Insider trading when there may not be an insider*

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Abstract

We study a continuous-time Kyle-Back model of insider trading with uncertainty about the existence of the insider. The market maker absorbs net market order flows and form beliefs about the asset's true value as well as the presence of the insider. When the insider may not exist, the results depend on the market maker's market power. The monopolistic market maker, in equilibrium, never updates his belief regarding the existence of the insider and hence his belief only converges to the truth conditional on the existence of the insider. This equilibrium does not exist with competitive market makers. Moreover, under additional conditions, there is no equilibrium with competitive market makers. When the insider exists for sure, both settings of market power generate essentially the same results that are consistent with the literature.

1 Introduction

Insider trading is the central topic in the literature of market micro-structure and attracts lots of attention from the practitioners. Pioneered by the seminal work of

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Kyle (1985), most of the papers focus on the situation where the insider exists for sure and trades on her private information (e.g., Glosten and Milgrom (1985), Back (1992) and Easley, Kiefer, O'Hara, and Paperman (1996), etc.). A more realistic question, however, is whether there exists an insider. After all, if the markets are somewhat efficient, not every stock is over/under valued. Even if some stock is over/under valued, it may not be recognized by some insider in a timely manner. This paper addresses insider trading when the insider may or may not exist.

We find that the market maker's market power plays an important role if there is uncertainty about the existence of the insider. We consider two settings of market power in this paper. The first setting takes the traditional assumption in the market micro-structure literature such as Kyle (1985), Back (1992) and Back and Baruch (2004), in which competitive market makers face perfect competition and have to set the price to the expected value of the asset. The second setting involves a monopolistic market maker that plays a zero-sum game with the potentially existing insider and sets the asset price to maximize his expected payoff.[1] When the insider exists for sure, the first and the second settings degenerate to those in Back and Baruch (2004) and Anderson and Smith (2013), respectively, and give rise to essentially the same equilibrium outcomes. When there is uncertainty about the existence of the insider, the first setting may not admit any equilibrium while the second setting leads to a unique equilibrium in which the market maker never update his belief regarding the insider's existence.

The story is as follows. A risky asset is traded indefinitely in a continuous-time market until a public announcement of its value at a random time. At the announcement, the true value of the asset is released to the public and all positions are liquidated at the price equal to the asset's true value. There may or may not exist a risk neutral informed insider. If she exists, she knows the asset's true value at time 0. There are also liquidity motivated noise trades arriving as a Brownian motion which camouflages the potentially existing insider's trading behavior. The market maker rationally anticipates the potentially existing insider's trading strategy, updates his beliefs based on the observed order flows and adjusts the price accordingly. The insider, if exists, understands that her trades affect the price of the asset through the

[1] The zero-sum feature captures the conflict of interests between the market maker and the insider. As well known in game theory, the game payoffs do not have to be exactly "zero-sum". The sum of the two players' payoffs can be any constant and all the analysis carries over.

market maker's Bayesian updating and chooses her optimal trading strategy, which, in equilibrium, coincides with the market maker's anticipation.

When the market maker is a monopolist, we obtain a unique equilibrium where the market maker never updates his belief regarding the existence of the insider. That is, from the market maker's perspective, the probability that there exists an insider is always a constant in equilibrium. Consequently, in contrast to the standard models in which the insider exists for sure, here the market maker's belief never converges to the truth. Instead, it converges to the conditional truth. That is, if there exists an insider, the market makers belief conditional on the existence of the insider converges to the true value of the asset. The intuition is as follows. In equilibrium, the market maker always rationally subtracts the potentially existing insider's expected trading from the net order flows, and adjusts the price according to the remaining innovation. The insider, if exists, understands this and tries to minimize the information revealed to the market maker by mixing with the other types. This results in a zero expected trading (from the market maker's perspective) in equilibrium. As an immediate implication, the expected trading conditional on the existence of the insider is also zero. Since the expected trading conditional on the absence of insider is zero by its nature, from the market maker's perspective there is no difference in terms of order flows between the presence and absence of the insider. Therefore, he will never be able to learn whether there is insider trading or not. Moreover, the market maker sets the price as if he is maximizing his conditional expected payoff given the existence of the insider. This is because, absent the insider, any price gives the market maker the same expected payoff. The resulted price in general differs from the "fair" value, the expected value of the asset, except when the insider exists for sure. This price-"fair" value discrepancy reflects the importance of the market power. In the setting where the market maker faces perfect competition, he has to price the asset at the "fair" value. This gives the potentially existing insider extra strategic advantages to achieve infinite expected payoff. In contrast to the monopolist setting, there is no equilibrium in which the market maker never updates his belief about the insider's existence. If we further assume that the potential insider only has one type (for example, the insider's existence implies that the asset value is one), there does not exist any (Markov) equilibrium.

Some previous papers addressed conceptually related topics under different settings. Recently, Back, Crotty, and Tao (2015) studies a setting in which the insider always exists but may or may not be informed about the true value of the asset.[2] In this setting, when the insider does not know the true value of the asset, he knows that he is uninformed and will rationally value the asset based on his prior expectation. This is equivalent to adding an additional possible value of the asset, which is the expected value of the asset under the prior belief, to the Kyle model of Back and Baruch (2004) and thus can be viewed as a special case of Back (1992). In other words, receiving no information is essentially equivalent to knowing that the true asset value is its expected value. Hence, it seems more interesting to address whether or not there exists an insider acting against the market maker rather than whether the always existing insider receives information or not. It is also not surprising that whether the market maker is monopolistic or perfect competitive does not matter in the latter case.

Several previous papers also addressed related topics. In quite a different setting, Chakraborty and Yilmaz (2004a) and Chakraborty and Yilmaz (2004b) show that the insider is bluffing in every equilibrium. That is, the insider may choose to trade against her information in order to hide her identity. This result partially follows the discrete nature (in terms of time and strategy space) of the model that makes it too easy for the insider to be "caught" by the market makers. In contrast, bluffing does not happen in our equilibrium due to the continuity nature (continuous time and Brownian noise trades) of our model. Foster and Viswanathan (1993) consider a situation where some market participant knows a noisy signal of the fundamental while the insider knows the true fundamental. In equilibrium, the insider trades on the less informed participant's information at beginning in order to exhaust his information advantage over the market makers, and then trades on her own information. Instead of uncertainty in the traders' side, Yueshen (2015) considers uncertain market making, which results in a random markup component being added to the pricing rule in a standard Kyle model. Collin-Dufresne and Fos (2016) considers a Kyle model with serially correlated noise trades. The insider strategically chooses to trade more aggressively when the volatility of noise trades is high and thus affects the informativeness of the net order flow. The market makers understand this and adjust the price accordingly, which in turn affects the insider's trading aggressiveness. This mechanism leads to a positive relation between price volatility and trading volume in equilibrium.

[2] Li (2013), which is later subsumed by Back, Crotty, and Tao (2015), also studies this problem.

We proceed as follows. Section 2 sets up the common framework for the analysis. Section 3 presents a special case when the potential insider can only be type-1. Subsection 3.1 considers the monopolistic market maker setting while subsection 3.2 considers the competitive market maker setting. Section 4 considers the general situation when the potential insider can be either type-0 or type-1. We characterize the equilibrium with monopolistic market maker and show that the equilibrium with competitive market makers does not have the similar belief updating features. Section 5 summarizes our main findings. All proofs are relegated to the appendix.

2 The Model Setup

We consider a continuous time insider trading model with uncertainty on the existence of the insider. A risky asset is traded continuously in the market from time 0. The true value of the asset is $\upsilon \in \{0, 1\}$, which is announced to the public at a random time τ, which follows an exponential distribution with parameter r. After the announcement, the price of the asset jumps to its true value and the game is essentially over.

Players There are three types of players in the model. The first is an insider, who may or may not exist in the market. The ex ante probability of her existence is $(1 - \hat{\pi}_u) \in [0, 1]$, where $\hat{\pi}_u$ is the prior probability that the insider does not exist. If exists, the insider knows the true value of the asset and enjoys a payoff according to the tables below from trading the asset strategically. Otherwise, no order flow is generated from insider trading. The second type of players are noise traders. Their order flows are modeled by $\sigma \cdot Z_t$, a Brownian motion with volatility σ, which can be interpreted as purely driven by their liquidity demands. The third is a market maker, who does not know the true value of the asset and starts with a belief $\Pr(\upsilon = 1) = \widehat{p} \in (0, 1)$ at time 0. The history of aggregate order flows from the noise traders and the potentially existing insider is public. The market maker updates his beliefs about the existence of the insider and the true value of the asset according to his observation of aggregate order flows. At each time instant, the insider, if exists, can submit market buy/sell orders, and the market maker sets the price after observing the aggregate order flows.

Market power We compare two settings of market power. In the setting of monopolistic market power, the market maker can freely set the price of the asset. This corresponds to the case in Anderson and Smith (2013). In the setting of competitive market power, the market maker faces perfect competition and has to price the asset at its expected value. This is the conventional setting in the insider trading literature (e.g., Kyle (1985), Back (1992) and Back and Baruch (2004)). These two settings deliver essentially the same equilibrium when the insider exists for sure, as shown by Back and Baruch (2004) and Anderson and Smith (2013). When the insider may not exist, however, the insider, if exists, behaves very differently in the two settings.

Model Setting With a Monopolistic Market Maker In this setting, the insider and the market maker play a zero-sum game. It is not surprising that the market maker's expected payoff from playing with the potential insider is negative due to the insider's information advantage. This does not violate the market maker's participation constraint because we can always shift up the sum of the payoffs to a positive constant so that the market maker's expected payoff becomes positive. The zero-sum feature, or essentially constant-sum feature, is meant to capture the conflict of interests between the two parties. In particular, we adopt a variation of the instantaneous stage game from Anderson and Smith (2013), as shown below. The instantaneous payoffs to the market maker are given by the entries of the matrices, the opposite of which are the insider's payoffs. At each instant, the insider, knowing which game she is playing, can choose actions B(market buy order) and S(market sell order) with respective intensities $\alpha \in [0, +\infty)$ and $\beta \in [0, +\infty)$. It is straightforward to see that the state contingent payoff of the insider at instant t only depends on $\theta_t = \alpha_t - \beta_t \in (-\infty, \infty)$, the net buying(selling) rate. The market maker can choose between two actions a and b, with a referring to setting price to 1 and b to setting price to 0. Restricting the market maker's action to a and b is without loss of generality as long as we allow him to mix between the two actions. Let p_t denote the probability weight on action a at time t. Then mixing with probability p_t can be equivalently interpreted as setting price to p_t.[3] If the insider does not exist, the market maker plays a trivial single player game. The payoffs of the possible games are given in the following matrices.[4]

[3]We can also equivalently assume the market maker to choose a price $p_t \in [0, 1]$, which seems more natural in an insider trading story. But we stick to the binary-action setting in order to make our model more comparable to Anderson and Smith (2013).

[4]In the payoff matrices of Anderson and Smith (2013) there is a parameter $1 \geq \xi > 0$ representing the insider's information advantage. We normalize ξ to 1 in our model.

Table: $v = 0$ with insider

	a(set $p = 1$)	b(set $p = 0$)
B(Buy)	1	0
S(Sell)	-1	0

Table: $v = 1$ with insider

	a(set $p = 1$)	b(set $p = 0$)
B(Buy)	0	-1
S(Sell)	0	1

Table: $v = 0$ no insider

	a(set $p = 1$)	b(set $p = 0$)
Null	0	0

Table: $v = 1$ no insider

	a(set $p = 1$)	b(set $p = 0$)
Null	0	0

Two underlying assumptions of the monopolistic market maker case are worth further elaboration. First, we model the stage game played at each instant as a simultaneous-move game. On one hand, as argued in Kyle (1985), if the insider can submit orders after observing the price, she would trade unbounded quantities as her quantity traded does not affect the immediate execution price. This trivializes the problem and leads to unrealistic predictions. On the other hand, allowing the market maker to set price after observing the instantaneous order flow $dY_t = \theta_t dt + \sigma dZ_t$ does not give him extra information for pricing the asset. This is because the potential insider's trading is of order dt, dominated by the magnitude of noise trading that is of order $dt^{\frac{1}{2}}$. As a result, at instant t, the market maker sets the price only based on the information conveyed by the entire order path up to time t. Hence, it is without loss of generality to adopt the simultaneous-move stage game in this continuous-time model. Second, we assume that the market maker's objective is to maximize his expected payoff from playing the zero-sum game with the potential insider rather than making profit from exploiting the noise traders. This is reasonable given his job of market making and further justifies the setting of simultaneous-move stage game. Otherwise, if the market maker maximizes his gains from both the insider and the (liquidity driven) noise traders, he would set price to 1 when observing $dY_t > 0$ and to 0 when observing $dY_t < 0$. In this way, he can achieve infinite expected payoff by essentially "front-running" the investors, an unethical practice prohibited by regulation. Interestingly, although the market maker is not allowed to intentionally make profit from the noise traders, he does gain from trading with the noise traders in equilibrium. A similar point is addressed in Section 4 of Kyle (1985). In addition, this assumption also makes the monopolistic market maker case more comparable to the case of competitive market making, because the pressure of competition automatically forces the market makers to only focus on the zero-sum game with the insider.

Model Setting With Competitive Market Makers When the market maker faces perfect competition, he has to set the price equal to the expected value of the asset. The insider, similar to the insider in the monopolistic setting, can choose actions B and S with respective intensities $\alpha \in [0, +\infty)$ and $\beta \in [0, +\infty)$. Again, her payoff only depends on $\theta = \alpha - \beta$, the net trading rate. In this setting, the market maker can only update his belief using anticipated insider's trading rate and observed order flow and set the price to be the "fair" value according to his belief. The insider, if exists, maximizes her expected payoff conditional on the current asset price and her anticipation of market maker's belief updating process. This setting generalizes the standard continuous time insider trading framework (e.g., Back and Baruch (2004)) by introducing the possibility of no insider.

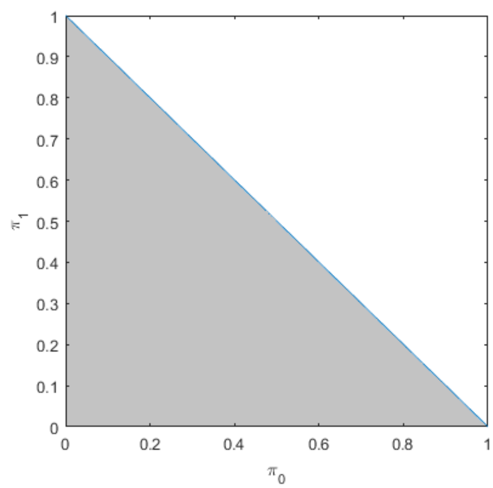
State space In principle, there are four possible scenarios: an insider exists and the true value $v = 0$, an insider exists and the true value $v = 1$, no insider and the true value $v = 0$, no insider and the true value $v = 1$. The last two scenarios are not distinguishable from the market maker's perspective. This is because there is no order flow from the insider no matter whether the true value is 0 or 1 and hence the observed order flow conveys no information. We treat the last two scenarios as a single one and denote it by u. We denote the first two scenarios by 0 and 1, respectively. In Scenario u, the market maker has to rely on his prior belief $\Pr(v = 1) = \widehat{p}$ to price the asset. Hence the price of the asset conditional on Scenario u is $\widehat{p}$. We denote the set of all scenarios by $\Omega = \{0, 1, u\}$. The market maker's belief over Ω can be represented by a pair (π_0, π_1), with π_0, π_1 and $\pi_u = 1 - \pi_0 - \pi_1$ referring to the probabilities assigned to Scenarios 0, 1 and u, respectively. Note that θ_t is the instantaneous (net) trading rate chosen by the insider (if she exists) at time t. In Scenario u, there is no insider and we can set $\theta_t = 0$ for all $t \geq 0$ in order to economize the notations. We denote the cumulative order flow at time t by $Y_t = \int_0^t \theta_s ds + \sigma Z_t$. We require that θ be adapted to the filtration generated by Y and the scenario $\omega \in \Omega$. The market maker's belief over Scenario $\omega \in \Omega$ at $t < \tau$ is given by

$$\pi_{\omega,t} = \mathbb{E}\left[1_\omega | \left(Y_s\right)_{s \leq t}\right]$$

where 1_ω is the indicator function for Scenario ω. The The equilibrium concept we consider here is Markov equilibrium, in which the probabilities π_0 and π_1 are the state variables of this economy.[5] Specifically, the state space is

$$\Pi = \{(\pi_0, \pi_1) : \pi_0 \geq 0,\ \pi_1 \geq 0, \text{ and } \pi_0 + \pi_1 \leq 1\}$$

as shown by the shaded area below. [6]



It is worth noting that the models of Back and Baruch (2004) and Anderson and Smith (2013) correspond to the special case $\{(\pi_0, \pi_1) : \pi_0 \geq 0,\ \pi_1 \geq 0, \text{ and } \pi_0 + \pi_1 = 1\}$ together with competitive and monopolistic market power, respectively. We also assume that the insider's cumulated trading volume is continuous with respect to time.[7]

[5]Note that the equilibrium concepts have slight difference under the monopolistic market maker setting and the competitive market maker setting. In the former, the insider, if exists, correctly anticipates the market maker's pricing strategy and chooses the optimal trading rate at each state; the market maker correctly anticipates the insider's trading strategy at each state and chooses the optimal pricing rule. In the equilibrium of the latter setting, the insider, if exists, does the same as in the former setting, while the market maker correctly anticipates insiders trading strategy but can only price the asset at its expected value. The latter could be micro-founded by Bertrand competition among market makers.

[6]Note that we do not have to include the market participants' positions into the set of state variables because of the risk neutrality assumption.

[7]We make this assumption to simplify the derivation and presentation. It is not essential since a

In a Markov equilibrium, the trading rate of the potentially existing insider at time t is a function of $(\pi_{0,t}, \pi_{1,t})$ and $\upsilon \in \{0, 1\}$. We denote the trading rate for type υ by $\theta_\upsilon(\pi_{0,t}, \pi_{1,t})$, with $\theta_\upsilon(\pi_{0,t}, \pi_{1,t}) > 0$ referring to buying and $\theta_\upsilon(\pi_{0,t}, \pi_{1,t}) < 0$ referring to selling, respectively. To economize the notations, we denote the trading rate in Scenario u by θ_u, which is zero for all $(\pi_{0,t}, \pi_{1,t}) \in \Pi$. In the rest of the paper, we will use the terms "belief" and "state" interchangeably.

We study four cases in this paper: i) a monopolistic market maker and only type-1 insider (i.e., the initial state is on the π_1 axis); ii) competitive market makers and only type-1 insider; iii) a monopolistic market maker and possibly both types of insider (i.e., the initial state is in the interior of Π); iv) competitive market makers and possibly both types of insider. The cases of monopolistic/competitive market makers with only type-0 insider (i.e., the initial state is on the π_0 axis) are similar to the first two cases and omitted.

3 Cases i) and ii): Only Type-1 Insider is Possible

This section studies cases i) and ii), where the initial state lies on the vertical boundary of state space Π excluding the origin. That is, if the insider exists, the true value of the asset is 1; otherwise, the true value follows the prior distribution and the fair value of the asset is $\hat{p} \in [0, 1)$. To provide an interpretation of this special case, think about a poorly managed firm that needs to be reformed. The market maker does not know whether or not there exists a capable activist that can take over and restructure the firm. The value of the firm is 1 if the activist exists. Otherwise, the value of the firm remains $\hat{p}$. In this case, π_0 is always zero and π_1 is the only state variable. Based on the filtering theory of Itô processes., the market maker updates π_1 according to

$$d\pi_1 = \lambda_1 \cdot (dY_t - \phi \cdot dt) \tag{1}$$

where

$$\lambda_1 = \frac{\pi_1(1 - \pi_1)\theta_1}{\sigma^2}$$

and $\phi = \pi_1\theta_1$.

jump of the order flow reveals the existence of the insider and is suboptimal for her.

3.1 The Monopolistic Market Maker Case

In this case, the market maker's expected payoff is 0 if the insider does not exist and is $-\int_0^{\tau}(1-p_t)\theta_{1,t}dt$ if the insider does exist. Note that any Markov Equilibrium can be uniquely determined by a pair $(\theta_1(\pi_1), p(\pi_1))$. In equilibrium, the market maker correctly anticipates $\theta_1(\pi_1)$, updates π_1 according to (1) and sets $p(\pi_1)$ to maximize his expected payoff. If the insider exists, she correctly anticipates the market maker's belief updating process given by (1) and optimally chooses trading rate $\theta_1(\pi_1) \in (-\infty, \infty)$ to maximize her expected payoff.

Proposition 1 *Any pair* $(\theta_1(\cdot), p(\cdot))$ *such that* $\theta_1(\pi_1) \geq 0$ *and* $p(\pi_1) = 1$ *for all* $\pi_1 \in (0, 1]$ *characterizes a Markov Equilibrium.*

The intuition of this proposition is straightforward. First, in equilibrium, the potential insider's trading rate must be always non-negative. Otherwise, anticipating the potentially existing insider to sell the asset at some certain time, the market maker will set the price to 0 to take advantage of the insider as the insider's existence implies that the true value is 1. Second, given that the potentially existing insider is always buying, the market maker must always set the price to 1, because this is the only price that can protect him from losing money to a buying insider whose existence implies the true value equal to 1. Finally, anticipating that the market maker always prices the asset at 1, the insider is indifferent to any buying strategy.

Proposition 1 implies multiple equilibria which differ in the potentially existing insider's trading strategies. We can significantly sharpen our prediction by introducing a natural ϵ-refinement. That is, the potential insider incurs an arbitrarily small flow trading cost $\epsilon > 0$ from choosing a non-zero trading rate at any instant.

Corollary 2 *There is a unique Markov Equilibrium surviving the* ϵ*-refinement. This equilibrium is characterized by* $p(\pi_1) = 1$ *and* $\theta_1(\pi_1) = 0$ *for all* $\pi_1 \in (0, 1]$.

In this equilibrium, since the potentially existing insider never trades, the market maker never updates his belief regarding the existence of the insider. Therefore, $\lambda_1(\pi_1) = 0$ for all $\pi_1 \in (0, 1]$. This "never learning the existence" feature remains in more general settings when both types of insider may exist, as will be shown in Subsection 4.2.

3.2 The Competitive Market Maker Case

Since the market maker is competitive, the asset is always priced at its expected value

$$p_t = \pi_{1,t} + \hat{p} \cdot (1 - \pi_{1,t}) \ .$$

In any Markov equilibrium, the market maker correctly anticipates the potentially existing insider's trading rate $\theta_1(\pi_1)$, updates his belief π_1 according to (1), and prices the asset at

$$p(\pi_1) = \pi_1 + \hat{p} \cdot (1 - \pi_1) \ .$$

The insider correctly anticipates the market maker's belief updating process and optimally chooses $\theta_1(\pi_1)$ to maximize his value function $V(\pi_1)$.

Using argument similar to Back and Baruch (2004), the insider's value function satisfies

$$rV = max_{\theta_1}(1-p)\theta_1 + \lambda_1(\theta_1 - \phi)V^{'} + \frac{1}{2}\lambda^2\sigma^2 V^{''} \ .$$

To attain a finite maximum, the coefficient in front of θ_1 must be 0, i.e.,

$$1 - p + \lambda V^{'} = 0 \ .$$

Then, the equilibrium, if exists, should be pinned down by boundary conditions $V(0) = \infty$, $V(1) = 0$ and the following five equations:

$$rV = -\lambda_1 \phi V^{'} + \frac{1}{2}\lambda_1^2 \sigma^2 V^{''}, \qquad (2)$$

$$1 - p + \lambda_1 V^{'} = 0 \ , \qquad (3)$$

$$\lambda_1 = \frac{\pi_1(1-\pi_1)\theta_1}{\sigma^2} \ , \qquad (4)$$

$$\phi = \pi_1 \theta_1 \ , \qquad (5)$$

and

$$p = \pi_1 + \hat{p}(1 - \pi_1) \ . \qquad (6)$$

The analysis of these equations leads to the following proposition.

Proposition 3 *Equations* (2)-(6) *with boundary conditions* $V(0) = \infty$, $V(1) = 0$ *do not admit a solution. That is, there does not exist any Markov equilibrium in this* *setting.*

This result is intuitive when compared with Proposition 1 for the case of the monopolistic market maker. Due to the competition, the market maker is no longer be able to set the price to 1 and hence unable to protect himself from losing money to the potentially existing insider. In turn, the insider, if exists, obtains strategic advantages to achieve infinite expected payoff. As a result, the equilibrium does not exist.

4 Cases iii) and iv): Both Types of Insider are possible

This section studies cases iii) and iv), where the initial state lies in the interior of state space Π. Hence, the existence of the insider does not pin down the true value of the asset.

4.1 The Market Maker's Belief Dynamics

Suppose the market maker believes that the potentially existing insider's trading strategy is described by some mapping $\theta_\upsilon : \Pi \to \mathbb{R}$, where $\upsilon \in \{0, 1\}$. Given the market makers' information at time t, the expected trading rate is

$$\phi(\pi_{0,t}, \pi_{1,t}) = \pi_{0,t} \cdot \theta_0(\pi_{0,t}, \pi_{1,t}) + \pi_{1,t} \cdot \theta_1(\pi_{0,t}, \pi_{1,t}) \tag{7}$$

and the innovation or surprise in the order flow at time t is

$$dY_t - \phi(\pi_{0,t}, \pi_{1,t})\, dt \ .$$

Standard filtering theory shows that the market makers' belief updating is proportional to the innovation in the observed order flow, which is given by

$$d\pi_{0,t} = \lambda_0(\pi_{0,t}, \pi_{1,t}) \cdot [dY_t - \phi(\pi_{0,t}, \pi_{1,t}) \cdot dt] \tag{8}$$

and

$$d\pi_{1,t} = \lambda_1(\pi_{0,t}, \pi_{1,t}) \cdot [dY_t - \phi(\pi_{0,t}, \pi_{1,t}) \cdot dt] \ , \tag{9}$$

where

$$\lambda_0(\pi_0, \pi_1) = \frac{\pi_0 \left[(1-\pi_0) \cdot \theta_0(\pi_0, \pi_1) - \pi_1 \cdot \theta_1(\pi_0, \pi_1) \right]}{\sigma^2} \tag{10}$$

and

$$\lambda_1(\pi_0, \pi_1) = \frac{\pi_1 \left[(1-\pi_1) \cdot \theta_1(\pi_0, \pi_1) - \pi_0 \cdot \theta_0(\pi_0, \pi_1) \right]}{\sigma^2} \ . \tag{11}$$

We need to augment (8) and (9) by defining values of λ_0 and λ_1 on the boundary of state space Π. Note that once the market maker puts zero probability on some scenario $\omega \in \{0, 1, u\}$, they will always put zero probability on that scenario forever regardless of the arrival of new information. Hence, we obtain

$$\lambda_0(0, \pi_1) = 0 \ , \tag{12}$$

$$\lambda_1(\pi_0, 0) = 0 \tag{13}$$

and

$$\lambda_0(\pi_0, \pi_1) + \lambda_1(\pi_0, \pi_1) = 0 \ , \text{ for all } \pi_0, \pi_1 \text{ such that } \pi_0 + \pi_1 = 1 \ . \tag{14}$$

Let $V_0(\pi_0, \pi_1), V_1(\pi_0, \pi_1)$ denote the value functions of the type-0 and type-1 insiders, respectively. Given θ_0 and θ_1, the market maker's beliefs on the two types of insider's trading strategies, as well as the insider's beliefs on the market maker's pricing strategy p, V_0 and V_1 satisfy the following HJB equations:

$$\begin{aligned} rV_0 = max_{\widetilde{\theta}_0} - p\widetilde{\theta}_0 + \lambda_0(\widetilde{\theta}_0 - \phi)\frac{\partial V_0}{\partial \pi_0} + \lambda_1(\widetilde{\theta}_0 - \phi)\frac{\partial V_0}{\partial \pi_1} + \\ \frac{1}{2}\lambda_0^2\sigma^2\frac{\partial^2 V_0}{\partial \pi_0^2} + \frac{1}{2}\lambda_1^2\sigma^2\frac{\partial^2 V_0}{\partial \pi_1^2} + \lambda_0\lambda_1\sigma^2\frac{\partial^2 V_0}{\partial \pi_0 \partial \pi_1} \end{aligned} \tag{15}$$

and

$$\begin{aligned} rV_1 = max_{\widetilde{\theta}_1}(1-p)\widetilde{\theta}_1 + \lambda_0(\widetilde{\theta}_1 - \phi)\frac{\partial V_1}{\partial \pi_0} + \lambda_1(\widetilde{\theta}_1 - \phi)\frac{\partial V_1}{\partial \pi_1} + \\ \frac{1}{2}\lambda_0^2\sigma^2\frac{\partial^2 V_1}{\partial \pi_0^2} + \frac{1}{2}\lambda_1^2\sigma^2\frac{\partial^2 V_1}{\partial \pi_1^2} + \lambda_0\lambda_1\sigma^2\frac{\partial^2 V_1}{\partial \pi_0 \partial \pi_1} \end{aligned} \tag{16}$$

The first term on the right hand side of (15) represents the type-0 insider's instantaneous gain (loss). The remaining terms on the right hand side reflects the long term effects on her value function if she chooses trading rate $\widetilde{\theta}_0$. The interpretation for equation (15) is similar.

4.2 The Monopolistic Market Maker Case

Definition of Equilibrium: A Markov equilibrium in this case is a five-tuple

$$(\theta_0(\pi_0, \pi_1), \theta_1(\pi_0, \pi_1), p(\pi_0, \pi_1), V_0(\pi_0, \pi_1), V_1(\pi_0, \pi_1))$$

satisfying the following conditions:

a) the market maker's beliefs π_0 and π_1 obey law of motions given by (8) and (9);

b) at each instant, p maximizes the market maker's instantaneous payoff;

c) given the market maker's pricing strategy p, θ_υ maximizes V_υ, the type-υ insider's value function;

d) the market maker correctly anticipates the insider's trading rate θ_υ at each instant;

e) $V_0(\pi_0, \pi_1)$ and $V_1(\pi_0, \pi_1)$ satisfy equations (15) and (16).

The definition is self-explanatory except condition b), which only requires the market maker to be a short-term maximizer. This is without loss of generality due to the fact that price setting has no direct impact on the belief updating process. In particular, on the one hand, in any equilibrium that satisfies the above definition, the market maker has no incentive to deviate to another price $\widetilde{p} \neq p$ at any instant. Recall that the market maker receives market orders and p has already maximizes his instantaneous payoff. Deviating to $\widetilde{p}$ does not influence the market order already submitted at this instant and thus potentially reduces his instantaneous payoff. Moreover, this "one-shot" deviation has no impact on the belief updating process, which is purely driven by the publicly observable order flows, and hence generates no future benefits either. Therefore, the market maker has no interest in any "one-shot" deviation and his short-term maximizing strategy also maximizes his long-run expected payoff. On the other hand, suppose we obtain an equilibrium in which the market maker is maximizing his long-run expected payoff. Then the market maker's instantaneous payoff should also always attains maximum. Otherwise, at some instant, the market maker can deviate to another price and achieve a higher instantaneous payoff without changing his belief updating process nor the future payoffs.

Lemma 4 *At any time instant t, if $\pi_{0,t} \cdot \theta_0 (\pi_{0,t}, \pi_{1,t}) + \pi_{1,t} \cdot \theta_1 (\pi_{0,t}, \pi_{1,t}) = 0$, then $d\pi_{u,t} = 0$.*

This lemma states that if in equilibrium the potential insider's expected trading rate is zero, the market maker does not update his beliefs regarding the insider's existence. This is because the expected trading rate conditional on the absence of insider is zero by its nature, there is no difference in terms of order flows between the presence and absence of the insider from the market maker's perspective. Therefore, he will never be able to learn whether there is insider trading or not. As a result, the belief pair (π_0, π_1) moves along a -45-degree line through the initial state $(\hat{\pi}_0, \hat{\pi}_1)$, and we can use π_1 solely as the state variable since $\pi_0 + \pi_1$ remains constant. In the next proposition, we conjecture and verify an equilibrium with this feature.

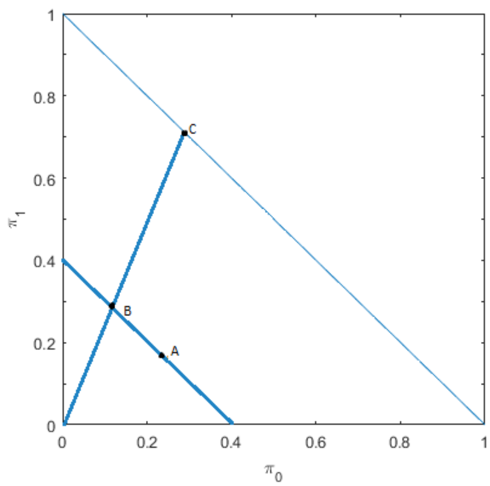
Proposition 5 *This game admits a unique equilibrium, the potentially existing insider's expected trading rate is always zero, i.e., $\pi_0\theta_0 + \pi_1\theta_1 = 0$ for all $t \geq 0$, and the market maker prices the asset at*

$$p(\pi_1) = \frac{\pi_1}{1 - \hat{\pi}_u} \ ,$$

where $\hat{\pi}_u = \pi_{u,0}$ is the probability that there is no insider at time 0. Moreover, the market maker's belief converges to the conditional truth as $t \to \infty$.

It is worth noting that the market maker sets price to $\frac{\pi_1}{1-\hat{\pi}_u}$, which differs from the asset's "fair" value $\pi_1 + \hat{p}\hat{\pi}_u$, also the price under perfect competition. When the insider exists for sure, $\hat{\pi}_u = 0$ and this difference disappears. Consequently, the market maker behaves essentially the same in the two market power settings, a result consistent with both Back and Baruch (2004) and Anderson and Smith (2013). When the insider may not exist, since the market maker is indifferent in setting any price in the scenario without the insider, he sets the price solely to maximize his conditional expected payoff in the scenario with the insider. Hence, he "rescales" the probability pair (π_0, π_1) to the conditional probabilities $\left(\frac{\pi_0}{1-\hat{\pi}_u}, \frac{\pi_1}{1-\hat{\pi}_u}\right)$ and price the asset at $\frac{\pi_1}{1-\hat{\pi}_u}$ accordingly. We use the following figure to illustrate this intuition.

In this picture, the initial state A lies on the line $\pi_0 + \pi_1 = 0.4$. In equilibrium, the belief (π_0, π_1) can only move along this line. If the type-1 (type-0) insider exists, as $t \to \infty$, belief converges to the point $(0, 0.4)$ ($(0.4, 0)$). To see how the market maker prices the asset at an arbitrary state B with coordinates (π_0^B, π_1^B), he first rescales state B to state C with coordinate $(2.5 \cdot \pi_0^B, 2.5 \cdot \pi_1^B)$ and sets price to $2.5 \cdot \pi_1^B$, which differs from the "fair" asset value $\pi_1^B + 0.6 \cdot \hat{p}$.



4.3 The Competitive Market Maker Case

This case is similar to that in 4.2 except that the market maker has to set price to $\pi_1 + \hat{p}(1 - \pi_0 - \pi_1)$, the expected value of the asset. We do not obtain a complete characterization of the equilibrium in this case. Instead, we show that if the equilibrium exists, it does not have the "non-learning" feature as in the equilibrium in 4.2.

Proposition 6 *There is no Markov equilibrium in which the market maker never updates his belief regarding the insider's existence.*

The intuition behind this non-existence result is straightforward. Suppose there is a Markov equilibrium in which the market maker never updates his belief regarding the insider's existence. Then even if the market maker is very sure that the insider is type-1 conditional on the insider's existence, the competition forces him to price the asset way below 1. Hence, no matter what the equilibrium trading rate of the type-1 insider is, she has an incentive to deviate by raising her trading rate. This is because her value function can be arbitrary large for states close to the vertical boundary. This contradicts to the existence of such equilibria. In contrast, if the market maker is a monopolist, he has no problem setting asset price close to 1 for states close to the vertical boundary. This market power deters the type-1 insider from raising her trading rate arbitrarily high as this hurts her future value from trading.

5 Conclusion

This paper contributes to the insider trading literature by introducing uncertainty about the insider's existence. When there may not exist an insider, the market power plays an important role. The monopolistic market maker will never update his belief about the insider's existence and set the price using a "rescaling rule". The competitive market makers have to set the price to the "fair" value of the asset and the setting might encounter an equilibrium existence problem. When the insider exists for sure, the monopolist setting and the traditional competitive market maker setting generate essentially the same equilibrium outcomes. How to model and solve an intermediate level of competition together with uncertainty on the insider's existence remains an open question for the future research.

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A Appendix

Proof of Proposition 1 and corollary 2

Proof. First note that in this zero sum game, the market maker's highest possible equilibrium payoff is 0 since the potentially existing insider can always guarantee herself a zero expected payoff by choosing a zero trading rate all the time. Setting $p = 1$ attains this upper bound for the market maker. Moreover, in responding to $p(\pi_1) < 1$ for some π_1, the potentially existing insider will pick $\theta_1(\pi_1) > 0$ to achieve a positive expected payoff, which is suboptimal for the market maker. Given that the asset is always priced at 1, the potentially existing insider, whose existence implies the true value being 1, is indifferent in choosing any trading rate. However, the trading rate must always be non-negative in equilibrium. Otherwise, anticipating $\theta_1(\pi_1) < 0$ for some π_1, the market maker will price the asset at 0 and achieve a positive expected payoff, which is suboptimal for the potentially existing insider. Hence, any pair $(\theta_1(\cdot), p(\cdot))$ such that $\theta_1(\pi_1) \geq 0$ and $p(\pi_1) = 1$ for all $\pi_1 \in (0, 1]$

is an Equilibrium. Once there is an ϵ participation cost, however, the potentially existing insider's only strategy to achieve a zero expected payoff is setting $\theta_1(\pi_1) = 0$ for all π_1. In this equilibrium, the market maker never updates π_1 because $\lambda_1 = 0$. ■

Proof of Proposition 3

Proof. For the ease of notation, set $\hat{p} = 0$, the proof for $\hat{p} \in (0, 1)$ is similar. We will also use π and θ, instead of π_1 and θ_1, to refer to the probability that the insider exists and his trading rate, respectively.

Note that $\lambda_1 = \frac{\pi(\theta - \pi\theta)}{\sigma^2}$ and $\phi = \pi\theta$. The belief updating process is $d\pi_t = \lambda_{1,t} \cdot [(\theta_t - \phi_t)dt + \sigma dZ_t]$. Thus, the HJB equation is (using π as the state variable):

$$rV = max_\theta(1 - \pi)\theta + \lambda_1(\theta - \phi)V^{'} + \frac{1}{2}\lambda_1^2\sigma^2V^{''}. \tag{17}$$

Since θ is unbounded and the right hand sider is linear in θ, to achieve a finite maximum, we obtain

$$1 - \pi + \lambda_1 V^{'} = 0.$$

Then, the equilibrium should be pinned down by the following equations:

$$rV = -\lambda_1\phi V^{'} + \frac{1}{2}\lambda_1^2\sigma^2V^{''}, \tag{18}$$

$$1 - \pi + \lambda_1 V^{'} = 0\ , \tag{19}$$

$$\lambda_1 = \frac{\pi(1 - \pi)\theta}{\sigma^2}\ , \tag{20}$$

and

$$\phi = \pi\theta\ . \tag{21}$$

From (19), (20) and (21), we obtain

$$\lambda_1 = -\frac{1 - \pi}{V'}$$

and

$$\phi = \frac{\sigma^2\lambda}{1 - \pi} = -\frac{\sigma^2}{V'}\ .$$

Plugging the expressions of λ_1 and ϕ into (18), we obtain

$$rV = -\frac{(1 - \pi)\sigma^2}{V'} + \frac{1}{2}\frac{(1 - \pi)^2\sigma^2}{(V')^2}V^{''} \tag{22}$$

with boundary conditions $V(0) = \infty$ and $V(1) = 0$.

To simplify the notations, we set $r = \sigma^2 = 1$. This has no effecct on the equilibrium existence result. Let $S(t) = V(1-t)$. Then (22) becomes

$$S = \frac{t}{S'} + \frac{1}{2}\frac{t^2}{(S')^2}S'' \tag{23}$$

for $t \in (0,1)$ with boundary conditions $S(0) = 0$ and $S(1) = \infty$. Note that the insider can guarantee herself a non-negative expected payoff and hence $S(t) \geq 0$ for $t \in (0,1)$. Then, if we can find a $k \geq 0$ such that initial conditions $S(0) = 0$ and $S'(0) = k$ induce $S(1) = \infty$, we obtain an equilibrium and the insider's strategy can be pinned down by (18) - (21). Transform (23) into

$$\frac{1}{2}t^2 \cdot S'' = S \cdot (S')^2 - t \cdot S' \ . \tag{24}$$

First note that $S(t) = t$ is a solution of this ODE with initial condition $S(0) = 0, S'(0) = 1$. But the boundary condition $S(1) = \infty$ does not hold. Next consider the case of $S'(0) = k \in (0,1)$. Then there exists a $t_0 \in (0,1)$, s.t. $S'(t) \in (0,1)$ for $t \in (0,t_0)$. Hence, $S(t) < t$ and $S(t) \cdot S'(t) < t \cdot 1 = t$ for $t \in (0,t_0)$. This implies $S''(t) < 0$ for $t \in (0,t_0)$, which further implies $S(t_0) < t_0$ and $S'(t_0) < 1$. Repeatedly using this argument we obtain that $S(t) < t$ for all $t \in (0,1)$. Thus the boundary condition $S(1) = \infty$ is violated and $S'(0) = k \in (0,1)$ can not be the right initial condition. Then consider the case of $S'(0) = k > 1$. Then there exists a $t_0 \in (0,1)$, s.t. $S'(t) > 1$ for $t \in (0,t_0)$. This implies $S(t) > t$ and $(S')^2 > S'$ in $(0,t_0)$. Then we have

$$\frac{1}{2}t^2 S''(t) > tS'(t)\left[S'(t) - 1\right] > tk(k-1)$$

for $t \in (0,t_0)$, where the second inequality holds because $S'(t) > k$, which stems from $S''(t) > 0$ as a result of the first inequality. Let $C = 2k(k-1)$, then the above inequalities imply that

$$S''(t) > \frac{C}{t}$$

for $t \in (0,t_0)$. Taking integral of both sides of this inequality with respect to t, we see that that $S(t)$ explodes at 0_+. Finally, consider the case of $S'(0) = 0$. Then by continuity, there exists a $t_0 \in (0,1)$, s.t. $0 < S'(t) < 1$ for $t \in (0,t_0)$. This implies $S(t) < t$ for $t \in (0,t_0)$. Following the same argument for the case of $S'(0) = k \in (0,1)$

we can see that this ODE cannot explode at $t = 1$ and the boundary condition $S(1) = \infty$ is violated. Therefore, the game does not admit an equilibrium. ■

Proof of Lemma 4.

Proof. If $\pi_0\theta_0 + \pi_1\theta_1 = 0$, we have

$$\begin{aligned} \lambda_0 &= \frac{\pi_0[(1-\pi_0)\theta_0 - \pi_1\theta_1]}{\sigma^2} \\ &= \frac{\pi_0[(\theta_0 - \pi_0\theta_0 - \pi_1\theta_1]}{\sigma^2} \\ &= \frac{\pi_0\theta_0}{\sigma^2} \end{aligned}$$

and

$$\begin{aligned} \lambda_1 &= \frac{\pi_1[(1-\pi_1)\theta_1 - \pi_0\theta_0]}{\sigma^2} \\ &= \frac{\pi_1[(\theta_1 - \pi_1\theta_1 - \pi_0\theta_0]}{\sigma^2} \\ &= \frac{\pi_1\theta_1}{\sigma^2}. \end{aligned}$$

Hence,

$$\begin{aligned} &\lambda_0 + \lambda_1 \\ = &\frac{\pi_0\theta_0 + \pi_1\theta_1}{\sigma^2} = 0 \ . \end{aligned}$$

Therefore,

$$\begin{aligned} d\pi_u &= -(d\pi_0 + d\pi_1) \\ &= -(\lambda_0 + \lambda_1) \cdot dY \\ &= 0 \ . \end{aligned}$$

■

Proof of Proposition 5

Proof. We guess and verify an equilibrium in which the potentially existing insider's expected trading rate is always $\phi = 0$. We then show that this equilibrium is essentially unique.

Suppose the expected trading rate is always $\phi = 0$. Lemma **4** suggests that the belief pair (π_0, π_1) always moves along a -45-degree line through the initial state $(\hat{\pi}_0, \hat{\pi}_1)$, and we can use π_1, the probability that the insider exists and the true value is 1, as the state variable. Then the HJB equation for the insider knowing true value equal to $v \in \{0, 1\}$ is

$$rV_v = sup_{\theta_v \in (-\infty,\infty)} \theta_v u_v(p) + \lambda_1 \cdot (\theta_v - \phi) V_v^{'} + \frac{1}{2}\sigma^2 \lambda_1^2 V_v^{''} \ , \tag{25}$$

with

$$\lambda_1 = \frac{\pi_1[(1-\pi_1)\theta_1 - (1-\hat{\pi}_u - \pi_1)\theta_0]}{\sigma^2} \ , \tag{26}$$

$$u_0(p) = -p \ , \tag{27}$$

and

$$u_1(p) = 1 - p \ . \tag{28}$$

Since the insider's trading rates are unconstrained in this model, to obtain a finite maximum of (25) in equilibrium, it must be

$$u_0(p) + \lambda_1 V_0^{'} = 0 \tag{29}$$

and

$$u_1(p) + \lambda_1 V_1^{'} = 0 \ . \tag{30}$$

Plugging $\phi = 0$, (29) and (30) into (25), we obtain

$$\frac{rV_0(\pi_1)}{\sigma^2} = \frac{1}{2}\lambda_1^2 V_0^{''}(\pi_1) \ , \tag{31}$$

and

$$\frac{rV_1(\pi_1)}{\sigma^2} = \frac{1}{2}\lambda_1^2 V_1^{''}(\pi_1) \ . \tag{32}$$

Define $\Lambda(\pi_1) = V_0(\pi_1) - V_1(\pi_1)$. From (29) $-$ (30), we get

$$\lambda_1 = \frac{1}{\Lambda'(\pi_1)} \ . \tag{33}$$

Let $\rho = \frac{4r}{\sigma^2}$. Substitute (33) into (31) and (32) to get

$$\rho V_0(\pi_1) = 2\frac{V_0^{''}(\pi_1)}{(\Lambda'(\pi_1))^2} \ , \tag{34}$$

and

$$\rho V_1(\pi_1) = 2\frac{V_1^{''}(\pi_1)}{(\Lambda^{'}(\pi_1))^2} \ . \tag{35}$$

From (34) and (35), we get

$$\rho \Lambda(\pi_1) = 2\frac{\Lambda^{''}(\pi_1)}{(\Lambda^{'}(\pi_1))^2} \ . \tag{36}$$

The general solution to (36) is

$$\Lambda(\pi_1) = \frac{2}{\sqrt{\rho}}\Phi^{-1}(2c \cdot (\pi_1 - \frac{1}{2}) + 2 \cdot c \cdot n) \ , \tag{37}$$

where

$$\Phi(s) = 2[\int_0^s e^{-t^2}dt]/\sqrt{\pi} \ . \tag{38}$$

By symmetry, $V_0(\frac{1-\hat{\pi}_u}{2}) = V_1(\frac{1-\hat{\pi}_u}{2})$, together with (37), this implies $n = \hat{\pi}_u/2$. To pin down c, notice that when $\pi_1 = 0$, the market maker sets $p = 0$ so that type-0 insider gets 0 expected payoff. This in turn implies that type-1 insider gains an infinite expected payoff. Hence, $\Lambda = V_0 - V_1$ goes to negative infinity as π_1 approaches 0. This implies $-c + \hat{\pi}_u c = -1$, i.e., $c = \frac{1}{1-\hat{\pi}_u}$. Thus,

$$\Lambda(\pi_1) = \frac{2}{\sqrt{\rho}}\Phi^{-1}(\frac{2}{1-\hat{\pi}_u} \cdot (\pi_1 - \frac{1-\hat{\pi}_u}{2})) \ . \tag{39}$$

We claim that The solution of V_0 is

$$V_0(\pi_1) = \frac{1}{1-\hat{\pi}_u}[\pi_1\Lambda(\pi_1) + \frac{2}{\rho\Lambda^{'}(\pi_1)}] \ . \tag{40}$$

To check that this is indeed the solution, take derivative on both sides to get

$$V_0^{'}(\pi_1) = \frac{1}{1-\hat{\pi}_u}[\pi_1\Lambda^{'}(\pi_1) + \Lambda(\pi_1) - \frac{2\Lambda^{''}(\pi_1)}{\rho(\Lambda^{'}(\pi_1))^2}] \ . \tag{41}$$

From (36), we get

$$V_0^{'}(\pi_1) = \frac{1}{1-\hat{\pi}_u}\pi_1\Lambda^{'}(\pi_1) \ . \tag{42}$$

Differentiate again and divide both sides by $(\Lambda^{'}(\pi_1))^2$ to get

$$\frac{V_0^{''}(\pi_1)}{(\Lambda^{'}(\pi_1))^2} = \frac{1}{1-\hat{\pi}_u}\left[\frac{\pi_1\Lambda^{''}(\pi_1)}{(\Lambda^{'}(\pi_1))^2} + \frac{1}{\Lambda^{'}(\pi_1)}\right] . \tag{43}$$

Plug the above expression in (36) to obtain

$$\frac{V_0^{''}(\pi_1)}{(\Lambda^{'}(\pi_1))^2} = \frac{1}{1-\hat{\pi}_u}\frac{\rho}{2}\left[\pi_1\Lambda(\pi_1) + \frac{2}{\rho\Lambda^{'}(\pi_1)}\right] . \tag{44}$$

By the definition of V_0, (44) is equivalent to (34). Moreover, when $\pi_1 \to 0$, we have $\pi_1\Lambda(\pi_1) \to 0$ and $\rho\Lambda^{'}(\pi_1) \to \infty$, which confirms $V_0 \to 0$; when $\pi_1 \to 1-\hat{\pi}_u$, we have $\pi_1\Lambda(\pi_1) \to \infty$, which confirms $V_0 \to \infty$. To check V_0 and V_1 are indeed symmetric, note that

$$V_0(1-\pi_1-\hat{\pi}_u) = \frac{1}{1-\hat{\pi}_u}[(1-\pi_1-\hat{\pi}_u)\Lambda(1-\pi_1-\hat{\pi}_u) + \frac{2}{\rho\Lambda^{'}(1-\pi_1-\hat{\pi}_u)}] . \tag{45}$$

The symmetry of Λ suggests $\Lambda(1-\pi_1-\hat{\pi}_u) = -\Lambda(\pi_1)$ and $\Lambda^{'}(1-\pi_1-\hat{\pi}_u) = \Lambda^{'}(\pi_1)$. This together with (45) implies

$$V_0(1-\pi_1-\hat{\pi}_u) = \frac{1}{1-\hat{\pi}_u}[\pi_1\Lambda(\pi_1)+\frac{2}{\rho\Lambda^{'}(\pi_1)}]-\Lambda(\pi_1) = V_0(\pi_1)-\Lambda(\pi_1) = V_1(\pi_1) . \tag{46}$$

Finally, we need to pin down $p(\pi_1)$. Plug (33) into (29) and (30), we get

$$p(\pi_1) = \frac{V_0^{'}(\pi_1)}{\Lambda^{'}(\pi_1)} , \tag{47}$$

which, together with (42), further implies

$$p(\pi_1) = \frac{\pi_1}{1-\hat{\pi}_u} . \tag{48}$$

It is straightforward to see that $p(\frac{1-\hat{\pi}_u}{2}) = \frac{1}{2}$, $p(0) = 0$ and $p(1-\hat{\pi}_u) = 1$. The above checking procedure confirms that the solutions of V_0, V_1 and p are indeed the value functions for the insider and the mixing probability for the market maker. Insiders equilibrium strategy θ_0 and θ_1 can be easily pinned down from $\phi = 0$ and $\lambda_1 = \frac{1}{\Lambda^{'}(\pi_1)}$, and are omitted here.

The proof of the uniqueness follows a similar argument in Anderson and Smith (2013). Define $V^*(\theta_0, \theta_1, p; \theta_0^M, \theta_1^M)$ to be the ex ante expected payoff for the potentially existing insider given the strategy tuple (θ_0, θ_1, p) and the market maker's anticipation of her trading rates (θ_0^M, θ_1^M). Suppose there are two equilibria characterized by $(\theta_0, \theta_1, p; \theta_0, \theta_1)$ and $(\bar{\theta}_0, \bar{\theta}_1, \bar{p}; \bar{\theta}_0, \bar{\theta}_1)$, respectively. From the market maker's perspective, given the insider's strategy, his equilibrium strategy should weakly dominate all other strategies, so

$$-V^*(\theta_0, \theta_1, p; \theta_0, \theta_1) \geq -V^*(\theta_0, \theta_1, \bar{p}; \theta_0, \theta_1) \ . \tag{49}$$

Similarly, from the insider's perspective,

$$V^*(\bar{\theta}_0, \bar{\theta}_1, \bar{p}; \bar{\theta}_0, \bar{\theta}_1) \geq V^*(\theta_0, \theta_1, \bar{p}; \bar{\theta}_0, \bar{\theta}_1) \ . \tag{50}$$

As argued in 4.2, it is sufficient for the market maker to maximize his instantaneous payoff, which is maximized only when he has a correct anticipation of the insider's trading rates. This implies

$$-V^*(\theta_0, \theta_1, \bar{p}; \theta_0, \theta_1) \geq -V^*(\theta_0, \theta_1, \bar{p}; \bar{\theta}_0, \bar{\theta}_1) \ . \tag{51}$$

Inequalities (49), (50) and (51) together imply that

$$V^*(\bar{\theta}_0, \bar{\theta}_1, \bar{p}; \bar{\theta}_0, \bar{\theta}_1) \geq V^*(\theta_0, \theta_1, p; \theta_0, \theta_1) \ .$$

Similar argument suggests that

$$V^*(\theta_0, \theta_1, p; \theta_0, \theta_1) \leq V^*(\bar{\theta}_0, \bar{\theta}_1, \bar{p}; \bar{\theta}_0, \bar{\theta}_1) \ .$$

The above two inequalities imply that

$$V^*(\theta_0, \theta_1, p; \theta_0, \theta_1) = V^*(\bar{\theta}_0, \bar{\theta}_1, \bar{p}; \bar{\theta}_0, \bar{\theta}_1) \ .$$

Therefore, the two equilibria are payoff-equivalent and the equilibrium we obtained is payoff-unique. ■

Proof of Proposition 6

Proof. We prove by contradiction. Suppose there exists an equilibrium with this feature. Again, Lemma **4** suggests that the belief pair (π_0, π_1) always moves along a -45-degree line through the initial state $(\hat{\pi}_0, \hat{\pi}_1)$, and we can use π_1, the probability that the insider exists and the true value is 1, as the state variable. Since the market maker's belief with respect to the insider's existence never changes, π_1 takes values in $(0, 1-\hat{\pi}_u)$, and $\lim_{\pi_1 \to 0} p(\pi_1) = \hat{\pi}_u \hat{p} > 0$ and $\lim_{\pi_1 \to 1-\hat{\pi}_u} p(\pi_1) = 1 - \hat{\pi}_u + \hat{\pi}_u \hat{p} < 1$. Then, the boundary conditions for V_1 are $\lim_{\pi_1 \to 0} V_1(\pi_1) = +\infty$ and $\lim_{\pi_1 \to 1-\hat{\pi}_u} V_1(\pi_1) = +\infty$. Similarly, the boundary conditions for V_0 are $\lim_{\pi_1 \to 0} V_0(\pi_1) = +\infty$ and $\lim_{\pi_1 \to 1-\hat{\pi}_u} V_0(\pi_1) = +\infty$. Follow the similar check procedure as in Back and Baruch (2004) to see that these boundary conditions together with the HJB equations do not admit finite V_0 and V_1 for $\pi_1 \in (0, 1-\hat{\pi}_u)$. ■