τ_I and that of others' signals τ_U, possibly different from τ_I as in Section 6. Then substitute $\tau_I = \tau_U$ as a special case and prove Lemma 1.

Assume for now $(N-1)\zeta^2\tau_U\tau_I \neq 0$. Group n's learning from price can be examined either by filtering the intercept of the residual supply schedule $p_{m,n}$ or by filtering the price P. Here we choose to filter the price P. First, subtract group n's own effect from prices, obtaining

$$\begin{aligned}\frac{P}{\sigma_V} - \alpha - \frac{\beta i_n}{N} + \frac{\delta\rho\sigma_V S_n}{N} - \frac{\eta o_n}{N\sigma_V} = \frac{\beta(N-1)}{N}\left(\frac{\Sigma_{n'\neq n} i_{n'}}{N-1}\right) + \gamma\rho\sigma_V z \\ - \frac{\delta\rho\sigma_V \Sigma_{n'\neq n} S_{n'}}{N} + \frac{\eta\Sigma_{n'\neq n} o_{n'}}{N\sigma_V}.\end{aligned} \tag{56}$$

Now define e_n^ζ as the error in the projection of the scaled average valuation of other traders $\sqrt{\tau_U}\left(\frac{\sum_{n'\neq n} v_{n'}}{(N-1)\sigma_V}\right)$ on group n's valuation v_n. This implies

$$\frac{\sum_{n'\neq n} i_{n'}}{N-1} = \sqrt{\tau_U}\left(\frac{\zeta v_n}{\sigma_V} + e_n^\zeta\right) + \frac{\sum_{n'\neq n} e_{n'}}{N-1}, \tag{57}$$

where

$$e_n^\zeta \equiv \frac{\sum_{n'\neq n} v_{n'}}{(N-1)\sigma_V} - \zeta\frac{v_n}{\sigma_V}, \tag{58}$$

with

$$\begin{aligned}\operatorname{var}\left\{e_n^\zeta\right\} &= \operatorname{var}\left\{\frac{\sum_{n'\neq n} v_{n'}}{(N-1)\,\sigma_V}\right\} - \operatorname{var}\left\{\zeta\frac{v_n}{\sigma_V}\right\} \\ &= \frac{\operatorname{cov}\left\{v_l, \sum_{n'\neq n}\sqrt{\tau_v}v_{n'}\right\}}{(N-1)\,\sigma_V^2} - \zeta^2 = \frac{\left(1+(N-2)\,\zeta-(N-1)\,\zeta^2\right)}{(N-1)} \geq 0.\end{aligned} \tag{59}$$

and

$$\mathrm{E}\left\{e_n^\zeta\right\} = \frac{(1-\zeta)\,v_0}{\sigma_V} \tag{60}$$

Let e_n^{ALL} denote all the noise in price resulting from exogenous noise z, other traders' supply shocks $\mathcal{S}_{n'}$, convenience yields $o_{n'}$, signal noise $e_{n'}$, and different valuations e_n^ζ. Then what is learned from prices can be expressed as a linear combination of v_n and this noise. We thus obtain

$$\frac{P}{\sigma_V} - \alpha - \frac{\beta i_n}{N} + \frac{\delta\rho\sigma_V S_n}{N} - \frac{\eta o_n}{N\sigma_V} = \frac{\beta\,(N-1)\,\zeta}{N}\left(\frac{\sqrt{\tau_U}v_n}{\sigma_V} + e_n^{\mathrm{ALL}}\right). \tag{61}$$

where the noise is given by

$$e_n^{\mathrm{ALL}} := \frac{\sqrt{\tau_U}}{\zeta}e_n^\zeta + \frac{\sum_{n'\neq n} e_{n'}}{(N-1)\,\zeta} + \frac{N}{\beta\,(N-1)\,\zeta}\left(\gamma\rho\sigma_V z - \frac{\delta\rho\sigma_V\Sigma_{n'\neq n}S_{n'}}{N} + \frac{\eta\Sigma_{n'\neq n}o_{n'}}{N\sigma_V}\right) \tag{62}$$

The variance of e_n^{ALL} is given by

$$\begin{aligned}\operatorname{var}\left\{e_n^{\mathrm{ALL}}\right\} &= \frac{1+\tau_U\,(1-\zeta)\,(1+(N-1)\,\zeta)}{\zeta^2\,(N-1)} \\ &\quad + \frac{1}{\zeta^2\,(N-1)}\left[\left(\frac{N^2}{N-1}\right)\left(\frac{\gamma\rho\sigma_V\sigma_Z}{\beta}\right)^2 + \left(\frac{\delta\rho\sigma_V\sigma_S}{\beta}\right)^2 + \left(\frac{\eta\sigma_O}{\beta\sigma_V}\right)^2\right].\end{aligned} \tag{63}$$

and

$$\mathrm{E}\left\{e_n^{\mathrm{ALL}}\right\} = \frac{\sqrt{\tau_U}\,(1-\zeta)}{\zeta}\frac{v_0}{\sigma_V}. \tag{64}$$

Since v_n, e_n^{ALL}, and e_n are independently distributed, it is straightforward to show that

$$\begin{aligned} \tau^* &= \sigma_V^2 \cdot \text{var}^{-1} \left\{ v_n \middle| \sqrt{\tau_I} \frac{v_n}{\sigma_V} + e_n = i_n, \sqrt{\tau_U} \frac{v_n}{\sigma_V} + e_n^{\text{ALL}} \right\} \\ &= 1 + \tau_I + \tau_U \text{var}^{-1} \left\{ e_n^{\text{ALL}} \right\} \end{aligned}. \tag{65}$$

and

$$\begin{aligned} &\text{E} \left\{ v_n \middle| \sqrt{\tau_I \tau_v} v_n + e_n = i_n, \sqrt{\tau_U \tau_v} v_n + e_n^{\text{ALL}} = \left(\frac{N}{\beta (N-1) \zeta} \right) \cdot \left(\frac{P}{\sigma_V} - \alpha - \frac{\beta i_n}{N} + \frac{\delta \rho \sigma_V S_n}{N} - \frac{\eta o_n}{N \sigma_V} \right) \right\} \\ &= \frac{1}{\tau^*} v_0 + \frac{\sqrt{\tau_I} \sigma_V}{\tau^*} i_n \\ &+ \frac{\sqrt{\tau_U} \sigma_V (N-1) \zeta^2 \varphi}{\tau^*} \left[\left(\frac{N}{\beta (N-1) \zeta} \right) \cdot \left(\frac{P}{\sigma_V} - \alpha - \frac{\beta i_n}{N} + \frac{\delta \rho \sigma_V S_n}{N} - \frac{\eta o_n}{N \sigma_V} \right) - \frac{\sqrt{\tau_U} (1-\zeta)}{\zeta} \frac{v_0}{\sigma_V} \right]. \end{aligned} \tag{66}$$

From equation (65), writing τ^* in the form of equation (22) implies

$$\begin{aligned} \frac{1}{\varphi} &= (N-1) \zeta^2 \text{var} \left\{ \epsilon_n^{\text{ALL}} \right\} \\ &= 1 + \tau_U (1-\zeta)(1 + (N-1)\zeta) + \left[\left(\frac{N^2}{N-1} \right) \left(\frac{\gamma \rho \sigma_V \sigma_Z}{\beta} \right)^2 + \left(\frac{\delta \rho \sigma_V \sigma_S}{\beta} \right)^2 + \left(\frac{\eta \sigma_O}{\beta \sigma_V} \right)^2 \right]. \end{aligned} \tag{67}$$

This is equivalent to equation (23). By allowing the right side of equation (23) to vary from zero to infinity, it is easy to see that φ satisfies inequality

$$0 \le \varphi \le \left[1 + \tau_U (1-\zeta)(1 + (N-1)\zeta) \right]^{-1}. \tag{68}$$

The conditional expectation (66) can be written

$$\begin{aligned}\mathrm{E}^n\left\{v_n|i_n, P, S_n, o_n\right\} &= \mathrm{E}^n\left\{v_n\middle|i_n, \left(\frac{N}{\beta\left(N-1\right)\zeta}\right)\cdot\left(\frac{P}{\sigma_V}-\alpha-\frac{\beta i_n}{N\sqrt{1+\tau_I}}+\frac{\delta S_n}{N\sigma_S}-\frac{\eta o_n}{N\sigma_O}\right)\right\}\\ &= \left[\frac{1-\tau_U\left(N-1\right)\zeta\left(1-\zeta\right)\varphi}{\tau^*}\right]v_0-\frac{\sqrt{\tau_U}N\zeta\varphi}{\tau^*\beta}\sigma_V\alpha+\left(\frac{\sqrt{\tau_I}-\sqrt{\tau_U}\zeta\varphi}{\tau^*}\right)\sigma_V i_n\\ &+\frac{\sqrt{\tau_U}N\zeta\varphi}{\tau^*\beta}P+\frac{\sqrt{\tau_U}\zeta\varphi\delta\rho\sigma_V^2}{\tau^*\beta}S_n-\frac{\sqrt{\tau_U}\zeta\varphi\eta}{\tau^*\beta}o_n.\end{aligned} \tag{69}$$

Now relax the assumption made initially and allow $(N-1)\,\zeta^2\tau_U = 0$. Then there is no learning from the price and the conditional expectation is simply

$$\mathrm{E}^n\left\{v_n|i_n, P, S_n, o_n\right\} = \frac{v_0}{\tau^*}+\frac{\sqrt{\tau_I}}{\tau^*}\sigma_V i_n, \tag{70}$$

which is consistent with (69) when $\varphi = 0$ is substituted. This completes the proof.

Proof of Theorem 2. Theorem 2 is a special case of Theorem 7. See proof of Theorem 7.

Proof of Theorem 3. Equation (31) is obtained directly by substituting equation (30) into the definition of χ in equation (29). Substituting equation (30) into the optimal demand (81) yields equation (32). Lastly, substituting equation (30) and equation (32) into the residual supply curve (19) yields equation(33).

Proof of Theorem 4. Equation (34) is obtained by substituting (93) into the definition of χ (29). An equilibrium exists if and only if (26). When (26) holds, some algebra allows us to write equation (34) as equation (35). Equation (26) guarantees that (35) is well defined.

Proof of Theorem 5. We prove a more general case of heterogeneous belief case, where traders believe the signal of their precision is τ_I and that of others' signals τ_U, possibly different from τ_I as in Section 6. Then substitute $\tau_I = \tau_U$ as a special case and prove Theorem 5.

Proving this theorem is accomplished by analyzing the two equations (45) and (46) determining endogenous parameter φ. Assume for now that $(N-1)\,\zeta^2\tau_I\tau_U\sigma_Z^2 > 0$. Rewrite (45) as

$$L(\varphi) = R(\varphi), \tag{71}$$

where we define $R(\varphi)$ and $L(\varphi)$ by

$$\begin{aligned} R(\varphi) :=& \left(\frac{1}{\varphi} - 1\right)\tau_I - \tau_I\tau_U(1-\zeta)(1+(N-1)\zeta) \\ & - (\rho\sigma_V\sigma_S)^2 - \left[1+\tau_I+(N-1)\tau_U\zeta^2\varphi\right]^2\left(\frac{\sigma_O}{\sigma_V}\right)^2, \end{aligned} \tag{72}$$

and

$$L(\varphi) := \frac{(\rho\sigma_V\sigma_Z)^2/(N-1)}{\left[\frac{MN-2}{MN-1} - \left(\frac{MN+N-2}{MN-1}\right)\frac{\zeta}{C}\varphi\right]^2}. \tag{73}$$

$R(\cdot)$ is monotonically decreasing everywhere with $\lim_{\varphi\to 0} R(\varphi) = \infty$ and $R(1) \leq 0$. Therefore, when there is no exogenous noise trading ($\sigma_Z^2 = 0$), there always exists a unique solution to (45). This implies that equilibrium is unique, if it exists. The solution indeed describes a unique symmetric linear equilibrium provided that the second order condition (46) holds, which is equivalent to (49) and (52).

On the other hand, when there is an exogenous noise trading ($\sigma_Z^2 > 0$), $L(\varphi)$ is monotonically increasing $\forall \varphi \in \left[0, \left(\frac{MN+N-2}{MN-2}\right)\frac{\zeta}{C}\right)$, with $\lim_{\varphi\to\left(\frac{MN+N-2}{MN-2}\right)\frac{\zeta}{C}} L(x) = \infty$. Therefore, there always exists a unique solution to (45) that is strictly less than $\left(\frac{MN+N-2}{MN-2}\right)\frac{\zeta}{C}$, hence satisfies the first inequality in the second order condition (46). However, this does not rule out the possibilities for another solution to (45) that satisfies the second inequality.

To finish the proof of Theorem 5, we need to show (1) there exists a unique equilibrium when $(N-1)\,\zeta^2\tau_I\tau_U\sigma_Z^2 = 0$ and $MN > 2$; (2) that there exists no equilibrium when $MN \leq 2$.

First, in the proof of Theorem (7) we haven shown that when $(N-1)\,\zeta^2\tau_I\tau_U\sigma_Z^2 = 0$, $\varphi = 0$ and this is an equilibrium if and only if $MN > 2$.

Second, to show there exists no equilibrium with $MN \leq 2$. We need to show there exists no equilibrium when $(N-1)\,\zeta^2\tau_I\tau_U\sigma_Z^2 > 0$ and $MN \leq 2$. This only happens when $N = 2$ and $M = 1$, when the S.O.C. (46) ($\varphi < 0$ or $\varphi \geq \infty$) cannot be satisfied.

This complete the proof of Theorem 5.

Proof of Theorem 6. We prove a more general case of heterogeneous belief case, where traders believe the signal of their precision is τ_I and that of others' signals τ_U, possibly different from τ_I as in Section 6. Then substitute $\tau_I = \tau_U$ (or $C = 1$) as a special case and prove Theorem 6.

For given l, economy $\mathcal{E}_l$ has an equilibrium because exogenous noise trading $\sigma_Z^2(l) > 0$. In equilibrium, φ is determined as a solution to (25), or equivalently as a solution to (71). Therefore $\forall l$, there exists unique φ_l that solves

$$\frac{(\rho\sigma_V\sigma_Z(l))^2}{(N-1)} = R(\varphi_l)\cdot\left[\frac{MN-2}{MN-1} - \left(\frac{MN+N-2}{MN-1}\right)\frac{\zeta}{C}\varphi_l\right]^2, \tag{74}$$

and satisfies the second order condition(46). As l approaches infinity, the left hand side of (74) approaches zero. Then possible solutions to (74) solve either $R(\varphi_l) \to 0$ or

$$\varphi_l \to \left(\frac{MN-2}{MN+N-2}\right)\frac{C}{\zeta}. \tag{75}$$

Consider a candidate solution $\varphi_{\text{candidate}}$ that solves $R(\varphi_{\text{candidate}}) = 0$. Since an equilibrium does not exist in economy $\mathcal{E}$ and more specifically (49) does not hold, $\varphi_{\text{candidate}}$ does not satisfy the second order condition (26). Therefore, we obtain (40).

Substituting (40) into equation (35) yields equation (41).

Proof of Theorem 7. To solve the trader's maximization problem, we use the no-regret approach of assuming that the trader observes his residual supply schedule (19) and picks the quantity $x_{m,n}$ which maximizes his expected utility. We then show that this choice can be implemented by submitting a demand schedule $X_{m,n}$.

Given the residual supply schedule (19), the trader's wealth equation (15) can be rewritten as a function of the quantity traded $x_{m,n}$:

$$W_{m,n}(x_{m,n}) := (v_n + o_n)\frac{\mathcal{S}_n}{M} + \left(v_n + o_n - \sigma_V p_{m,n} - \lambda\rho\sigma_V^2 x_{m,n}\right)x_{m,n}. \tag{76}$$

The random variables are jointly normally distributed and trading strategies are linear. Thus, the trading strategy which maximizes exponential utility in equation (16) is the same as the trading strategy which solves the quadratic maximization problem

$$\max_{x_{m,n}}\left\{\mathrm{E}^{m,n}\left\{W_{m,n}(x_{m,n}) \mid p_{m,n}, \mathcal{S}_n, i_n, o_n\right\} - \frac{M\rho}{2}\mathrm{var}^{m,n}\left\{W_{m,n}(x_{m,n}) \mid p_{m,n}, \mathcal{S}_n, i_n, o_n\right\}\right\}. \tag{77}$$

Since the conditional variance of v_n is a constant given by σ_V^2/τ^* in equation (22), the objective function to be maximized can be written as

$$\left(\mathrm{E}^{m,n}\left\{v_n \mid p_{m,n}, i_n, S_n, o_n\right\} + o_n\right)\frac{\mathcal{S}_n}{M} - \frac{\rho \mathcal{S}_n^2 \sigma_V^2/\tau^*}{2M}$$
$$+\left[\mathrm{E}^{m,n}\left\{v_n \mid p_{m,n}, i_n, S_n, o_n\right\} + o_n - \sigma_V p_{m,n} - \rho\sigma_V^2 \mathcal{S}_n/\tau^*\right] x_{m,n} - \frac{(2\lambda + M/\tau^*)\,\rho\sigma_V^2}{2} x_{m,n}^2 \tag{78}$$

Suppose the second order condition holds with strict inequality

$$2\lambda + M/\tau^* > 0. \tag{79}$$

Then the first order condition

$$\mathrm{E}^{m,n}\left\{v_n \mid p_{m,n}, i_n, S_n, o_n\right\} + o_n - \sigma_V p_{m,n} - \frac{\rho\sigma_V^2 \mathcal{S}_n}{\tau^*} - (2\lambda + M/\tau^*)\,\rho\sigma_V^2 x_{m,n} = 0, \tag{80}$$

implies that optimal demand is given by

$$\rho\sigma_V x_{m,n} = \frac{1}{2\lambda + M/\tau^*}\left[\mathrm{E}^{m,n}\left\{\frac{v_n}{\sigma_V} \mid p_{m,n}, i_n, S_n, o_n\right\} + \frac{o_n}{\sigma_V} - p_{m,n} - \rho\sigma_V \mathcal{S}_n/\tau^*\right]. \tag{81}$$

Using $\frac{P}{\sigma_V} = p_{m,n} + \lambda \cdot \rho\sigma_V \cdot x_{m,n}$ to replace the intercept of the residual supply schedule $p_{m,n}$ with the market-clearing price P, the demand (81) can be more conveniently written

$$\rho\sigma_V x_{m,n} = \frac{1}{\lambda + M/\tau^*}\left[\mathrm{E}^{m,n}\left\{\frac{v_n}{\sigma_V} \mid P, i_n, \mathcal{S}_n, o_n\right\} + \frac{o_n}{\sigma_V} - \frac{P}{\sigma_V} - \rho\sigma_V \mathcal{S}_n/\tau^*\right]. \tag{82}$$

If the second order condition holds with an equality ($2\lambda + M/\tau^* = 0$), the maximization problem becomes linear. Later we show that in this case there is no finite optimal demand for any finite M, and thus an equilibrium does not exist.

In moving from the demand schedule (81) to demand schedule (82), we have shown that traders in a symmetric linear equilibrium can implement a strategy which picks the best point on the linear supply schedule by submitting a demand schedule which is a function of price. This demand schedule will be linear because Lemma 1 shows that the conditional expectation in the demand schedules (81) and (82) are linear functions.

The remainder of the proof equilibrates the market clearing price to the clearing condition clears the market and the initial conjecture that is used by traders is indeed correct, rational expectation.

Substituting the conditional expectation (69) into the demand function (82) allows the demand function to be written as

$$\begin{aligned}(\lambda + M/\tau^*)\,\rho\sigma_V^2 X_{m,n}^*(P) = {} & \left[\frac{1-\tau_U(N-1)\zeta(1-\zeta)\varphi}{\tau^*}\right]v_0 - \frac{\sqrt{\tau_U}N\zeta\varphi}{\tau^*\beta}\sigma_V\alpha - \left(1-\frac{\sqrt{\tau_U}N\zeta\varphi}{\tau^*\beta}\right)P \\ & + \left(\frac{\sqrt{\tau_I}-\sqrt{\tau_U}\zeta\varphi}{\tau^*}\right)\sigma_V i_n + \left(1-\frac{\sqrt{\tau_U}\zeta\varphi\eta}{\tau^*\beta}\right)o_n + \left(\frac{\sqrt{\tau_U}\zeta\varphi\delta\rho\sigma_V^2}{\tau^*\beta}-\frac{\rho\sigma_V^2}{\tau^*}\right)S_n.\end{aligned} \tag{83}$$

Using the optimal demand function (83), the market clearing condition (14) can be written

$$\frac{(\lambda + M/\tau^*)\,\rho\sigma_V^2\sum_{m=1}^M\sum_{n=1}^N X_{m,n}^*(P)}{MN} + \frac{(\lambda + M/\tau^*)\,\rho\sigma_V^2}{MN}z = 0 \tag{84}$$

and then

$$\begin{aligned}0 = {} & \left[\frac{1-\tau_U(N-1)\zeta(1-\zeta)\varphi}{\tau^*}\right]v_0 - \frac{\sqrt{\tau_U}N\zeta\varphi\sigma_V\alpha}{\tau^*\beta} - \left(1-\frac{\sqrt{\tau_U}N\zeta\varphi}{\tau^*\beta}\right)P + \left(\frac{\sqrt{\tau_I}-\sqrt{\tau_U}\zeta\varphi}{\tau^*}\right)\frac{\sigma_V\Sigma_{n=1}^N i_n}{N} \\ & + \left(1-\frac{\sqrt{\tau_U}\zeta\varphi\eta}{\tau^*\beta}\right)\frac{\Sigma_{n=1}^N o_n}{N} + \left(\frac{\sqrt{\tau_U}\zeta\varphi\delta\rho\sigma_V^2}{\tau^*\beta}-\frac{\rho\sigma_V^2}{\tau^*}\right)\frac{\Sigma_{n=1}^N S_n}{N} + \frac{(\lambda + M/\tau^*)\,\rho\sigma_V^2}{MN}z.\end{aligned} \tag{85}$$

From the fact that $\lambda + M/\tau^* > 0$ and $\pi_P \neq 0$, it follows that the coefficient on $P\left(1 - \frac{\sqrt{\tau_U}N\zeta\varphi}{\tau^*\beta}\right)$ is nonzero. By dividing equation (85) by the coefficient on P, the function P can be written

$$\begin{aligned}\frac{P}{\sigma_V} &= \frac{\left[1-\tau_U\left(N-1\right)\zeta\left(1-\zeta\right)\varphi\right]}{\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta}\frac{v_0}{\sigma_V} - \frac{\sqrt{\tau_U}N\zeta\varphi/\beta}{\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta}\alpha \\ &+ \frac{\left(\sqrt{\tau_I} - \sqrt{\tau_U}\zeta\varphi\right)}{\left(\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta\right)}\frac{\Sigma_{n=1}^N i_n}{N} + \left(\frac{\tau^* - \sqrt{\tau_U}\zeta\varphi\eta/\beta}{\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta}\right)\frac{\Sigma_{n=1}^N o_n}{N\sigma_V} \\ &+ \frac{\left(\sqrt{\tau_U}\zeta\varphi\delta/\beta - 1\right)}{\left(\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta\right)}\frac{\rho\sigma_V\Sigma_{n=1}^N S_n}{N} + \frac{\tau^*\left(\lambda + M/\tau^*\right)}{MN\left(\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta\right)}\rho\sigma_V z.\end{aligned} \tag{86}$$

Equating the five coefficients on the constant term, $\sum_{n=1}^N S_n$, $\sum_{n=1}^N o_n$, $\sum_{n=1}^N i_n$, and z, respectively, in equation (86) to the corresponding coefficients in equation (20) yields the five equations

$$\alpha = \frac{\left[1-\tau_U\left(N-1\right)\zeta\left(1-\zeta\right)\varphi\right]v_0/\sigma_V - \sqrt{\tau_U}N\zeta\varphi\alpha/\beta}{\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta}, \quad \beta = \frac{\sqrt{\tau_I} - \sqrt{\tau_U}\zeta\varphi}{\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta}, \tag{87}$$

$$\gamma = \frac{\tau^*\left(\lambda + M/\tau^*\right)}{MN\left(\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta\right)}, \quad \delta = \frac{1 - \sqrt{\tau_U}\zeta\varphi\delta/\beta}{\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta}, \quad \eta = \frac{\tau^* - \sqrt{\tau_U}\zeta\varphi\eta/\beta}{\tau^* - \sqrt{\tau_U}N\zeta\varphi/\beta}. \tag{88}$$

Our goal is to solve for α, β, γ, δ, and η as functions of exogenous parameters and φ. To do this, first notice that the only endogenous parameters in equation for β are φ and β itself. This implies either $\beta = 0$ or

$$\beta = \left(1 + \left(N-1\right)\frac{\zeta}{C}\varphi\right)\frac{\sqrt{\tau_I}}{\tau^*}, \tag{89}$$

which implies that if $\tau_I \neq 0$, (89) is the unique solution.

Notice that equation for α can be simplified to an equation with endogenous parameters ϕ and α itself as well. We solve for α to obtain

$$\alpha = \left[\frac{1-\tau_U\left(N-1\right)\zeta\left(1-\zeta\right)\varphi}{\tau^*}\right]\frac{v_0}{\sigma_V}. \tag{90}$$

To solve for δ and η, we plug in the solution for β from equation (89) to equations for δ and η respectively. The results are

$$\delta = \frac{1 + (N-1)\frac{\zeta}{C}\varphi}{\tau^*}, \quad \eta = 1 + (N-1)\frac{\zeta}{C}\varphi. \tag{91}$$

The parameter γ measures the price impact of exogenous noise trading, and the parameter λ measures the price impact of trader (m, n)'s trading. Since all $M \times N$ traders provide liquidity to exogenous noise traders and to one another, but trader (m, n), does not provide liquidity to himself, the price impact parameter γ is slightly smaller than λ. This can be also directly inferred from comparing λ and γ as functions of π_P in (21). Then follows

$$\gamma = \left(\frac{MN-1}{MN}\right)\lambda. \tag{92}$$

Substitute equation (92) and (89) into equation (88) to obtain the desired solution

$$\lambda = \left(\frac{1 + (N-1)\frac{\zeta}{C}\varphi}{MN - 2 - (MN + N - 2)\frac{\zeta}{C}\varphi}\right)\frac{M}{\tau^*}. \tag{93}$$

Substituting (93) into equation (21) yields

$$\pi_P = \frac{\left(MN - 2 - (MN + N - 2)\frac{\zeta}{C}\varphi\right)}{M(MN-1)\left(1 + (N-1)\frac{\zeta}{C}\varphi\right)}\tau^*. \tag{94}$$

To exclude trivial no-trade equilibrium, we require $(MN-1)\pi_P \neq 0$, that is,

$$\left(\frac{MN - 2 - (MN + N - 2)\frac{\zeta}{C}\varphi}{M}\right)\left(\frac{\tau^*}{1 + (N-1)\frac{\zeta}{C}\varphi}\right) \neq 0 \tag{95}$$

Substituting equation (93) into the second order condition (79) yields the second order condition

$$\left(M-(M-1)\frac{\zeta}{C}\varphi\right)\left(\frac{MN-2-(MN+N-2)\frac{\zeta}{C}\varphi}{M}\right)^{-1}\left(\frac{\tau^*}{N}\right)^{-1}>0 \tag{96}$$

Since $\frac{MN-2}{MN+N-2}\leq 1\leq\frac{M}{M-1}$, this combined with (95) implies that

$$\varphi<\left(\frac{MN-2}{MN+N-2}\right)\frac{C}{\zeta}\qquad\text{or}\qquad\varphi>\left(\frac{M}{M-1}\right)\frac{C}{\zeta}. \tag{97}$$

Substituting equations (89)-(93) into equation (23) yields

$$\begin{aligned}\left(\frac{1-\varphi}{\varphi}\right)\tau_I &= \tau_I\tau_U\left(1-\zeta\right)\left(1+(N-1)\zeta\right)\\ &+\frac{\left(\rho\sigma_V\sigma_Z\right)^2/\left(N-1\right)}{\left[\frac{MN-2}{MN-1}-\left(\frac{MN+N-2}{MN-1}\right)\frac{\zeta}{C}\varphi\right]^2}+\left(\rho\sigma_V\sigma_S\right)^2+\left(\frac{\tau^*\sigma_O}{\sigma_V}\right)^2,\end{aligned} \tag{98}$$

which is well-defined since $\varphi\neq\left(\frac{MN-2}{MN+N-2}\right)\frac{C}{\zeta}$.

The parameters π_{const}, π_S, π_O and π_I defining the linear demand functions can be obtained by substituting (90) - (93) into equation (21).

$$\pi_{\text{const}}=\left(\frac{MN-2-(MN+N-2)\frac{\zeta}{C}\varphi}{MN-1}\right)\frac{\left[1-\tau_U\left(N-1\right)\zeta\left(1-\zeta\right)\varphi\right]v_0/\sigma_V}{1+(N-1)\frac{\zeta}{C}\varphi}, \tag{99}$$

$$\pi_I=\left(\frac{MN-2-(MN+N-2)\frac{\zeta}{C}\varphi}{MN-1}\right)\sqrt{\tau_I}, \tag{100}$$

$$\pi_S=\left(\frac{MN-2-(MN+N-2)\frac{\zeta}{C}\varphi}{MN-1}\right), \tag{101}$$

$$\pi_O=\left(\frac{MN-2-(MN+N-2)\frac{\zeta}{C}\varphi}{MN-1}\right)\tau^*, \tag{102}$$

which are all well-defined because $N \neq 1$.

Substituting (90) - (93) into (69), (82) and (20) yields a trader's demand schedule and the equilibrium price as (27) and (28) respectively.

The Case $\varphi = 0$. Now, we relax the assumption that $(N-1)\zeta^2\tau_I\tau_U = 0$. First, if $(N-1)\zeta^2\tau_U = 0$, trader's view that there is no information to learn from other traders. Therefore, we can set $\varphi = 0$ and substitute it into (69) and (24) to get the the conditional expectation and variance. Second, if $\tau_I = 0$, we know that β in (20) has to satisfy $\beta = 0$. With $\beta = 0$, traders cannot learn others' information from the price. Thus, we can also set $\varphi = 0$ to get the conditional expectation and variance. Moreover, $(N-1)\zeta^2\tau_I\tau_U = 0$ is the only case when the price is completely uninformative ($\varphi = 0$). Similarly, the market clearing condition (14) and matching the price conjecture to the market clearing price result in the demand schedule and the price as in (27) and (28) with $\varphi = 0$. For this to be an equilibrium, we need to ensure (1) the S.O.C. (96) holds, (that is, $MN \geq 2$) and (2) the resulting equilibrium is not a trivial no-trade equilibrium. From (27) with $\varphi = 0$ substituted, ruling out trivial no-trade equilibria implies that $MN \neq 2$. Therefore, there exists an equilibrium if and only if $MN > 2$.

Proof of Theorem 8. An upside-down equilibrium corresponds to φ that satisfies the second inequality in (46). Substituting this into (94), (100) and (47) yields equation (53) and $\chi < 0$. Moreover, the second order condition (79) guarantees that at any equilibrium $\chi > -\frac{1}{2}$, thus we get equation (54).

When $\sigma_Z^2 = 0$, equation (52) implies that there is unique φ that satisfies the second inequality in (46). For any $\sigma_Z^2 > 0$, there exists $\bar{\sigma}_Z$ such that for all $\sigma_Z \leq \bar{\sigma}_Z$, there exist multiple equilibria provided that $(N-1)\zeta^2\tau_I\tau_U\sigma_Z^2 > 0$ and (52) holds. When rewriting (45) as (71) with

$R(\varphi)$ defined as (72), we have

$$0 < R\left(\left(\frac{M-1}{M}\right)\frac{\zeta}{C}\right) < R\left(\left(\frac{MN+N-2}{MN-2}\right)\frac{\zeta}{C}\right). \tag{103}$$

Moreover, $L(\varphi)$ defined as (73) is monotonically decreasing for all $\varphi > \left(\frac{MN+N-2}{MN-2}\right)\frac{\zeta}{C}$ with $\lim_{\varphi \to \left(\frac{MN+N-2}{MN-2}\right)\frac{\zeta}{C}^+} L(\varphi)$ ∞ and $L\left(\left(\frac{M-1}{M}\right)\frac{\zeta}{C}\right) > 0$. Since σ_Z^2 only affects $L(\cdot)$, but not $R(\cdot)$, there exists $\bar{\sigma}_Z$ such that $\forall \sigma_Z < \bar{\sigma}_Z$, $L\left(\left(\frac{M-1}{M}\right)\frac{\zeta}{C}\right) < R\left(\left(\frac{M-1}{M}\right)\frac{\zeta}{C}\right)$ and by the Intermediate Value Theorem, there exists a solution to (45) that satisfies the second inequality of (46).

Proof of Theorem 9. When $\sigma_Z^2 = 0$, equation (45) has a unique solution and there cannot be multiple equilibria. $\sigma_Z^2 > 0$ guarantees a unique φ that satisfies the first inequality of (46), which we denote by subscript "N." With $\sigma_Z^2 > 0$, there can be multiple equilibria provided that $\sigma_Z^2 < \bar{\sigma}_Z^2$ from Theorem 8. Whenever there are multiple equilibria, equation (55) follows directly from (46).

We want to compare the ex-ante expected utility of traders under the assumption $\mathcal{S}_n = o_n = 0, \ \forall n$. Then the paper trading profit is given by

$$\begin{aligned} &\mathrm{E}^n\left\{\frac{v_n}{\sigma_V} + \frac{o_n}{\sigma_V} - p_{m,n} | i_n, P, S_n, o_n\right\} \\ &\quad = \frac{\left(M - (M-1)\frac{\zeta}{C}\varphi\right)}{MN-1}\left(\frac{\sqrt{\tau_I}}{\tau^*}\left(N i_n - \Sigma_{n'} i_{n'}\right) - \frac{\rho \sigma_V z}{\frac{MN-2}{MN-1} - \left(\frac{MN+N-2}{MN-1}\right)\frac{\zeta}{C}\varphi}\right). \end{aligned} \tag{104}$$

Substituting this into (78) the objective function yields

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{M\rho\sigma_V/\tau^*}\right)\frac{\left(\mathrm{E}^n\left\{\frac{v_n}{\sigma_V}|i_n, P, S_n, o_n\right\}+\frac{o_n}{\sigma_V}-p_{m,n}\right)^2}{1+2\chi}\\
=&\frac{1}{2}\frac{(MN-1)^2}{MN\rho\sigma_V}\left(M-(M-1)\frac{\zeta}{C}\varphi\right)\left(MN-2-(MN+N-2)\frac{\zeta}{C}\varphi\right)\\
&\times\left(\sqrt{\tau_I}\,(Ni_n-\Sigma_{n'}i_{n'})-\frac{\tau^*\rho\sigma_V z}{\frac{MN-2}{MN-1}-\left(\frac{MN+N-2}{MN-1}\right)\frac{\zeta}{C}\varphi}\right)^2.
\end{aligned} \tag{105}$$

The implications for the certainty equivalent are ambiguous.

Tables

Table 1: Exogenous Parameters

Parameter	Units	Description
N	1	Number of groups, indexed $n = 1, \ldots, N$
M	1	Number of traders per group, indexed $m = 1, \ldots, M$
τ_I	1	Trader's belief about precision of own group's signal
τ_U	1	Trader's belief about precision of other groups' signals
ζ	1	Correlation of of signals v_n and $v_{n'}$, $n' \neq n$
ρ	1/\$	Risk aversion of group n; risk aversion of trader (m, n) is ρM
σ_V^2	$\$^2/\text{Share}^2$	Variance of group n's asset fundamental value v_n is $\sigma_V^2 > 0$
σ_O^2	$\$^2/\text{Share}^2$	Variance of group n's purely private convenience yield
σ_Z^2	Share^2	Variance of exogenous demand noise
σ_S^2	Share^2	Variance of group n's aggregate endowment supply shock
C	1	Defined by $C :\equiv \tau_I/\tau_U$. $C > 1$ implies overconfidence, $C < 1$ implies underconfidence, $C = 1$ implies common prior.
θ	1	Defined by $\theta := \tau_I\tau_U\,(1-\zeta)\,(1+(N-1)\,\zeta) + (\rho\sigma_V\sigma_S)^2$.
τ_n	1	Precision of signal of group n, equal to either τ_I or τ_U

This table describes the exogenous parameters.

Table 2: Endogenous Parameters

Parameter	Units	Description
π_{const}	1	Constant term in demand function
π_S	1	Negative of coefficient on endowment shock S_n in demand function
π_O	1	Coefficient on convenience yield o_n in demand function
π_I	1	Coefficient on private signal i_n in demand function
π_P	1	Coefficient on price P in demand function
α	1	Constant term in linear price function
β	1	Coefficient on private signal i_n in linear price function
γ	1	Coefficient on exogenous noise z in linear price function
δ	1	Negative of coefficient on endowment shock S_n in linear price function
η	1	Coefficient on convenience yield o_n in linear price function
λ	1	Slope of trader's linear residual supply schedule $\partial P/\partial x_{m,n}$
φ	1	Fraction of private information revealed in prices
χ	1	Scaled illiquidity $(\lambda\tau^*/M)$.
τ^*	1	Reciprocal of error variance (precision) of trader's estimate of asset value v_n/σ_v.

This table describes the endogenous parameters.

Table 3: Random Variables

Parameter	Units	Description
z	Share	Exogenous demand by noise traders
v_n	\$/Share	Correlated fundamental value of asset to traders in group n
o_n	\$/Share	Uncorrelated private value (convenience yield) of asset to traders in group n
$\mathcal{S}_n$	Share	Aggregate endowment (demand) shock to traders in group, $\mathcal{S}_n/N$ per trader n
i_n	1	Common private signal to all traders in group n
P	\$/Share	Market-clearing price
$P(X)$	\$/Share	Market-clearing price as function of matrix of demand schedules X
$W_{m,n}$	\$	Wealth of trader (m, n).
$X_n(P)$	Share	Demand schedule for trader in group n. Same as $X_n(P \mid \mathcal{S}_n, o_n, i_n)$
$x_{m,n}$	Share	Quantity traded by trader (m, n)
$p_{m,n}$	1	Intercept of trader (m, n)'s linear residual supply schedule
ϵ_n^{ζ}	1	Group n's valuation error based on other traders' valuations $v_{n'}$, $n' \neq n$
ϵ_n^{ALL}	1	Group n's scaled sum of all noise in price

This table describes the random variables.