Liquidity and Risk Management: Coordinating Investment and Compensation Policies*

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Abstract

We study the corporate finance implications of risky inalienable human capital for a risk-averse entrepreneur. We show how liquidity and risk management policies coordinate investment and executive compensation policies to efficiently retain managerial talent and honor corporate liabilities. The firm optimally balances the goal of attaining mean-variance efficiency for the entrepreneur's net worth and that of preserving financial slack. The former is the main consideration when the firm is flush with liquidity and the latter is the only consideration when the firm has depleted its financial slack. We show that relative to the first-best, the entrepreneur's net worth is over-exposed to idiosyncratic risk and under-exposed to systematic risk. These distortions are greater the more financially constrained the firm is.

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1 Introduction

The general problem addressed in this paper is how firms' financing policies are affected by inalienability of human capital, or what is also commonly referred to as *key-man risk.* This term describes investors' general concern with the possibility that key talent could at any time leave the firm, significantly reducing its value. A firm's ability to retain talent is obviously driven by its capacity to offer adequate present and future state-contingent compensation to its employees. Our main contribution is to show how this key-man risk problem has critical implications for the firm's liquidity and risk management policies. The more liquidity or spare borrowing capacity the firm has the greater is the credibility of its future compensation promises. In addition, by managing the firm's exposures to idiosyncratic and aggregate risk the firm can reduce the cost of retaining talent.

In sum, our paper offers a new theory of corporate liquidity and risk management based on key-man risk. Even when there are no capital market frictions, corporations add value by optimally managing risk and liquidity because this allows them to reduce the cost of key-man risk to investors. This rationale for corporate risk and liquidity management is particularly relevant for technology firms where key-man risk is acute.

The main building blocks of our model are as follows. We consider the problem of a risk-averse entrepreneur, who cannot irrevocably commit her human capital to the firm. The entrepreneur has constant relative risk-averse preferences and seeks to smooth consumption. The firm's operations are exposed to both idiosyncratic and aggregate risk. The firm's capital is illiquid and is exposed to stochastic depreciation. It can be accumulated through investments that are subject to adjustment costs. The entrepreneur faces risk with respect to both the firm's performance and her outside options, which are more valuable the larger is the firm's capital stock. To best retain the entrepreneur, the firm optimally compensates her by smoothing her consumption and limiting her risk exposure. But to be able to do so the firm must engage in liquidity and risk management. The firm's optimized balance sheet is composed of illiquid capital, K, and cash or marketable securities, S, on the asset side. The liability side is composed of equity and a line of credit, with a limit that depends on the entrepreneur's outside option.

The solution of this problem has the following key elements. The entrepreneur manages the firm's risk by choosing optimal loadings on the idiosyncratic and market risk factors. The firm's liquidity is augmented through retained earnings from operations and through returns from its portfolio of marketable securities, including its hedging and insurance positions. The unique state variable is the firm's liquidity-to-capital ratio $s = S/K$. When liquidity is abundant (s is large) the firm is essentially unconstrained and can choose its policies to maximize its market value (or equivalently the entrepreneur's net worth.) The firm's investment policy then approaches the Hayashi (1982) risk-adjusted first-best benchmark, and its consumption and asset allocations approach the generalized Merton (1971) consumption and mean-variance portfolio choices. In particular, the firm then completely insulates its market value from idiosyncratic risk and retains no net idiosyncratic risk exposure for the entrepreneur's net worth.

In contrast, when the firm exhausts its credit limit, its objective essentially becomes maximizing survival by preserving liquidity s and eliminating the volatility of s at the endogenously determined debt limit $\underline{s}$. As one would expect, preserving liquidity requires cutting investment and consumption, engaging in asset sales, and lowering the systematic risk exposure of the entrepreneur's net worth. More surprisingly, preserving financial slack also involves retaining a significant net worth exposure to idiosyncratic risk. That is, relative to the first-best, the entrepreneur's net worth is over-exposed to idiosyncratic risk and under-exposed to systematic risk.

In short, the risk management problem for the firm boils down to a compromise between achieving mean-variance efficiency for the entrepreneur's net worth and preserving the firm's financial slack. The latter is the dominant consideration when liquidity s is low.

The first model to consider the corporate finance consequences of inalienable human capital is Hart and Moore (1994). They propose a theory of debt as an optimal financial contract between a firm seeking financing for a single project with a finite horizon and no cash-flow uncertainty and outside investors. Both the entrepreneur and investors are assumed to have linear utility functions. We generalize the Hart and Moore model in several important directions. It is helpful to consider in turn our two main generalizations to better understand which assumptions underpin our key results.

Our first generalization is to consider an infinitely-lived firm, with ongoing investment subject to adjustment costs, and an entrepreneur with a strictly concave utility function. The firm's financing constraint is always binding in Hart and Moore (1994), but in our model the financing constraint is generically non-binding. Because it is optimal to smooth investment and consumption, the firm does not want to run through its stock of liquidity in one go. This naturally gives rise to a theory of liquidity management even when there is no uncertainty. We describe this special case in Section 8.

Our second generalization is to introduce both idiosyncratic and aggregate risk, which leads to a theory of corporate risk management that ties together classical intertemporal asset pricing and portfolio choice theory with corporate liquidity demand. The distiction between diversifiable and undiversifiable risk is only meaningful if investors are risk averse. Investors set the market price of risk, which the entrepreneur takes as given to determine the firm's optimal risk exposures. All in all, by generalizing the Hart and Moore model to include ongoing investment, consumption smoothing, uncertainty, and risk aversion for both entrepreneur and investors, we are able to show that inalienability of human capital not only gives rise to a theory of debt capacity, but also a theory of liquidity and risk management.

Corporate risk management in our analysis is not about achieving an optimal risk-return profile for investors, they can do that on their own, but about offering optimal risk-return profiles to risk-averse, under-diversified, key employees (the entrepreneur in our setting) under an inalienability of human capital constraint. In our setup the firm is, in effect, both the employer and the asset manager for its key employees. This perspective on corporate risk management is consistent with Duchin et al. (2016), who find that non-financial firms invest 40% of their liquid savings in risky financial assets. More strikingly, they find that the less constrained firms invest more in the market portfolio, which is consistent with our predictions. In addition, when firms are severely financially constrained, we show that they cut compensation, reduce corporate investment, engage in asset sales, and reduce hedging positions, with the primary objective of surviving by honoring liabilities and retaining key employees. These latter predictions are in line with the findings of Rampini, Sufi, and Viswanathan (2014), Brown and Matsa (2016), and Donangelo (2016).

Furthermore, corporate liquidity management in our model is not about avoiding costly external financing, but about compensation smoothing, which requires in particular maintaining liquidity buffers in low productivity states. This motive is so strong that it generally outweighs the countervailing investment financing motive of Froot, Scharfstein, and Stein (1993), which prescribes building liquidity buffers in high productivity states, where investment opportunities are good. If the firm has been unable to build a sufficient liquidity buffer in the low productivity state, we show that it is optimal for the entrepreneur to take a pay cut, consistent with the evidence on executive compensation and corporate cash holdings (e.g. Ganor, 2013). It is possible for the firm to impose a pay cut because in a low productivity state the entrepreneur's outside options are also worth less. Most remarkably, it is also optimal to sell insurance in a low productivity state to generate valuable liquidity. While asset sales in response to a negative productivity shock (also optimal in our setting) are commonly emphasized (Campello, Giambona, Graham, and Harvey, 2011), our analysis can further explain why it maybe optimal to sell insurance and moderate pay in low productivity states.

Our model is particularly relevant for human-capital intensive, high-tech, firms. These firms often hold substantial cash pools. We explain that these pools may be necessary to make future compensation promises credible and thereby retain highly valued employees who naturally have attractive alternative job opportunities. Indeed, employees in these firms are largely paid in the form of deferred stock compensation. When their stock options vest and are exercised, the companies generally engage in stock repurchases so as to avoid excessive stock dilution. But such repurchase programs require funding, which could partly explain why these companies hold such large liquidity buffers.

The firm's optimal liquidity and risk management problem can also be reformulated as a dual optimal contracting problem between an investor and an entrepreneur with inalienable human capital. The dual problem is, in other words, the equivalent contracting-based planning problem that corresponds to the decentralized complete financial markets problem that the entrepreneurial firm faces under inalienability of human capital. More concretely, in the contracting problem the state variable is the certainty-equivalent wealth that the investor promises to the entrepreneur per unit of capital, w, and the investor's value is $p(w)$.

As Table 1 below summarizes, this dual contracting problem is equivalent to the entrepreneur's liquidity and risk management problem with $s = -p(w)$ and the entrepreneur's certainty-equivalent wealth $m(s) = w$. The key observation here is that the firm's endogenously determined credit limit is the outcome of an optimal financial contracting problem. In other words, the firm's financial constraint is the optimal credit limit that reflects the entrepreneur's inability to irrevocably commit her human capital to the firm.

Table 1: Equivalence: Primal optimization and dual contracting problems

	Primal Optimization	Dual Contracting
State Variable	s	w
Value Function	$m(s)$	$p(w)$

Ai and Li (2015) consider a closely related contracting problem. They characterize optimal CEO compensation and corporate investment under limited commitment, but they do not consider the implementation of the contract through corporate liquidity and risk management policies. Their formulation differs from ours in two important respects. First, they assume that investors are risk neutral, so that they cannot make a meaningful distinction between idiosyncratic and aggregate risk. Second, their limited-commitment assumption does not take the form of an inalienability-of-human-capital constraint. In their setup, the entrepreneur can abscond with the firm's capital. When she does so, she can only continue operating under autarky, while in our setup the entrepreneur offers her human capital to another firm under an optimal contract. These different assumptions are critical and give rise to substantially different predictions, as we show in the body of the paper.

Other Related Literature. Rampini and Viswanathan (2010, 2013) develop a limited-commitment-based theory of risk management that focuses on the tradeoff between exploiting current versus future investment opportunities. If the firm invests today it may exhaust its debt capacity and thereby forego future investment opportunities. If instead the firm foregoes investment and hoards its cash it is in a position to be able to exploit potentially more profitable investment opportunities in the future. The difference between our theory and theirs is mainly due to our assumptions of risk aversion for the entrepreneur and investors, our modeling of limited commitment in the form of risky inalienable human capital, and our assumption of physical capital illiquidity. We focus on a different aspect of corporate liquidity and risk management, namely the management of risky human capital and key-man risk. In particular, we emphasize the benefits of risk management to help smooth consumption of the firm's stakeholders (entrepreneur, managers, key employees).

Lambrecht and Myers (2012) consider an intertemporal model of a firm run by a risk-averse entrepreneur with habit formation and derive the firm's optimal dynamic corporate policies. They show that the firm's optimal payout policy resembles the famous Lintner (1956) payout rule of thumb. Building on Merton's intertemporal portfolio choice framework, Wang, Wang, and Yang (2012) study a risk-averse entrepreneur's optimal consumption-savings, portfolio choice, and capital accumulation decisions when facing uninsurable capital and productivity risks. Unlike Wang, Wang, and Yang (2012), our model features optimal liquidity and risk management policies that arise endogenously from an underlying financial contracting problem.

Our theory has elements in common with the microeconomics literature on contracting under limited commitment following Harris and Holmstrom (1982). They analyze a model of optimal insurance for a risk-averse worker, who is unable to commit to a long-term contract. Berk, Stanton, and Zechner (2010) generalize Harris and Holmstrom (1982) by incorporating capital structure and human capital bankruptcy costs into their setting. In terms of methodology, our paper builds on the dynamic contracting in continuous time work of Holmstrom and Milgrom (1987), Schaettler and Sung (1993), and Sannikov (2008), among others.

Our model is evidently related to the dynamic corporate security design literature in the vein of DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), and DeMarzo and Fishman (2007b).[1] These papers also focus on the implementation of the optimal contracting solution via corporate liquidity (cash and credit line.) Two key differences are: (1) risk aversion and (2) systematic and idiosyncratic risk, which together lead to a theory of the "marketable securities" entry on corporate balance sheets and the firm's off-balance-sheet (zero-NPV) futures and insurance positions. A third difference is the focus on moral hazard, which is different from our focus on inalienability of risky human capital. A fourth difference is our generalization of the q-theory of investment to settings

[1]See also Biais, Mariotti, Rochet, and Villeneuve (2010), and Piskorski and Tchistyi (2010), among others. Biais, Mariotti, and Rochet (2013) and Sannikov (2013) provide recent surveys of this literature. For static security design models, see Townsend (1979) and Gale and Hellwig (1985), Innes (1990), and Holmstrom and Tirole (1997).

with inalienable human capital.[2]

Our theory is also related to the liquidity asset pricing theory of Holmstrom and Tirole (2001). We significantly advance their agenda of developing an asset pricing theory based on corporate liquidity. They consider a three-period model with risk neutral agents, where firms are financially constrained and therefore have higher value when they hold more liquidity. Their assumptions of risk neutrality and no consumption smoothing limit the integration of asset pricing and corporate finance theories.

There is also an extensive macroeconomics literature on limited commitment.[3] Green (1987), Thomas and Worrall (1990), Marcet and Marimon (1992), Kehoe and Levine (1993) and Kocherlakota (1996) are important early contributions on optimal contracting under limited commitment. Alvarez and Jermann (2000) extend the first and second welfare theorems to economies with limited commitment. Our entrepreneur's optimization problem is closely related to the agent's dynamic optimization problem in Alvarez and Jermann (2000). While their focus is on optimal consumption allocation, we focus on both consumption and corporate investment. As with DeMarzo and Sannikov (2006), our continuous-time formulation allows us to provide sharper closed-form solutions for consumption, investment, liquidity and risk management policies, up to an ordinary differential equation (ODE) for the entrepreneur's certainty equivalent wealth $m(s)$.

Albuquerque and Hopenhayn (2004), Quadrini (2004), Clementi and Hopenhayn (2006), and Lorenzoni and Walentin (2007) characterize financing and investment decisions under limited commitment or asymmetric information. Kehoe and Perri (2002) and Albuquerque (2003) analyze the implications of limited commitment for international business cycles and foreign direct investments. Miao and Zhang (2015) develop a duality-based solution method for limited-commitment problems.

Our analysis also contributes to the executive compensation literature, which typically abstracts from financial constraints (see Frydman and Jenter, 2010, and Edmans and Gabaix, 2016, for recent surveys). Our model brings out an important positive link between compensation and corporate liquidity, and helps explain why companies typically cut compensation, investment, and risk exposures when liquidity is tight (See Stulz (1984, 1996), Smith and Stulz (1985), and Tufano (1996) for early work on the link between corporate hedging and executive compensation.)

Finally, our paper is clearly related to the voluminous economics literature on human capital that builds on Ben-Porath (1967) and Becker (1975).

[2]DeMarzo and Fishman (2007a), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo, Fishman, He and Wang (2012) generalize the moral hazard model of DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007b) by adding investment.

[3]See Ljungqvist and Sargent (2004) Part V for a textbook treatment of limited-commitment models.

2 Model

We consider an intertemporal optimization problem faced by a risk-averse entrepreneur, who cannot irrevocably promise to operate the firm indefinitely under all circumstances. This inalienability problem for the entrepreneur results in *endogenous* financial constraints distorting her consumption, savings, capital investment, and exposures to both systematic and idiosyncratic risks. To best highlight the central economic mechanism arising from the inalienability of human capital, we remove all other financial frictions from the model and assume that financial markets are otherwise fully competitive and dynamically complete.

2.1 Production Technology and Preferences

Production and Capital Accumulation. We adopt the capital accumulation specification of Cox, Ingersoll, and Ross (1985) and Jones and Manuelli (2005). The firm's capital stock K evolves according to a controlled Geometric Brownian Motion (GBM) process:

$$dK_t = (I_t - \delta_K K_t)dt + \sigma_K K_t \left(\sqrt{1-\rho^2}d\mathcal{Z}_{h,t} + \rho d\mathcal{Z}_{m,t}\right), \tag{1}$$

where I is the firm's rate of gross investment, $\delta_K \geq 0$ is the expected rate of depreciation, and σ_K is the volatility of the capital depreciation shock. Without loss of generality, we decompose risk into two orthogonal components: an idiosyncratic shock represented by the standard Brownian motion $\mathcal{Z}_h$ and a systematic shock represented by the standard Brownian motion $\mathcal{Z}_m$. The parameter ρ measures the correlation between the firm's capital risk and systematic risk, so that the firm's systematic volatility is equal to $\rho\sigma_K$ and its idiosyncratic volatility is given by

$$\epsilon_K = \sigma_K \sqrt{1-\rho^2}\,. \tag{2}$$

The capital stock includes physical capital as traditionally measured and intangible capital (such as, patents, know-how, brand value, and organizational capital).

Production requires combining the entrepreneur's inalienable human capital with the firm's capital stock K_t, which together yield revenue AK_t. Without the entrepreneur's human capital the capital stock K_t does not generate any cash flows.[4] Investment involves both a direct purchase and an adjustment cost as in the standard q-theory of investment, so that the firm's *free cash flow* (after all capital costs but before consumption) is given by:

$$Y_t = AK_t - I_t - G(I_t, K_t), \tag{3}$$

[4] An implication of our assumptions is that managerial retention is always optimal.

where the price of the investment good is normalized to one and $G(I,K)$ is the standard adjustment cost function. Note that Y_t can take negative values, which simply means that additional financing may be needed to close the gap between contemporaneous revenue, AK_t, and capital expenditures.

We further assume that the adjustment cost $G(I,K)$ is homogeneous of degree one in I and K (a common assumption in the q-theory of investment) and express $G(I,K)$ as follows:

$$G(I,K) = g(i)K, \tag{4}$$

where $i = I/K$ denotes the investment-capital ratio and $g(i)$ is increasing and convex in i. As Hayashi (1982) has shown, given this homogeneity property Tobin's average and marginal q are equal in the first-best benchmark.[5] However, under inalienability of human capital an endogenous wedge between Tobin's average and marginal q will emerge in our model.[6]

Preferences. The infinitely-lived entrepreneur has a standard concave utility function over positive consumption flows $\{C_t; t \geq 0\}$ given by:

$$J_t = \mathbb{E}_t \left[\int_t^{\infty} \zeta e^{-\zeta(v-t)} U(C_v) dv \right], \tag{5}$$

where $\zeta > 0$ is the entrepreneur's subjective discount rate, $\mathbb{E}_t[\,\cdot\,]$ is the time-t conditional expectation, and $U(C)$ takes the standard constant-relative-risk-averse utility (CRRA) form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \tag{6}$$

with $\gamma > 0$ denoting the coefficient of relative risk aversion. We normalize the flow payoff with ζ in (5), so that the utility flow is given by $\zeta U(C)$.[7]

2.2 Complete Financial Markets

We assume that financial markets are perfectly competitive and complete. By using essentially the same argument as in the Black-Merton-Scholes option pricing framework, we can

[5] Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average q. Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

[6] An endogenous wedge between Tobin's average and marginal q also arises in cash-based models such as Bolton, Chen, and Wang (2011) and optimal contracting models such as DeMarzo, Fishman, He, and Wang (2012).

[7] This normalization is convenient in contracting models (see Sannikov, 2008). We can generalize these preferences to allow for a coefficient of relative risk aversion that is different from the inverse of the elasticity of intertemporal substitution, à la Epstein and Zin (1989). Indeed, as Epstein-Zin preferences are homothetic, allowing for such preferences in our model will not increase the dimensionality of the optimization problem.

dynamically complete markets with three long-lived assets (Harrison and Kreps, 1979 and Duffie and Huang, 1985): Given that the firm's production is subject to two shocks, $\mathcal{Z}_h$ and $\mathcal{Z}_m$, financial markets are dynamically complete if the following three non-redundant financial assets can be dynamically and frictionlessly traded:

a. A risk-free asset that pays interest at a constant risk-free rate r;

b. A hedging contract that is perfectly correlated with the idiosyncratic shock $\mathcal{Z}_h$. There is no up-front cost for enter this hedging contract as the risk involved is purely idiosyncratic and thus the counter-party earns no risk premium. The transaction at inception is therefore off-the-balance sheet. The instantaneous payoff for each unit of the contract is $\epsilon_K d\mathcal{Z}_{h,t}$.

c. A stock market portfolio. The incremental return $dR_{m,t}$ of this asset is

$$dR_{m,t} = \mu_m dt + \sigma_m d\mathcal{Z}_{m,t}\,, \tag{7}$$

where μ_m and σ_m are constant drift and volatility parameters. As this risky asset is only subject to the systematic shock we refer to it as the *market portfolio*.

Dynamic and frictionless trading with these three securities implies that the following unique stochastic discount factor (SDF) exists (e.g., Duffie, 2001):

$$\frac{d\mathbb{M}_t}{\mathbb{M}_t} = -rdt - \eta d\mathcal{Z}_{m,t}\,, \tag{8}$$

where $\mathbb{M}_0 = 1$ and η is the Sharpe ratio of the market portfolio given by:

$$\eta = \frac{\mu_m - r}{\sigma_m}\,. \tag{9}$$

Note that the SDF $\mathbb{M}$ follows a geometric Brownian motion with the drift equal to the negative risk-free rate, as required under no-arbitrage. By definition the SDF is only exposed to the systematic shock $\mathcal{Z}_m$. Fully diversified investors will only demand a risk premium for their exposures to systematic shocks.

Dynamic Trading. Let $\{S_t; t \geq 0\}$ denote the entrepreneur's liquid wealth process. When $S_t > 0$, the entrepreneur's savings is positive and when $S_t < 0$, she is a borrower. The entrepreneur continuously allocates S_t between the risk-free asset and the stock market portfolio $\Phi_{m,t}$ whose return is given by (7). Moreover, the entrepreneur chooses a pure idiosyncratic-risk hedging position $\Phi_{h,t}$. Therefore, her liquid wealth S_t evolves as follows:

$$dS_t = (rS_t + Y_t - C_t)dt + \Phi_{h,t}\epsilon_K d\mathcal{Z}_{h,t} + \Phi_{m,t}[(\mu_m - r)dt + \sigma_m d\mathcal{Z}_{m,t}] \,. \tag{10}$$

The first term in (10), $rS_t + Y_t - C_t$, is simply the sum of the interest income rS_t and net operating cash flows, $Y_t - C_t$, the second term, $\Phi_{h,t}\epsilon_K d\mathcal{Z}_{h,t}$, is the exposure to the idiosyncratic shock $\mathcal{Z}_h$, which earns no risk premium, and the third term, $\Phi_{m,t}[(\mu_m - r)dt + \sigma_m d\mathcal{Z}_{m,t}]$, is the excess payoff from the market portfolio.

In the absence of any risk exposure $rS_t + Y_t - C_t$ is simply the rate at which the entrepreneur saves when $S_t \geq 0$ or dissaves. In general, saving all liquid wealth S at the risk-free rate is sub-optimal. By dynamically engaging in risk taking and risk management, through the risk exposures Φ_h and Φ_m, the entrepreneur will do better, as we show next.

2.3 Inalienable Human Capital and Endogenous Debt Capacity

The entrepreneur has an option to walk away at any time from her current firm of size K_t, thereby leaving behind all her liabilities. Her next-best alternative is to manage a firm of size αK_t, where $\alpha \in (0, 1)$ is a constant. That is, under this alternative, her talent creates less value as $\alpha < 1$. Therefore, as long as the current firm's liabilities are not too large the entrepreneur prefers to stay with the firm.[8]

Let $J(K_t, S_t)$ denote the entrepreneur's value function at time t. The inalienability of her human capital gives rise to an endogenous debt capacity, denoted by $\underline{S}_t$, that satisfies:

$$J(K_t, \underline{S}_t) = J(\alpha K_t, 0). \tag{11}$$

That is, $\underline{S}_t$ equates the value for the entrepreneur $J(K_t, \underline{S}_t)$ of remaining with the firm and her outside option value $J(\alpha K_t, 0)$ associated with managing a smaller firm of size αK_t and no liabilities. Given that it is never efficient for the entrepreneur to quit on the equilibrium path, $J(K, S)$ must satisfy the following condition:

$$J(K_t, S_t) \geq J(K_t, \underline{S}_t) \,. \tag{12}$$

We can equivalently express the inalienability constraint given by (11) and (12) as:

$$S_t \geq \underline{S}_t = \underline{S}(K_t) \,, \tag{13}$$

[8]In practice entrepreneurs can sometimes partially commit themselves and lower their outside options by signing non-compete clauses. This possibility can be captured in our model by lowering the parameter α, which relaxes the entrepreneur's inalienability-of-human-capital constraints.

where $\underline{S}(K_t)$ defines the endogenous credit capacity as a function of the capital stock K_t. When $S_t < 0$, the entrepreneur draws on a line of credit (LOC) and services her debt at the risk-free rate r up to $\underline{S}(K_t)$. Note that debt is risk-free because (13) ensures that the entrepreneur does not walk away from the firm in an attempt to evade her debt obligations.

3 Liquidity and Risk Management

In this section we characterize the firm's liquidity and risk management policies.

3.1 Dynamic Programming in the Interior Region

The entrepreneur's *liquid* wealth S and *illiquid* productive capital K play different roles and accordingly both serve as natural state variables. By the standard dynamic programming argument, the solution in the interior region where $S \geq \underline{S}$ is characterized by the Hamilton-Jacobi-Bellman (HJB) equation for $J(K,S)$:

$$\begin{aligned}\zeta J(K,S) = \max_{C,I,\Phi_h,\Phi_m} \; & \zeta U(C) + (rS + \Phi_m(\mu_m - r) + AK - I - G(I,K) - C)J_S(K,S) \\ & + (I - \delta_K K)J_K(K,S) + \frac{\sigma_K^2 K^2}{2} J_{KK}(K,S) \\ & + \left(\epsilon_K^2 \Phi_h + \rho\sigma_K\sigma_m\Phi_m\right) K J_{KS}(K,S) + \frac{(\epsilon_K\Phi_h)^2 + (\sigma_m\Phi_m)^2}{2} J_{SS}(K,S)\,. \quad (14)\end{aligned}$$

The first term on the right side of (14) represents the entrepreneur's normalized flow utility of consumption; the second term (involving $J_S(K,S)$) represents the marginal value of incremental liquidity; the third term (involving $J_K(K,S)$) represents the marginal value of net investment $(I - \delta_K K)$; the last three terms (involving J_{KK}, J_{KS} and J_{SS}) correspond to the changes resulting from idiosyncratic and systematic shocks.

The entrepreneur chooses consumption C, investment I, idiosyncratic-risk hedge Φ_h, and market-portfolio allocation Φ_m, to maximize her lifetime utility. With a concave utility function $U(C)$, optimal consumption is determined by the first-order condition (FOC):

$$\zeta U'(C) = J_S(K,S)\,, \quad (15)$$

which equates the marginal utility of consumption $\zeta U'(C)$ with J_S, the marginal value of liquid savings. The optimality condition for investment I is somewhat less obvious:

$$\left(1 + G_I(I,K)\right) J_S(K,S) = J_K(K,S)\,. \quad (16)$$

It equates: (a.) the marginal cost of investing in illiquid capital, given by the product of the marginal cost of investing $(1+G_I)$ and the marginal value of liquid savings J_S, with (b.) the entrepreneur's marginal value of investing in illiquid capital J_K.

To optimal hedge against idiosyncratic risk Φ_h is given by the FOC:

$$\Phi_h = -\frac{K J_{KS}}{J_{SS}}\,, \tag{17}$$

and the optimal stock-market portfolio allocation Φ_m satisfies the FOC:

$$\Phi_m = -\frac{\eta}{\sigma_m}\frac{J_S}{J_{SS}} - \frac{\rho\sigma_K}{\sigma_m}\frac{K J_{KS}}{J_{SS}}. \tag{18}$$

The first term in (18) is in the spirit of Merton's mean-variance demand and the second term is the hedge against the firm's systematic-risk exposure. Equations (14), (15), (16), (17), and (18) jointly characterize the interior solution of the firm's optimization problem.

The Entrepreneur's Certainty-Equivalent Wealth $M(K,S)$. A key step in our derivation is to establish that the entrepreneur's value function $J(K,S)$ takes the following form:[9]

$$J(K,S) = \frac{(bM(K,S))^{1-\gamma}}{1-\gamma}\,, \tag{19}$$

where $M(K,S)$ is naturally interpreted as the entrepreneur's certainty-equivalent wealth, and where b is the constant:[10]

$$b = \zeta\left[\frac{1}{\gamma} - \frac{1}{\zeta}\left(\frac{1-\gamma}{\gamma}\right)\left(r + \frac{\eta^2}{2\gamma}\right)\right]^{\frac{\gamma}{\gamma-1}}\,. \tag{20}$$

In words, $M(K,S)$ is the dollar amount the entrepreneur would demand to permanently give up her productive human capital and retire as a Merton-style consumer living under complete markets. By linking the entrepreneur's value function $J(K,S)$ to her certainty-equivalent wealth $M(K,S)$ we are able to transform the problem from utility to wealth space.

This transformation is conceptually important, as it allows us to measure payoffs in dollars and thereby to make the economics of the entrepreneur's problem more explicit. In particular, it is only possible to determine the marginal dollar-value of liquidity, $M_S(K,S)$, after making the transformation from $J(K,S)$ to $M(K,S)$. As we will show, the economic-

[9]Our conjecture is guided by the twin observations that: i) the value function for the standard Merton portfolio-choice problem (without illiquid assets) inherits the CRRA form of the agent's utility function $U(\cdot)$ and, ii) the entrepreneur's problem is homogeneous in S and K.

[10]We infer the value of b from the solution of Merton (1971)'s closely related consumption and portfolio choice problem under complete markets. Note also that for the special case where $\gamma = 1$ we have $b = \zeta \exp\left[\frac{1}{\zeta}\left(r + \frac{\eta^2}{2} - \zeta\right)\right]$.

s of the entrepreneur's problem and the solution of the entrepreneur's liquidity and risk management problem are closely tied to the marginal dollar-value of liquidity $M_S(K,S)$.

Simplifying the Problem by Using the Homogeneity with Respect to K. An additional simplifying step is to exploit the model's homogeneity property with respect to K to reduce the entrepreneur's problem to one dimension and express all control variables per unit of capital. We denote the per-unit-of-capital solution with the following lower-case variables: consumption $c_t = C_t/K_t$, investment $i_t = I_t/K_t$, liquidity $s_t = S_t/K_t$, idiosyncratic-risk hedge $\phi_{h,t} = \Phi_{h,t}/K_t$, and market-portfolio position $\phi_{m,t} = \Phi_{m,t}/K_t$. We also express the entrepreneur's certainty equivalent wealth $M(K,S)$ as follows:

$$M(K,S) = m(s) \cdot K. \tag{21}$$

Endogenous Effective Risk Aversion γ_e. To better interpret our solution it is helpful to introduce the following measure of endogenous relative risk aversion for the entrepreneur, denoted by $\gamma_e(s)$ and defined as follows:

$$\gamma_e \equiv -\frac{J_{SS}}{J_S} \times M(K,S) = \gamma m'(s) - \frac{m(s)m''(s)}{m'(s)}\,. \tag{22}$$

In (22) the first identity sign gives the definition of γ_e and the second equality follows from homogeneity in K. What economic insights does γ_e capture and why do we introduce γ_e? First, inalienability of human capital results in a form of endogenous market incompleteness. Therefore, the entrepreneur's *effective* risk aversion is captured by the curvature of her value function $J(K,S)$ rather than her utility function $U(\,\cdot\,)$. We can characterize the entrepreneur's coefficient of endogenous absolute risk aversion with the standard definition: $-J_{SS}/J_S$. But how do we link this absolute risk aversion measure to a relative risk aversion measure? We need to multiply absolute risk aversion $-J_{SS}/J_S$ with an appropriate measure of the entrepreneur's wealth. There is no well-defined market measure of the entrepreneur's total wealth under inalienability. However, the entrepreneur's certainty equivalent wealth $M(K,S)$ is a natural measure. This motivates our definition of γ_e in (22).[11] We will show that the inalienability of human capital causes the entrepreneur to be under-diversified and hence effectively more risk averse, so that $\gamma_e(s) > \gamma$.[12] The second equality in (22) confirms this intuition, as her certainty equivalent wealth $m(s)$ is concave in s with $m'(s) > 1$, which we establish below.

[11] See Wang, Wang, and Yang (2012) for a similar definition in a different setting where markets are exogenously incomplete.

[12] We will establish that under the first-best we have $\gamma_e(s) = \gamma$.

3.2 Optimal Policy Rules

Substituting $J(K,S) = \frac{(bm(s)\cdot K)^{1-\gamma}}{1-\gamma}$ into the optimality conditions (15), (16), (17), and (18), we obtain the following policy functions in terms of the liquidity ratio s.

Consumption C_t and Corporate Investment I_t. The consumption policy is given by:

$$c(s) = \chi m'(s)^{-1/\gamma} m(s)\,, \tag{23}$$

where $\chi = \zeta^{\frac{1}{\gamma}} b^{\frac{\gamma-1}{\gamma}}$ denotes the marginal propensity to consume (MPC) under the first-best. Under inalienability, consumption is nonlinear and depends on both the entrepreneur's wealth $m(s)$ and the marginal value of liquidity $m'(s)$. Similarly, investment $i(s)$ is given by

$$1 + g'(i(s)) = \frac{m(s)}{m'(s)} - s\,, \tag{24}$$

which also depends on $m(s)$ and $m'(s)$.

Idiosyncratic Risk Hedge $\Phi_{h,t}$ and Market Portfolio Allocation $\Phi_{m,t}$. Simplifying (17) and (18) gives in turn the following optimal idiosyncratic risk hedge $\phi_h(s)$:

$$\phi_h(s) = -\left(\frac{\gamma\, m(s)}{\gamma_e(s)} - s\right), \tag{25}$$

and the optimal market portfolio allocation $\phi_m(s)$:

$$\phi_m(s) = \frac{\eta}{\sigma_m}\frac{m(s)}{\gamma_e(s)} - \beta^{FB}\left(\frac{\gamma\, m(s)}{\gamma_e(s)} - s\right), \tag{26}$$

where $\gamma_e(\,\cdot\,)$ is the entrepreneur's effective risk aversion given by (22). The first term in (26) reflects the mean-variance demand for the market portfolio, which differs from the standard Merton model in two ways: (1.) risk aversion γ is replaced by the effective risk aversion $\gamma_e(s)$ and (2.) net worth is replaced by certainty equivalent wealth $m(s)$. The second term in (26) captures the hedge with respect to systematic risk $\mathcal{Z}_m$.

3.3 Dynamics of the Liquidity Ratio $\{s_t : t \geq 0\}$

Using Ito's formula, we can show that the liquidity ratio s_t follows:

$$ds_t = d(S_t/K_t) = \mu^s(s_t)dt + \sigma_h^s(s_t)d\mathcal{Z}_{h,t} + \sigma_m^s(s_t)d\mathcal{Z}_{m,t}\,, \tag{27}$$

where the endogenous idiosyncratic volatility of scaled liquidity s_t, $\sigma_h^s(\,\cdot\,)$, and the endogenous systematic volatility of s_t, $\sigma_m^s(\,\cdot\,)$, are respectively given by:

$$\sigma_h^s(s) \;=\; -\epsilon_K \frac{\gamma}{\gamma_e(s)}\, m(s)\;, \tag{28}$$

$$\sigma_m^s(s) \;=\; \left(\frac{\eta}{\gamma} - \rho\sigma_K\right) \frac{\gamma}{\gamma_e(s)}\, m(s)\,. \tag{29}$$

The systematic volatility $\sigma_m^s(s)$ and the idiosyncratic volatility $\sigma_h^s(s)$ are perfectly correlated, as they are both proportional to $\gamma m(s)/\gamma_e(s)$.[13] This property is very helpful when we determine the endogenous debt limit $\underline{s}$. The drift $\mu^s(\,\cdot\,)$ of s_t is given by:

$$\begin{aligned}\mu^s(s_t) \;=\;& A - i(s_t) - g(i(s_t)) + \phi_m(s_t)(\mu_m - r) - c(s_t)\\ &+(r + \delta_K - i(s_t))s_t - (\epsilon_K \sigma_h^s(s_t) + \rho\sigma_K \sigma_m^s(s_t)),\end{aligned} \tag{30}$$

where all the terms in the first line of (30) derive from the drift of S_t, the term, $(r + \delta_K - i(s_t))s_t$, derives from the drift of K_t, and the remaining term, $-(\epsilon_K \sigma_h^s(s_t) + \rho\sigma_K \sigma_m^s(s_t))$, is due to the quadratic covariation between S_t and K_t.

3.4 The Endogenous Credit Limit

In the interior region the credit constraint

$$s_t \geq \underline{s} \tag{31}$$

does not bind. As in the household buffer-stock savings literature (e.g., Deaton (1991) and Carroll (1997)), the *risk-averse* entrepreneur manages her liquid holdings s with the objective of smoothing her consumption. Setting $s_t = \underline{s}$ for all t is too costly and suboptimal in terms of consumption smoothing. Although the credit constraint (31) rarely binds, it has to be satisfied with probability one. Only then can we ensure that the entrepreneur always stays with the firm. Given that $\{s_t : t \geq 0\}$ is a diffusion process and hence is continuous, both the idiosyncratic and systematic volatility at $\underline{s}$ must equal zero:

$$\sigma_h^s(\underline{s}) = 0 \;\text{ and }\; \sigma_m^s(\underline{s}) = 0\,. \tag{32}$$

Otherwise, the probability of crossing a candidate debt limit of $\underline{s}$ to its left is strictly positive, violating the credit constraint (31). From (28) and (29) it is straightforward to see that (32)

[13] When $\eta/\gamma = \rho\sigma_K$, the mean-variance demand and the hedging demand exactly offset each other giving $\sigma_m^s(s) = 0$. To avoid this degenerate case for systematic risk exposure, we require $\eta/\gamma \neq \rho\sigma_K$.

is equivalent to:

$$\frac{m(\underline{s})}{\gamma_e(\underline{s})} = 0\,. \tag{33}$$

In other words, at the endogenously determined $\underline{s}$, either $m(\underline{s}) = 0$ or $\gamma_e(\underline{s}) = \infty$.[14] As we will show, under the first-best solution we have $m(\underline{s}^{FB}) = 0$ and $\underline{s}^{FB} = -q^{FB}$. But with inalienable human capital we have $m(\underline{s}) > 0$, so that it must be the case that $\gamma_e(\underline{s}) = \infty$. That is, the entrepreneur's effective risk aversion $\gamma_e(s)$ approaches ∞ when she runs out of liquidity, which is equivalent to $m''(\underline{s}) = -\infty$.

3.5 ODE for $m(s)$

Substituting the policy rules for c, i, ϕ_h, and ϕ_m and the value function (19) into the HJB equation (14) and using the homogeneity property, we obtain the following ODE for $m(s)$:

$$\begin{aligned} 0 \;=\; & \frac{m(s)}{1-\gamma}\left[\gamma\chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta\right] + \left[rs + A - i(s) - g(i(s))\right] m'(s) + (i(s)-\delta)(m(s) - sm'(s)) \\ & + \left(\frac{\gamma\sigma_K^2}{2} - \rho\eta\sigma_K\right)\frac{m(s)^2 m''(s)}{\gamma_e(s)\, m'(s)} + \frac{\eta^2 m'(s) m(s)}{2\gamma_e(s)}\,, \end{aligned} \tag{34}$$

where δ is the risk-adjusted depreciation rate: $\delta = \delta_K + \rho\eta\sigma_K$.[15]

To summarize, the optimal policy functions for c, i, ϕ_h, and ϕ_m and the ODE for $m(s)$ (34) describe both the solutions for the inalienability of human capital problem and the first-best problem. The only difference between the two problems is reflected in the endogenous credit limit $\underline{s}$, which is given by the condition $m(\underline{s}) = 0$ under the first-best problem, and by $\lim_{s\to\underline{s}} \gamma_e(s) = \infty$ under inalienability.

3.6 First Best

Under the first-best, we conjecture and verify that the entrepreneur's net worth is given by

$$M_t^{FB} = S_t + Q_t^{FB} = (s_t + q^{FB})K_t = m^{FB}(s_t)K_t, \tag{35}$$

where $Q_t^{FB} = q^{FB}K_t$ is the market value of capital, and q^{FB} is the endogenously determined Tobin q. As net worth must be positive at all times, we must require $s \geq -q^{FB}$, which implies that the first-best debt capacity is q^{FB} per unit of capital: $\underline{s}^{FB} = -q^{FB}$.

By granting the entrepreneur a credit line up to q^{FB} per unit of capital at the risk-free rate r, the entrepreneur can achieve first-best consumption smoothing and investment,

[14] We verify that the drift $\mu^s(s)$ given in (30) is non-negative at $\underline{s}$, so that s is weakly increasing at $\underline{s}$.
[15] Footnote 16 further elaborates on this standard risk adjustment.

attaining the maximal value of capital at $q^{FB}K_t$ and the maximal net worth $m^{FB}(s)$ given in (35). In a first-best world, the certainty-equivalent wealth coincides with the mark-to-market valuation of net worth.

Substituting (35) into the FOC for consumption (23) yields the following $c^{FB}(s_t)$:

$$c^{FB}(s) = \chi m^{FB}(s) = \chi \left(s + q^{FB} \right) , \tag{36}$$

where χ is the marginal propensity to consume (MPC) under the first-best given by

$$\chi = r + \frac{\eta^2}{2\gamma} + \gamma^{-1} \left(\zeta - r - \frac{\eta^2}{2\gamma} \right) , \tag{37}$$

as in Merton (1971).

Substituting (35) into the FOC for investment (24) yields the following i^{FB}:

$$q^{FB} = 1 + g'(i^{FB}), \tag{38}$$

which equates Tobin's q to the marginal cost of investing, $1 + g'(i)$. As in the q-theory of investment, adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that $q^{FB} \neq 1$. We can show that q^{FB} also satisfies the following formula:

$$q^{FB} = \max_i \quad \frac{A - i - g(i)}{r_K - (i - \delta_K)} , \tag{39}$$

where $r_K = r + \rho\eta\sigma_K$. Equation (39) is simply the Gordon growth formula with endogenously determined i^{FB}. The numerator is the free-cash flow and the denominator is given by the difference between the growth rate $(i^{FB} - \delta_K)$ and the cost of capital r_K. We can further write $r_K = r + \beta^{FB} \times (\mu_m - r)$, where β^{FB} is the CAPM β given by

$$\beta^{FB} = \frac{\rho\sigma_K}{\sigma_m} . \tag{40}$$

We can equivalently write the formula (39) as follows:

$$q^{FB} = \max_i \quad \frac{A - i - g(i)}{r - (i - \delta)} . \tag{41}$$

That is, (41) is the Gordon growth formula under the risk-neutral measure.[16] Note that the production side of our model generalizes the Hayashi (1982) model to situations where

[16]By that we mean that δ is the capital depreciation rate under the risk-neutral measure: The gap $\delta - \delta_K$ is equal to the risk premium $\rho\eta\sigma_K$ for capital shocks. The two Gordon growth formulae (39) and (41) are equivalent: The CAPM, implied by no arbitrage and the unique SDF given in (8), connects the two formulae under the physical and the risk-neutral measures.

a firm faces both idiosyncratic and systematic risk, and where systematic risk commands a risk premium.

Next, substituting the net worth given in (35) into (25), the FOC for $\phi_h^{FB}(s_t)$, yields:

$$\phi_h^{FB}(s) = -q^{FB} . \tag{42}$$

The entrepreneur optimally chooses to completely neutralize her idiosyncratic risk exposure (due to her long position in the business venture) by going short and setting $\phi_h^{FB}(s) = -q^{FB}$, leaving her net worth M^{FB} with a zero net exposure to idiosyncratic risk $\mathcal{Z}_h$.

Similarly, substituting the net worth given in (35) into (26), the FOC for $\phi_m^{FB}(s_t)$, yields:

$$\phi_m^{FB}(s) = -\beta^{FB} q^{FB} + \frac{\eta}{\gamma\sigma_m} m^{FB}(s) . \tag{43}$$

The first term in (43), $-\beta^{FB} q^{FB}$, fully offsets the entrepreneur's exposure to the aggregate shock through the firm's operations, and the second term achieves the target mean-variance aggregate risk exposure for her net worth M^{FB}. As in Merton (1971), the entrepreneur's net worth then follows the process:

$$dM_t^{FB} = M_t^{FB} \left[\left(r - \chi + \frac{\eta^2}{\gamma} \right) dt + \frac{\eta}{\gamma} d\mathcal{Z}_{m,t} \right] , \tag{44}$$

which is a GBM process. Note the zero net exposure of net worth to idiosyncratic risk $\mathcal{Z}_{h,t}$.

3.7 Inalienable Human Capital

At the credit limit $\underline{S}_t$ the entrepreneur is indifferent between staying with the firm and taking her human capital to be employed elsewhere, as shown in (11). Substituting $J(K, S)$ given in (11) and simplifying, we obtain the following value-matching condition for $m(s)$ at $s = \underline{s}$:

$$m(\underline{s}) = \alpha m(0) . \tag{45}$$

Note that when $\alpha = 0$ the entrepreneur has no outside option, so that $m(\underline{s}) = 0$ and (45) reduces to the boundary condition (33) for the first-best problem. By optimally setting $\underline{s} = -q^{FB}$ we attain the first-best outcome where the entrepreneur can potentially pledge the entire market value of capital. At the other extreme, when $\alpha = 1$, the entrepreneur's outside option is as good as her current employment. No long-term contract can then retain the entrepreneur, so that the model has no solution.

Therefore, in order for the inalienability of human capital problem to have an interesting and non-degenerate solution, we restrict attention to the range $0 < \alpha < 1$. For these values of α, (45) implies that $m(\underline{s}) > 0$.[17] The volatility boundary conditions (33) can then only be satisfied if

$$m''(\underline{s}) = -\infty\,. \tag{46}$$

That is, the inalienability condition (45) implies that the curvature of the certainty-equivalent wealth function approaches infinity at the endogenous boundary $s = \underline{s}$. We summarize the solution for the inalienability case in the theorem below.

Theorem 1 *When $0 < \alpha < 1$, the solution to the inalienability problem is such that $m(s)$ solves the ODE (34) subject to the FOCs (23) for consumption, (24) for investment, (25) for idiosyncratic risk hedge ϕ_h, (26) for market portfolio allocation ϕ_m, and the boundary conditions (45) and (46) at the endogenously determined $\underline{s}$.*

4 Equivalent Optimal Contract

We consider next the long-term contracting problem between an infinitely-lived, fully diversified, investor (*the principal*) and an infinitely-lived, financially constrained, risk-averse, entrepreneur (*the agent*). The output process Y_t is publicly observable and verifiable. In addition, the entrepreneur cannot privately save.[18] The contract specifies an investment $\{I_t; t \geq 0\}$ and compensation $\{C_t; t \geq 0\}$ policy, both of which depend on the entire history of idiosyncratic and aggregate shocks $\{\mathcal{Z}_{h,t},\ \mathcal{Z}_{m,t};\ t \geq 0\}$.

Because the risk-averse investor is fully diversified and markets are complete, the investor chooses investment $\{I_t; t \geq 0\}$ and compensation $\{C_t; t \geq 0\}$ to maximize the risk-adjusted discounted value of future cash flows net of the agent's compensation:

$$F(K_0, V_0) = \max_{C,\,I}\ \mathbb{E}_0 \left[\int_0^\infty \mathbb{M}_t (Y_t - C_t) dt \right] , \tag{47}$$

where K_0 is the initial capital stock and V_0 is the entrepreneur's reservation utility at time 0. Given that the investor is fully diversified, we use the same SDF $\mathbb{M}$ given in (8) to evaluate the present value of cash flows $(Y_t - C_t)$. The contracting problem is subject to the entrepreneur's inalienability constraints at all future dates $t \geq 0$ and the participation constraint at time 0. We denote by $\underline{V}(K_t)$ the entrepreneur's endogenous outside payoff, so that the inalienability constraint at time t is given by:

$$V_t \geq \underline{V}(K_t)\,, \quad t \geq 0, \tag{48}$$

[17] Otherwise $m(0) = m(\underline{s}) = 0$, which does not make economic sense.

[18] This is a standard assumption in the dynamic moral hazard literature (Ch. 10 in Bolton and Dewatripont, 2005). Di Tella and Sannikov (2016) develop a contracting model with hidden savings for asset management.

where V_t is the entrepreneur's promised utility specified under the contract.

4.1 Recursive Formulation

We proceed in three steps to transform the optimal contracting problem into a recursive form: (1.) we define the entrepreneur's promised utility V and the principal's value $F(K,V)$ in recursive form; (2.) we map promised utility V into promised certainty-equivalent wealth W; and (3.) we use homogeneity to reduce the contracting problem to a one-dimensional problem. While Step (1.) is standard in the recursive contracting literature, steps (2) and (3) are less common but are essential to allow us to connect the contracting problem to the liquidity and risk management problem analyzed before.

The investor's value function $F(K,V)$. The dynamics of the entrepreneur's promised utility can be defined in the recursive form:

$$\mathbb{E}_t\left[\zeta U(C_t)dt + dV_t\right] = \zeta V_t dt\,, \tag{49}$$

where $\zeta U(C_t)dt$ is the (normalized) utility of current compensation and dV_t is the change in promised utility. Furthermore, the stochastic differential equation (SDE) for dV implied by (49) can be written as the sum of: *i*) the expected change $\mathbb{E}_t\left[dV_t\right]$ (the drift term); *ii*) a martingale term driven by the Brownian motion $\mathcal{Z}_h$; and *iii*) a martingale term driven by the Brownian motion $\mathcal{Z}_m$:

$$dV_t = \zeta(V_t - U(C_t))dt + z_{h,t}V_t d\mathcal{Z}_{h,t} + z_{m,t}V_t d\mathcal{Z}_{m,t}\,, \tag{50}$$

where $\{z_{h,t}; t \geq 0\}$ and $\{z_{m,t}; t \geq 0\}$ respectively control the idiosyncratic and systematic volatilities of the entrepreneur's promised utility V.

We can then write the investor's value function $F(K,V)$ in terms of: *i*) the entrepreneur's promised utility V; and, *ii*) the venture's capital stock K. The contracting problem specifies investment I, compensation C, idiosyncratic risk exposure z_h and systematic risk exposure z_m to maximize the investor's risk-adjusted discounted value of net cash flows. Applying Ito's Lemma to $F(K,V)$ a recursive formulation for the contracting problem can be obtained, which is given by the following HJB equation for the investor's value $F(K,V)$:

$$\begin{aligned} rF(K,V) = \max_{C,\,I,\,z_h,\,z_m} \; &\Big\{(Y-C) + (I-\delta K)F_K + [\zeta(V-U(C)) - z_m\eta V]F_V \\ &+\frac{\sigma_K^2 K^2 F_{KK}}{2} + \frac{(z_h^2+z_m^2)V^2F_{VV}}{2} + (z_h\epsilon_K + z_m\rho\sigma_K)KVF_{VK}\Big\} \end{aligned} \tag{51}$$

From Promised Utility V_t To Promised Certainty-Equivalent Wealth W_t. To link the optimal contract to the optimal liquidity and risk management policies, we need to express the entrepreneur's promised utility in dollars (units of consumption) rather than in utils. This involves mapping the entrepreneur's promised utility V into promised (certainty-equivalent) wealth W. As before, we define W as the solution to the equation:

$$U(bW) = V \quad \text{and equivalently} \quad W = U^{-1}(V)/b\,, \tag{52}$$

where b is the constant given in (20). We show in the Appendix that the following SDE for W obtains by using the transformation (52) and applying Ito's formula to V_t:

$$\begin{aligned} dW_t &= \frac{1}{V_W}\left[\zeta(V - U(C_t))dt + z_h V d\mathcal{Z}_{h,t} + z_m V_t d\mathcal{Z}_{m,t}\right] - \frac{(z_h^2 + z_m^2)V^2 V_{WW}}{2V_W^3}dt \\ &= \left[\frac{\zeta(V - U(C_t))}{V_W} - \frac{(x_h^2 + x_m^2)K^2 V_{WW}}{2V_W}\right]dt + x_h K_t d\mathcal{Z}_{h,t} + x_m K_t d\mathcal{Z}_{m,t}\,, \end{aligned} \tag{53}$$

where $x_m = \frac{z_m V}{KV_W}$ and $x_h = \frac{z_h V}{KV_W}$. Note that x_h and x_m control the idiosyncratic and systematic volatilities of W_t, respectively. As will become clear, x_h and x_m are closely tied to the firm's optimal risk management policies ϕ_h and ϕ_m analyzed earlier.

Reduction to One Dimension. We can reduce the contracting problem to one dimension, with the scaled promised certainty-equivalent wealth $w = W/K$ as the unique state variable, by writing the investor's value $F(K, V)$ as:

$$F(K, V) \equiv F(K, U(bW)) = P(K, W) = p(w) \cdot K. \tag{54}$$

We then only need to solve for $p(w)$ and characterize the scaled consumption, investment, idiosyncratic risk, and stock market allocation rules as functions of w.

The Principal's Endogenous Risk Aversion γ_p. As with our analysis for the previous optimization problem, it is helpful to introduce an endogenous measure of risk aversion for the principal. Accordingly, let γ_p denote the principal's risk-aversion under the contract:

$$\gamma_p \equiv \frac{WP_{WW}(K, W)}{P_W(K, W} = \frac{wp''(w)}{p'(w)}\,. \tag{55}$$

The identity sign gives the definition of γ_p, and the equality sign follows from the homogeneity property.[19] As w is a liability for the investor we have $p'(w) < 0$. This is why, unlike in the

[19]Here, the subscript p refers to the principal, while the subscript e in γ_e refers to the entrepreneur's endogenous effective risk aversion in the liquidity and risk management problem analyzed earlier.

standard definition of risk aversion, there is no minus sign in (55).

Under the first-best, the investor's value is linear in w, so that $p''(w) = 0$ and the principal's effective risk aversion $\gamma_p^{FB}(w)$ is zero for all w. Under inalienability, we can show that the investor's endogenous risk aversion $\gamma_p(w) > 0$ since $p(w)$ is decreasing and concave.

4.2 Optimal Policy Functions

Substituting (54) into the optimality conditions, we obtain the following policy functions.

Consumption C_t and Corporate Investment I_t. The consumption policy is given by:

$$\mathfrak{c}(w) = \chi \left(-p'(w)\right)^{1/\gamma} w \,, \tag{56}$$

where again χ is the MPC under the first-best solution given in (37). Under inalienability, consumption depends on both w and the investor's marginal value of liquidity $p'(w)$. Similarly, investment $\mathfrak{i}(w)$ depends on $p(w)$ and $p'(w)$ and is given by the following FOC:

$$1 + g'(\mathfrak{i}(w)) = p(w) - wp'(w) \,. \tag{57}$$

Idiosyncratic Risk Exposure $x_h(w)$ and Systematic Risk Exposure $x_m(w)$. Using the principal's endogenous coefficient of risk aversion $\gamma_p(w)$ given in (55) we obtain the following simple and economically transparent expression for the optimal idiosyncratic risk exposure $x_h(w_t)$:

$$x_h(w) = \frac{\gamma_p(w)}{\gamma_p(w) + \gamma}\, \epsilon_K w. \tag{58}$$

Under the first-best, $\gamma_p^{FB}(w) = 0$ so that $x_h^{FB}(w) = 0$ for all $w \geq 0$, implying that the entrepreneur's promised net worth W_t has no net exposure to idiosyncratic risk $\mathcal{Z}_{h,t}$.

We also obtain the following expression for the systematic risk exposure $x_m(w_t)$:

$$x_m(w) = \frac{\eta w}{\gamma_p(w) + \gamma} + \rho \sigma_K w \frac{\gamma_p(w)}{\gamma_p(w) + \gamma} \,, \tag{59}$$

where the first term gives the mean-variance demand and the second term gives the hedging demand. Under the first-best, since $\gamma_p^{FB}(w) = 0$, we have $x_m^{FB}(w) = \eta\, w/\gamma$, which is the standard mean-variance demand for the entrepreneur's net worth W. In contrast, under inalienability we see that $x_h(w)$ given in (58) and $x_m(w)$ given in (59) involve optimal co-insurance between an endogenously risk-averse principal with relative risk aversion $\gamma_p(w)$ and the risk-averse agent.[20]

[20]Note that the coinsurance weight $\frac{\gamma_p(w)}{\gamma_p(w)+\gamma}$ appears in (58) and (59).

4.3 Dynamics of Promised Certainty-Equivalent Wealth w

Applying Ito's formula to $w_t = W_t/K_t$, we obtain the following dynamics for w:

$$dw_t = d\left(W_t/K_t\right) = \mu^w(w_t)dt + \sigma_h^w(w_t)d\mathcal{Z}_{h,t} + \sigma_m^w(w_t)d\mathcal{Z}_{m,t}\,. \tag{60}$$

where the idiosyncratic and systematic volatilities for w, $\sigma_h^w(\,\cdot\,)$ and $\sigma_m^w(\,\cdot\,)$, are given by

$$\sigma_h^w(w) \;=\; -\epsilon_K \frac{\gamma w}{\gamma_p(w)+\gamma} \;<0\,, \tag{61}$$

$$\sigma_m^w(w) \;=\; \left(\frac{\eta}{\gamma} - \rho\sigma_K\right)\frac{\gamma w}{\gamma_p(w)+\gamma}\,. \tag{62}$$

Again, $\sigma_h^w(w)$ and $\sigma_m^w(w)$ are perfectly correlated, as they are both proportional to $w/(\gamma_p(w)+\gamma)$. Finally, the drift function $\mu^w(\,\cdot\,)$ of w_t is given by:

$$\mu^w(w) = \frac{\zeta}{1-\gamma}\left(w + \frac{\mathfrak{c}(w)}{\zeta p'(w)}\right) - w(\mathfrak{i}(w)-\delta_K) + \frac{\gamma(x_h^2(w)+x_m^2(w))}{2w} - (\epsilon_K\sigma_h^w(w) + \rho\sigma_K\sigma_m^w(w))\,. \tag{63}$$

4.4 ODE for $p(w)$

Substituting $F(K,V) = p(w)\cdot K$ into the HJB equation (B.4), solving for $p(w)$, and substituting for the policy functions $\mathfrak{c}(w)$, $\mathfrak{i}(w)$, $x_h(w)$ and $x_m(w)$, we obtain the following ODE for the investor's value $p(w)$:

$$\begin{aligned} rp(w) = {} & A - \mathfrak{i}(w) - g(\mathfrak{i}(w)) + \frac{\chi\gamma}{1-\gamma}\left(-p'(w)\right)^{1/\gamma} w + (\mathfrak{i}(w)-\delta)(p(w)-wp'(w)) \\ & + \frac{\zeta}{1-\gamma} wp'(w) + \left(\frac{\gamma\sigma_K^2}{2} - \rho\eta\sigma_K\right)\frac{w^2 p''(w)}{\gamma_p(w)+\gamma} - \frac{\eta^2}{2}\frac{wp'(w)}{\gamma_p(w)+\gamma}\,, \end{aligned} \tag{64}$$

where $\mathfrak{i}(w)$ is given by (57) and $\gamma_p(w)$ is given by (55). This ODE for $p(w)$ characterizes the interior solution for both the first-best and inalienability cases. The only difference between the two problems is reflected in the inalienability constraint to which we turn next.

4.5 Inalienability Constraint

The entrepreneur's outside option at any time is to manage a new firm with effective size αK_t but with no legacy liabilities. Other than the size of the capital stock K, the new firm's production technology is identical to the one that she has just abandoned. Let $\widetilde{V}(\,\cdot\,)$ and $\widetilde{W}(\,\cdot\,)$ be the entrepreneur's utility and the corresponding certainty-equivalent wealth in this new firm, and suppose as before that investors in the new firm make zero net profits under competitive markets. Then, from (54) we obtain the following condition:

$$F(\alpha K_t, \widetilde{V}(\alpha K_t)) = P(\alpha K_t, \widetilde{W}(\alpha K_t)) = 0\,. \tag{65}$$

When the entrepreneur is indifferent between leaving her employer or not we have

$$\underline{W}(K_t) = \widetilde{W}(\alpha K_t). \tag{66}$$

Dividing by K_t the entrepreneur's indifference condition is:

$$\underline{w}_t \equiv \underline{W}(K_t)/K_t = \widetilde{W}(\alpha K_t)/K_t = \alpha \widetilde{W}(\alpha K_t)/(\alpha K_t) = \alpha \widetilde{w}_t\,, \tag{67}$$

where the second equality follows from the continuity of W in (66), and the third equality follows from the assumption that the new firm's capital is a fraction α of the original firm's capital stock. The homogeneity property and (65) together imply that $p(\widetilde{w}) = 0$. Thus, substituting $\underline{w}_t = \alpha \widetilde{w}_t$ into $p(\widetilde{w}_t) = 0$ we obtain the following simple expression for the inalienability constraint when $0 < \alpha < 1$:

$$p(\underline{w}/\alpha) = 0\,. \tag{68}$$

Note that the entrepreneur's outside option implies that her minimum certainty-equivalent wealth must be positive $\underline{w} > 0$. In the first-best, when $\alpha = 0$, however, the entrepreneur does not have a valuable outside option, so that $\underline{w} = 0$.

In both the first-best and inalienability cases we require that the volatility functions $\sigma_h^w(w)$ and $\sigma_m^w(w)$ are equal zero at $\underline{w}$ to ensure that w never crosses $\underline{w}$ to the left ($w \geq \underline{w}$):

$$\sigma_h^w(\underline{w}) = 0 \;\text{ and }\; \sigma_m^w(\underline{w}) = 0\,. \tag{69}$$

Equations (61) and (62) imply that the volatility conditions given in (69) are equivalent to:

$$\frac{\gamma\, \underline{w}}{\gamma_p(\underline{w}) + \gamma} = 0\,. \tag{70}$$

Equation (70) holds when either $\underline{w} = 0$ (for the first-best case) or $\gamma_p(\underline{w}) = \infty$ (under inalienability), which is equivalent to

$$p''(\underline{w}) = -\infty\,. \tag{71}$$

Again, our contracting analysis reveals that the boundary conditions under inalienability are fundamentally different from those for the first-best: under inalienability $\gamma_p(\underline{w}) = \infty$, while under the first-best $\gamma_p(w) = 0$ for all w. The first-best solution confirms the conventional wisdom for hedging, which calls for the complete elimination of idiosyncratic risk exposures for the risk-averse entrepreneur's net worth. This conventional wisdom applies only to a complete-markets, Arrow-Debreu, world. In general, with financial imperfections such as inalienability, there is no reason to expect this conventional wisdom to hold.

We summarize the contracting solution under inalienability in the theorem below.

Theorem 2 *When $0 < \alpha < 1$, the optimal contract under inalienable human capital is such that $p(w)$ solves the ODE (64) subject to the FOCs (56) for consumption, (57) for investment, (58) for idiosyncratic risk exposure x_h, (59) for systematic risk exposure x_m, and the boundary conditions (68) and (71).*[21]

Finally, to complete the characterization of the optimal contracting solution we set the entrepreneur's initial reservation utility V_0^* such that $F(K_0, V_0^*) = 0$ to be consistent with the general assumption that capital markets are competitive.

4.6 Equivalence

By equivalence, we mean that the resource allocations $\{C_t, I_t; t \geq 0\}$ under the two problem formulations are identical for any path $\{\mathcal{Z}_h, \mathcal{Z}_m\}$. We demonstrate this equivalence in Appendix (B.2), by verifying that the following holds:

$$s = -p(w) \;\text{ and }\; w = m(s)\,, \tag{72}$$

implying that $-p \circ m(s) = s$. In other words, the state variable s in the primal problem is shown to be equal to $-p(w)$, the negative of the value function in the dual contracting problem, and correspondingly the value function $m(s)$ in the primal problem is shown to equal w, the state variable in the contracting problem.

Table 2 provides a detailed side-by-side comparison of the two problem formulations along all three relevant dimensions of the model: (a.) the state variable, (b.) the policy rules, and (c.) the value functions for both inalienability and first-best cases. Panels A, B, and C offer a side-by-side mapping for the state variable, value function, and policy rules under the two formulations. These apply to both inalienability and first-best cases. The differences between the inalienability and first-best cases are entirely driven by the conditions pinning down the firm's borrowing capacity, as we highlight in Panels D and E.

[21] We also require that the drift $\mu^w(w)$ given in (63) is non-negative at $\underline{w}$, so that w is weakly increasing at $\underline{w}$ with probability one.

Panel D describes the conditions characterizing the borrowing capacity for the inalienability case, where $0 < \alpha < 1$. The entrepreneur's inalienability of human capital implies that $m(\underline{s}) = \alpha m(0)$ given in (45) and $p(\underline{w}/\alpha) = 0$ given in (68) have to be satisfied at the respective free boundaries $\underline{s}$ and $\underline{w}$ in the two formulations. Given these inalienability constraints, the volatility conditions can only be satisfied if the curvatures of the value functions, $m(s)$ and $p(w)$, approach $-\infty$ at the left boundaries.

Panel E summarizes the first-best case, where $\alpha = 0$. The investor's value is given by the difference between the market value of capital, q^{FB}, and the promised wealth to the entrepreneur, w_t: $p^{FB}(w_t) = q^{FB} - w_t$. Equivalently, $w_t = m^{FB}(s_t) = s_t + q^{FB}$. The first-best policy rules under the two formulations are thus consistent. The same is true for the optimal consumption rule: $\mathfrak{c}^{FB}(w_t) = \chi w_t = \chi m^{FB}(s_t) = c^{FB}(s_t)$. The investment-capital ratio is also consistent: under both formulations it equals the same constant i^{FB}. The optimal idiosyncratic risk exposure $x_h^{FB}(w) = 0$ shuts down the idiosyncratic risk exposure of W_t, which is equivalent to setting the idiosyncratic risk hedge $\phi_h^{FB}(s) = -q^{FB}$ in the primal formulation, thus eliminating idiosyncratic risk for M_t. The optimal systematic risk exposure $x_m^{FB}(w) = \frac{\eta}{\gamma} w$ yields the aggregate volatility of η/γ for W_t, which is consistent with the fact that $\phi_m^{FB}(s)$ given in (43) implies an aggregate volatility of η/γ for M_t. Last but not the least, the borrowing limits in the two formulations are also consistent, in that $\underline{w}^{FB} = 0$ if and only if $\underline{s}^{FB} = -q^{FB}$, which means that the entrepreneur can at any time t borrow up to the entire market value of capital $q^{FB} K_t$.

5 Quantitative Analysis

In this section, we present our main qualitative and quantitative results. For simplicity, we choose the widely-used quadratic adjustment cost function, $g(i) = \theta i^2/2$, for which we have explicit formulae for Tobin's q and optimal i under the first-best:[22]

$$q^{FB} = 1 + \theta i^{FB}, \text{ and } i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - 2\frac{A - (r+\delta)}{\theta}}. \tag{73}$$

Our model is relatively parsimonious with eleven parameters. We set the entrepreneur's coefficient of relative risk aversion to $\gamma = 2$, the equity risk premium $(\mu_m - r)$ to 6%, and the annual volatility of the market portfolio return to $\sigma_m = 20\%$, implying a Sharpe ratio of $\eta = (\mu_m - r)/\sigma_m = 30\%$. We choose the annual risk-free rate to be $r = 5\%$ and set the entrepreneur's discount rate $\zeta = r = 5\%$. These are standard parameter values in the asset pricing literature.

For the production-side parameters, we take the estimates in Eberly, Rebelo, and Vincent (2009)

[22]The necessary convergence condition is $(r+\delta)^2 - 2\frac{A-(r+\delta)}{\theta} \geq 0$.

Table 2: COMPARISON OF PRIMAL AND DUAL OPTIMIZATION PROBLEMS.

	Primal Optimization	**Dual Contracting**
A. State Variable	s	w
Drift	$\mu^s(s)$ given in (30)	$\mu^w(w)$ given in (63)
Idiosyncratic Volatility	$\sigma_h^s(s)$ given in (28)	$\sigma_h^w(w)$ given in (61)
Systematic Volatility	$\sigma_m^s(s)$ given in (29)	$\sigma_m^w(w)$ given in (62)
Admissible Range	$s \geq \underline{s}$	$w \geq \underline{w}$
B. Value Function	$m(s)$	$p(w)$
Interior Region	ODE given in (34)	ODE given in (64)
C. Policy Rules		
Compensation	$c(s)$ given in (23)	$\mathfrak{c}(w)$ given in (56)
Corporate Investment	$i(s)$ given in (24)	$\mathfrak{i}(w)$ given in (57)
Idiosyncratic Risk Hedge	$\phi_h(s)$ given in (25)	$x_h(w)$ given in (58)
Systematic Risk Exposure	$\phi_m(s)$ given in (26)	$x_m(w)$ given in (59)
D. Inalienability Case: $0 < \alpha < 1$		
Inalienability Constraint	$m(\underline{s}) = \alpha m(0)$	$p(\underline{w}/\alpha) = 0$
Curvature Condition	$m''(\underline{s}) = -\infty$	$p''(\underline{w}) = -\infty$
E. First-Best case: $\alpha = 0$		
Borrowing Limit	$\underline{s} = -q^{FB}$	$\underline{w} = 0$

and set the annual productivity A at 20% and the annual volatility of capital shocks at $\sigma_K = 20\%$. We set the correlation between the market portfolio return and the firm's depreciation shock at $\rho = 0.2$, which implies that the idiosyncratic volatility of the depreciation shock is $\epsilon_K = 19.6\%$. We fit the first-best values of q^{FB} and i^{FB} to the sample averages by setting the adjustment cost parameter at $\theta = 2$ and the (expected) annual capital depreciation rate at $\delta_K = 11\%$, both of which are in line with estimates in Hall (2004) and Riddick and Whited (2009). These parameters imply that $q^{FB} = 1.264$, $i^{FB} = 0.132$, and $\beta^{FB} = 0.2$. Finally, we set the inalienability parameter $\alpha = 0.8$. The parameter values for our baseline calculation are summarized in Table 3.

5.1 Firm Value and Endogenous Debt Capacity

We begin by linking the value functions of the two optimization problems, $p(w)$ and $m(s)$.

Table 3: PARAMETER VALUES

This table summarizes the parameter values for our baseline analysis in Section 5. Whenever applicable, parameter values are annualized.

Parameters	Symbol	Value
Risk-free rate	r	5%
The entrepreneur's discount rate	ζ	5%
Correlation	ρ	20%
Excess market portfolio return	$\mu_m - r$	6%
Volatility of market portfolio	σ_m	20%
The entrepreneur's relative risk aversion	γ	2
Capital depreciation rate	δ_K	11%
Volatility of capital depreciation shock	σ_K	20%
Quadratic adjustment cost parameter	θ	2
Productivity parameter	A	20%
Inalienability parameter	α	80%

Scaled Liquidity s and Scaled Certainty-Equivalent Wealth $m(s)$. Panels A and C of Figure 1 plot $m(s)$ and the marginal value of liquidity $m'(s)$, respectively. Under the first-best, the entrepreneur's scaled net worth is simply given by the sum of her financial wealth s and the market value of the capital stock: $m^{FB}(s) = s + q^{FB} = s + 1.264$. Note that $m^{FB}(s) \geq 0$ implies $s \geq -q^{FB}$, so that the debt limit under the first-best is $\underline{s}^{FB} = -q^{FB}$.

As one would expect, $m(s) < m^{FB}(s) = q^{FB} + s$ due to inalienability. Moreover, $m(s)$ is increasing and concave. The higher the liquidity s the less constrained is the entrepreneur, so that $m'(s)$ decreases. In the limit, as $s \to \infty$, $m(s)$ approaches $m^{FB}(s) = q^{FB} + s$ and $m'(s) \to 1$. The equilibrium credit limit under inalienability is $\underline{s} = -0.208$, meaning that the entrepreneur's maximal borrowing capacity is 20.8% of the contemporaneous capital stock K, which is as little as one-sixth of the first-best debt capacity. The corresponding scaled certainty-equivalent wealth is $m(-0.208) = 0.959$. When the endogenous financial constraint binds at $\underline{s} = -0.208$, the marginal value of liquidity $m'(s)$ is highest and is equal to $m'(-0.208) = 1.394$. Figure 1 clearly illustrates that the first-best and inalienability cases are fundamentally different.[23]

Promised Scaled Wealth w and Investors' Scaled Value $p(w)$. Panels B and D of Figure 1 plot $p(w)$ and $p'(w)$, respectively. Under the first-best, compensation to the entrepreneur

[23]The first-best case is degenerate because the entrepreneur's indifference condition $m(-q^{FB}) = 0$ implies zero-volatility of s at $s = -q^{FB}$. But this is not true for the inalienability case. Besides the indifference condition $m(\underline{s}) = \alpha m(0)$, we also need to provide incentives for the entrepreneur to choose zero volatility for s at the credit limit $\underline{s}$, which requires the entrepreneur to be endogenously infinitely risk averse at $\underline{s}$, $\gamma_e(\underline{s}) = \infty$, meaning that $m''(\underline{s}) = -\infty$.

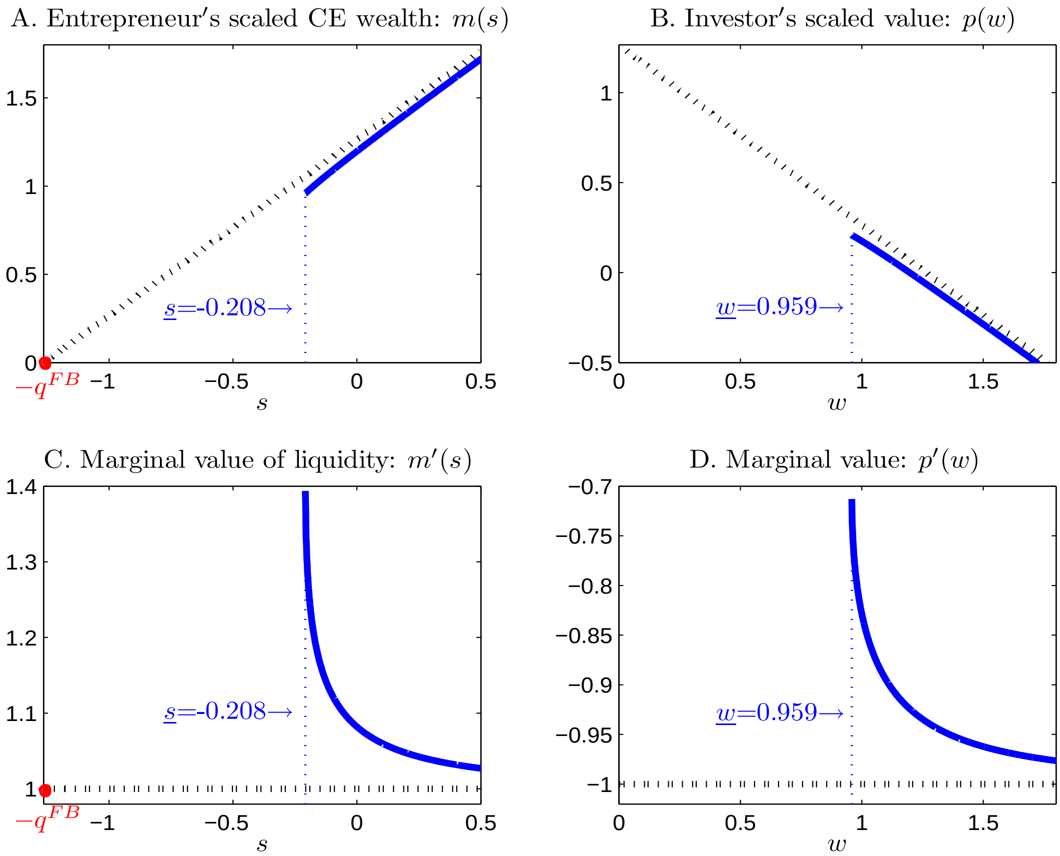

Figure 1: **Certainty equivalent wealth $m(s)$ and investors' value $p(w)$.** The dotted lines depict the first-best results: $m(s) = q^{FB} + s$ and $m'(s) = 1$ for $s \geq -q^{FB} = -1.264$, $p(w) = q^{FB} - w$ and $p'(w) = -1$ for $w \geq \underline{w}^{FB} = 0$. The solid lines depict the inalienability case: $m(s)$ is increasing and concave where $s \geq \underline{s} = -0.208$, and $p(w)$ is decreasing and concave where $w \geq \underline{w} = 0.959$. The debt limit $\underline{s}$ is determined by $m(\underline{s}) = \alpha m(0)$ and $m''(\underline{s}) = -\infty$, and $\underline{w}$ is determined by $p(\underline{w}/\alpha) = 0$ and $p''(\underline{w}) = -\infty$.

is simply a one-to-one transfer from investors: $p^{FB}(w) = q^{FB} - w = 1.264 - w$. With inalienable human capital, $p(w) < q^{FB} - w$, and $p(w)$ is decreasing and concave. As w increases the entrepreneur is less constrained. In the limit, as $w \to \infty$, $p(w)$ approaches $q^{FB} - w$, and $p'(w) \to -1$. The entrepreneur's inability to fully commit not to walk away *ex post* imposes a lower bound $\underline{w}$ on w. For our parameter values, $\underline{w} = 0.959$. Note that $\underline{w} = 0.959 = m(\underline{s}) = m(-0.208)$. This is no coincidence and is implied by our equivalence result between the two optimization problems. The entrepreneur receives at least 95.9% in promised certainty-equivalent wealth for every unit of capital stock, which is strictly greater than $\alpha = 0.8$ since the capital stock generates strictly positive net present value under the entrepreneur's control.

Panels A and B of Figure 1 illustrate how $(s, m(s))$ is the "mirror-image" of $(-p(w), w)$. To be precise, rotating Panel B counter-clock-wise by 90° (turning the original x-axis (for w) into the new y-axis $m(s)$) and adding a minus sign to the horizontal x-axis (setting $-p(w) = s$), produces Panel A. Panel C shows that the entrepreneur's marginal value of liquidity $m'(s)$ is greater than 1, which means that the liquid asset is valued more than its face value by the financially constrained entrepreneur. Panel D illustrates the same idea viewed from the investor's perspective: the marginal cost of a monetary transfer to the entrepreneur is less than one for the investor, $-1 < p'(w) < 0$, because the relaxation of the entrepreneur's financial constraint generates value. Despite being fully diversified the investor behaves in an under-diversified manner due to the entrepreneur's inalienability constraints. This is reflected in the concavity of both the investor's value function $p(w)$ and the entrepreneur's certainty-equivalent wealth function $m(s)$.

5.2 Idiosyncratic Risk Management

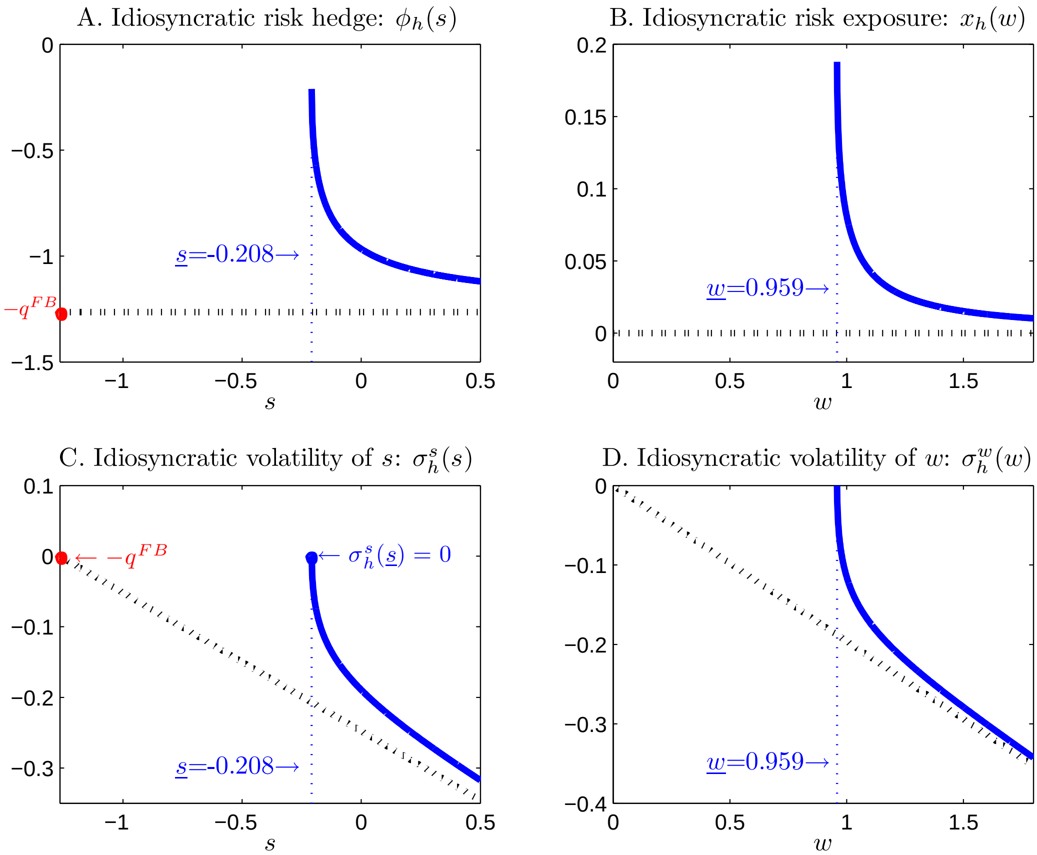

Figure 2: **Idiosyncratic risk management policies, $\phi_h(s)$ and $x_h(w)$, and volatilities for s and w, $\sigma_h^s(s)$ and $\sigma_h^w(w)$.** The dotted lines depict the first-best results: $\phi_h^{FB}(s) = -q^{FB} = -1.264$ and $x_h^{FB}(w) = 0$. The solid lines depict the inalienability case: the idiosyncratic risk hedge $\phi_h(s) < 0$, $|\phi_h(s)| < |\phi_h^{FB}(s)| = q^{FB}$, and $|\phi_h(s)|$ is increasing in s. The idiosyncratic risk exposure of the entrepreneur's certainty equivalent wealth W is positive, decreasing in w.

Panels A and B of Figure 2 plot the idiosyncratic-risk hedge rules $\phi_h(s)$ and $x_h(w)$ in the two problem formulations. Note that ϕ_h and x_h respectively control the idiosyncratic volatilities of *total* liquid wealth S and certainty equivalent wealth W, as seen in (10) and (53). In Panels C and D of Figure 2, we plot the idiosyncratic volatilities of respectively *scaled* liquidity s, $\sigma_h^s(s)$, and *scaled* wealth w, $\sigma_h^w(w)$, which are directly linked to the risk management policies ϕ_h and x_h. A key observation is that the volatility of S is different from the volatility of scaled liquidity, $s = S/K$. Making this observation explicit, we apply Ito's formula to $s_t = S_t/K_t$ and rewrite the instantaneous idiosyncratic volatility $\sigma_h^s(s_t)$ as follows:[24]

$$\sigma_h^s(s_t) = (\phi_h(s_t) - s_t)\,\epsilon_K\,. \tag{74}$$

This expression makes clear that $\sigma_h^s(s_t)$ is affected by the hedging position $\phi_h(s_t)\epsilon_K$, which drives changes in S, and by $-s_t\epsilon_K$, through the idiosyncratic risk exposure of K. Proceeding in the same way for the contracting formulation, we obtain the following expression linking $x_h(w)$ and $\sigma_h^w(w)$:[25]

$$\sigma_h^w(w_t) = -\frac{\gamma}{\gamma_p(w_t)}x_h(w_t)\,. \tag{75}$$

This expression encapsulates the optimal co-insurance of key-man risk.

Consider now the first-best solution given by the dotted lines in Figure 2. Panel A shows that the first-best idiosyncratic-risk hedge is constant: $\phi_h(s_t) = -q^{FB} = -1.264$. Panel B confirms the same result, as $x_h(w_t) = 0$ for all w_t. This hedging policy completely insulates the entrepreneur's net worth $M_t^{FB} = S_t + q^{FB}K_t$ from the idiosyncratic shock $\mathcal{Z}_h$, as one can see from the dynamics of M given in (44). This result confirms the conventional wisdom that optimal hedging for a *risk-averse* entrepreneur involves a zero net exposure of her net worth to idiosyncratic shocks.

Panels C and D reveal a less obvious but important insight, namely that complete idiosyncratic risk hedging of net worth implies neither zero volatility for s nor for w. It is only when the entrepreneur has fully exhausted her debt capacity, when $s_t = -q^{FB}$ (and equivalently $w_t = 0$), that we have $\sigma_h^s(s_t) = \sigma_h^w(w_t) = 0$. When $s_t > -q^{FB}$ (and $w_t > 0$), the first-best solution is such that $|\sigma_h^s(s_t)|$ and $|\sigma_h^w(w_t)|$ strictly increase with respectively $s_t = S_t/K_t$ and $w_t = W_t/K_t$, because of the impact of $\mathcal{Z}_h$ on the firm's capital stock.

Consider next the inalienability case. Panels A and B strikingly reveal how different the hedging policy under inalienability is from the first-best. Because the endogenous debt limit $|\underline{s}| = 0.208$ (and $\underline{w} = 0.959$) under inalienability is much tighter than the first-best limit, $|\underline{s}^{FB}| = q^{FB} = 1.264$ (and $\underline{w}^{FB} = 0$), the entrepreneur is severely constrained in her ability to hedge out the idiosyncratic risk exposure of her net worth M.

A general optimality condition is that the entrepreneur has to honor her liabilities with probability one, meaning that $\sigma_h^s(\underline{s}) = 0$ and $\sigma_h^w(\underline{w}) = 0$. This equilibrium condition of zero volatility

[24] See Appendix A.

[25] See Appendix B.1.

together with the indifference conditions $m(\underline{s}) = \alpha m(0)$ and $p(\underline{w}/\alpha) = 0$ imply endogenous infinite 'key-man' risk aversion at $\underline{s}$ and $\underline{w}$, meaning that $\gamma_e(\underline{s}) = \infty$ and $\gamma_p(\underline{w}) = \infty$.[26]

Zero idiosyncratic volatility for scaled s (and w) is achieved by setting the hedging position to $\phi_h(\underline{s}) = \underline{s}$ (and $x_h(\underline{w}) = \epsilon_K \underline{w}$). These expressions encapsulate the following general insight about hedging key-man risk. Suppose that the entrepreneur's liquidity is at its limit, $s_t = \underline{s}$, and consider the consequences of a positive idiosyncratic shock $dZ_{h,t}$. Among other effects, such a shock increases the outside value of the entrepreneur's human capital and increases the entrepreneur's incentives to leave the firm.[27] How can the entrepreneur hedge against this risk so as to continue honoring her outstanding debt liabilities? By setting $\phi_h(s) = s$ at the credit limit $\underline{s}$, as we explain next. Let $Z_{h,t+\Delta} = Z_{h,t} + \sqrt{\Delta}$ denote the outcome of a positive shock over a small time increment Δ. We can calculate the resulting liquidity ratio $s_{t+\Delta}$ as follows:[28]

$$s_{t+\Delta} \equiv \frac{S_{t+\Delta}}{K_{t+\Delta}} \approx \frac{S_t + \phi_{h,t} K_t \, \epsilon_K \sqrt{\Delta}}{(1+\epsilon_K \sqrt{\Delta}) K_t} = \frac{s_t + \phi_{h,t} \epsilon_K \sqrt{\Delta}}{(1+\epsilon_K \sqrt{\Delta})}, \tag{76}$$

where the numerator uses (10) for dS and the denominator uses (1) for dK. To ensure that the credit constraint is satisfied at $t + \Delta$ we have to set $s_{t+\Delta} = s_t = \underline{s}$ in (76), which means that $\phi_h(\underline{s}) = \underline{s} < 0$. Had the entrepreneur chosen a larger hedging position, say $|\phi_h(\underline{s})| > |\underline{s}|$, or in the extreme scenario $|\phi_h(\underline{s})| = |\phi_h^{FB}| = q^{FB}$, we would have $s_{t+\Delta} < s_t = \underline{s} < 0$, violating the equilibrium condition that $\underline{s}$ is the debt limit. Following essentially the same argument for $w = W/K$, we can verify that $x_h(\underline{w}) = \epsilon_K \underline{w} > 0$, which implies that the entrepreneur's net worth W is overexposed to idiosyncratic risk.

In words, the hedges at $\underline{s}$ and $\underline{w}$ are set so as to exactly offset the impact of the idiosyncratic shock $\mathcal{Z}_h$ on K_t in $s_t = S_t/K_t$ and $w_t = W_t/K_t$ and thereby turn off the volatilities of s and w. These hedging positions in turn significantly expose the entrepreneur's net worth W to idiosyncratic risk.

Turning now to the right end of the support for s and w, we observe that as $s \to \infty$ (and equivalently $w \to \infty$), the inalienability constraint becomes irrelevant. As a result, the entrepreneur achieves perfect risk sharing: $\lim_{s\to\infty} \phi_h(s) = \phi_h^{FB} = -q^{FB}$ and $\lim_{w\to\infty} x_h(w) = x_h^{FB} = 0$.

With inalienability, the idiosyncratic risk hedge $|\phi_h(\underline{s})| = |\underline{s}|$ at the debt limit is much lower than when the entrepreneur is unconstrained. More generally, when s moves away from the debt

[26] This result can be seen from Panels B and D in Figure 1 where the slopes of $m'(s)$ and $p'(w)$ approach $-\infty$ at $\underline{s}$ and $\underline{w}$. Mathematically, this follows from the definition of γ_e given in (22), $\sigma_h^s(s)$ given in (28), and $m(\underline{s}) = 0.207$. Similar mathematical reasoning applies for $\gamma_p = \frac{wp''(w)}{p'(w)}$ in (55).

[27] A negative shock has the opposite effect on the entrepreneur's human capital and relaxes the inalienability constraint. Hence, we focus on the positive shock.

[28] The (diffusion) risk term for any stochastic process locally dominates its drift effect as the former is of order $\sqrt{\Delta}$ and the latter is of order Δ. We thus can drop the drift term in the limit for this calculation.

limit $\underline{s}$, $|\phi_h(s)|$ in effect becomes a 'weighted average' of the first-best policy of maximizing net worth and the zero-volatility policy for s at the debt limit, with an increasing weight put on the first-best policy as s increases. Correspondingly, as $x_h(w)$ decreases with w (See Panel B,) the entrepreneur's certainty equivalent wealth W becomes less exposed to idiosyncratic risk.[29] To summarize, the 'key-man' risk management problem for the firm boils down to a compromise between the maximization of the entrepreneur's net worth, which requires full insurance against idiosyncratic risk, and the maximization of the firm's financing capacity, which involves reducing the volatility of scaled liquidity and hence exposing the entrepreneur to idiosyncratic risk. This compromise can be seen as a general principle of idiosyncratic risk management for financially constrained firms that emerges from our analysis.[30]

5.3 Optimal Equity Market Exposure

Panels A and B of Figure 3 plot the entrepreneur's market portfolio allocation $\phi_m(s)$ and the agent's systematic risk exposure $x_m(w)$ in the two formulations. Recall that ϕ_m and x_m respectively control the systematic volatilities of *total* liquid wealth S and certainty equivalent wealth W, as seen in the last terms of (10) and (53). Panels C and D of Figure 3 plot the systematic volatility of *scaled* liquidity s, $\sigma_m^s(s)$, and of *scaled* w, $\sigma_m^w(w)$, respectively.

Again, the policies ϕ_m and x_m, plotted in Panels A and B, are directly linked to the corresponding volatilities, $\sigma_m^s(s)$ and $\sigma_m^w(w)$, plotted in Panels C and D. Applying Ito's formula to $s_t = S_t/K_t$ as before, we obtain:

$$\sigma_m^s(s) = (\phi_m(s) - s\beta^{FB})\sigma_m \,. \tag{77}$$

Note that $\sigma_m^s(s_t)$ contains both the market allocation term $\phi_m(s_t)\sigma_m$ and $-s_t\beta^{FB}\sigma_m = -s_t\rho\sigma_K$, which comes from the systematic risk exposure of K. Proceeding in the same way for the contracting problem, we obtain the following expression linking $x_m(w)$ and $\sigma_m^w(w)$:

$$\sigma_m^w(w_t) = x_m(w_t) - \rho\sigma_K w_t \,. \tag{78}$$

Again, the key observation is that the systematic volatility of W, which is equal to $x_m(w_t)K_t$, is

[29] There is a natural analogy here with the general principle in moral hazard theory that the agent's compensation trades off incentive and risk sharing considerations. Following Holmstrom (1979), this literature assumes that the agent's utility function is separable in effort and wealth (or consumption.) In our framework, exerting effort means staying with the firm. With this analogy, we note that our model does not assume the standard separability as that the severity of the agency problem depends on the distance of w to the debt limit $\underline{w}$. We therefore obtain a sharper result, namely that the more severe the agency problem is the less the agent is insured against idiosyncratic risk. See Sannikov (2008) for a continuous-time version of the classical moral hazard problem.

[30] Rampini, Sufi, and Viswanathan (2014) provide empirical evidence showing that more financially constrained firms hedge less. However, our analysis implies that more constrained firms have less volatile s.

different from $\sigma_m^w(w_t)$, the systematic volatility for $w = W/K$.

Consider now the first-best solution given by the dotted lines in Figure 3. Panels A and B plot the classic Merton (1971) portfolio allocation result, which is linear in s and w. Panels C and D reveal a less emphasized insight, which however is important for our risk management analysis, namely that the systematic volatilities for scaled s and w, $\sigma_m^s(s_t)$ and $\sigma_m^w(w_t)$, are also linear to s and w, respectively. It is only when the entrepreneur has fully exhausted her debt capacity at $s_t = -q^{FB}$ (and $w_t = 0$), that we have $\sigma_m^s(s_t) = \sigma_m^w(w_t) = 0$.

Consider next the inalienability case. Panels A and B again reveal how different the risk exposures are from the first-best. Recall that the debt limit under inalienability $|\underline{s}| = 0.208$ (and $\underline{w} = 0.959$) is much tighter than the first-best debt limit, $|\underline{s}^{FB}| = q^{FB} = 1.264$ (and $\underline{w}^{FB} = 0$). As a result, the entrepreneur is endogenously more risk averse, $\gamma_e(s) > \gamma$, and $m(s)$ is lower than the first-best level for all s. Equivalently, in the contracting problem the principal is also endogenously more risk averse, $\gamma_p(w) > 0$, and $p(w)$ is lower than the first-best level for all w. It follows that the entrepreneur allocates less of her net worth to the stock market for any s, and equivalently the principal exposes the agent to less systematic risk for any w. At the debt limit, in particular, the endogenous risk aversion for both the entrepreneur and the principal approach infinity, $\gamma_e(\underline{s}) = \infty$ and $\gamma_p(\underline{w}) = \infty$, so that the systematic volatilities for both s and w approach zero: $\sigma_m^s(\underline{s}) = \sigma_m^w(\underline{w}) = 0$.

It is important to note that zero systematic volatilities are achieved by setting $\phi_m(\underline{s}) = \beta^{FB}\underline{s}$ and $x_m(\underline{w}) = \rho\sigma_K\underline{w}$, as can be seen from (77) and (78). Remarkably, while the mean-variance term vanishes at the debt limit, the hedging term does not, because the entrepreneur still needs to immunize the systematic risk exposures of s and w that comes from K.[31]

At the other end of the support, as $s \to \infty$ (and equivalently as $w \to \infty$) and the inalienability constraint becomes irrelevant, we see that the entrepreneur achieves the first-best: $\lim_{s\to\infty}\phi_m(s) = \phi_m^{FB}$ and $\lim_{w\to\infty} x_m(w) = x_m^{FB} = \eta w/\gamma$. In general, for any given s, $|\phi_m(s)|$ is like a 'weighted average' of the first-best policy of maximizing net worth and the zero-volatility policy for s at the debt limit, with an increasing weight being put on the first-best policy as s increases (the same is true for $x_m(w)$ as w increases.)

In sum, the risk management problem for the firm boils down to a compromise between achieving mean-variance efficiency for the entrepreneur's net worth M and maximizing the firm's financing capacity. To expand its financing capacity the firm must reduce the volatility of s when s is low, which involves scaling back $|\phi_h(s)|$ and $|\phi_m(s)|$. Overall, this amounts to both reducing the systematic risk exposure and increasing the idiosyncratic risk exposure of the entrepreneur's net

[31]Note that the zero systematic volatility condition for s (and equivalently for w) turns out to be identical to the zero idiosyncratic volatility condition for s (and equivalently for w).

worth. This last result can be seen more directly from the risk exposures of the agent's net worth under the optimal contract. Indeed, the optimal contract requires that $x_m(w) < x_m^{FB}(w) = \eta\, w/\gamma$ and $x_h(w) > x_h^{FB}(w) = 0$, as can be seen from Panels B in Figures 2 and 3.

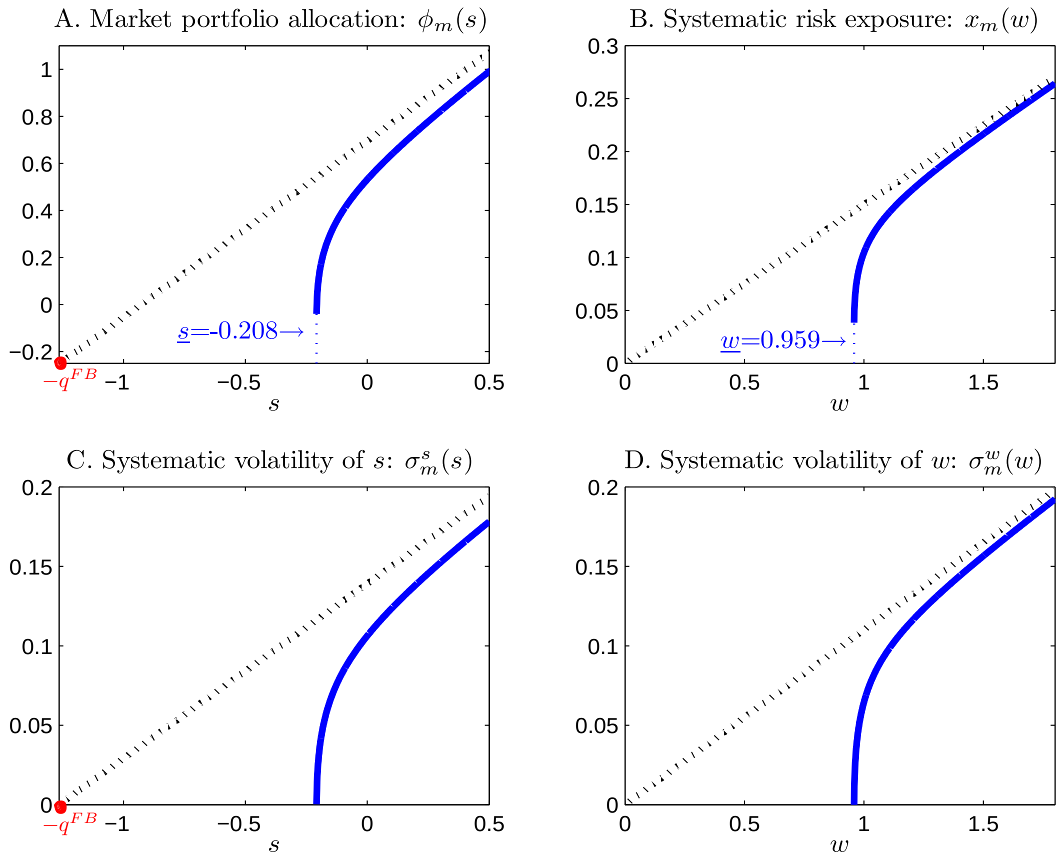

Figure 3: **Systematic risk exposure, $\phi_m(s)$ and $x_m(w)$, the systematic volatility, $\sigma_m^s(s)$ and $\sigma_m^w(w)$.** The dotted lines depict the first-best results: $\phi_m^{FB}(s) = -\beta^{FB}q^{FB} + \frac{\eta}{\gamma\sigma_m}m(s)$ and $x_h^{FB}(w) = \eta w/\gamma$. The solid lines depict the inalienability case: the systematic risk exposure $|\phi_m(s)|$ is increasing in s and $|\phi_m(s)| < |\phi_m^{FB}(s)|$. The systematic risk exposure of the entrepreneur's certainty equivalent wealth W is positive, increasing in w, and $|x_m(w)| < |x_m^{FB}(w)|$.

5.4 Investment and Compensation

Investment and its Sensitivity to Liquidity. Figure 4 plots corporate investment and its sensitivity. Panels A and B plot $i(s)$ and $i'(s)$ for the primal problem, and Panels C and D plot $\mathsf{i}(w)$ and $\mathsf{i}'(w)$ for the contracting problem, respectively. The dotted lines describe the constant $i^{FB} = 0.132$ under the first-best benchmark. The investment-capital ratio i_t is lower than $i^{FB} = 0.132$ under all circumstances, and i_t increases from -0.043 to $i^{FB} = 0.132$ when s increases from $\underline{s} = -0.208$ towards ∞, or equivalently when w increases from $\underline{w} = 0.959$ towards ∞, as can be seen from Panels A and C, respectively. As the firm's financial slack s (and equivalently

w) increases, under-investment distortions are reduced. This result can be derived analytically by differentiating $i(s)$ given in (24):

$$i'(s) = -\frac{1}{\theta}\frac{m(s)m''(s)}{m'(s)^2} > 0. \tag{79}$$

The positive investment-liquidity sensitivity follows from the concavity of $m(s)$, implied by the inalienability constraint. Note also that a sufficiently constrained firm optimally sells assets, $i_t < 0$, in order to replenish valuable liquidity.

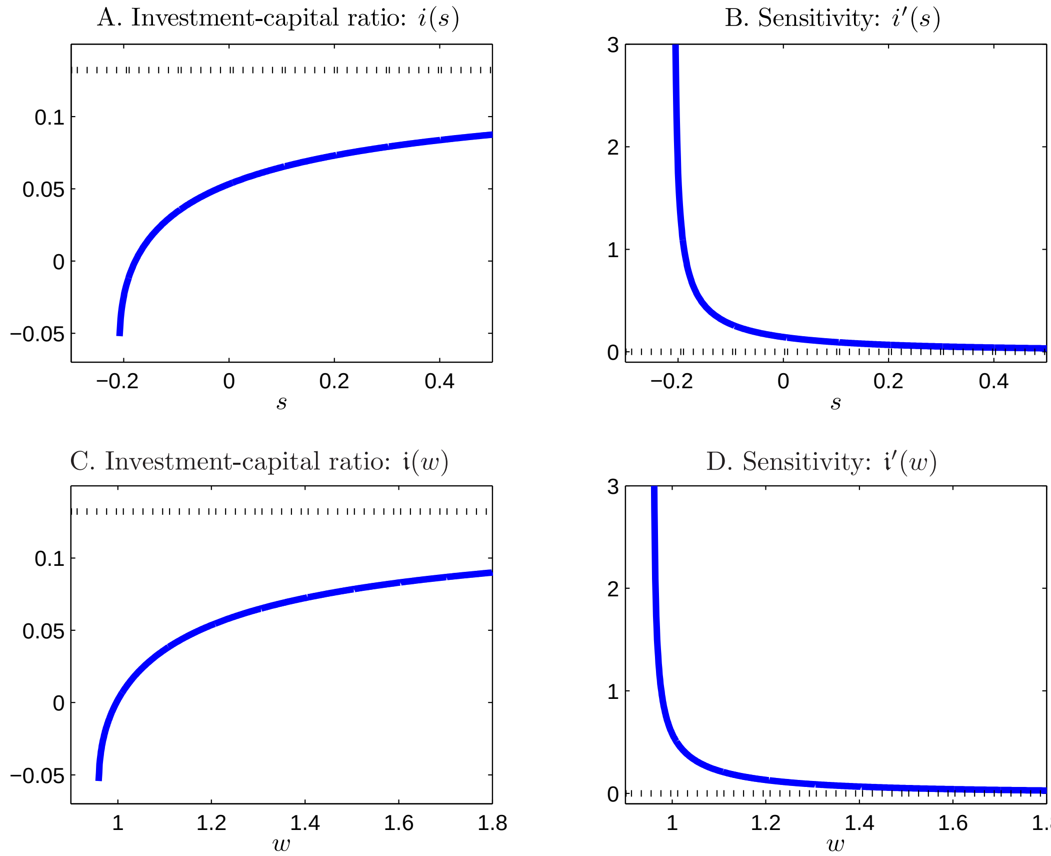

Figure 4: **Investment-capital ratio and its sensitivity.** The dotted lines depict the first-best results: $q^{FB} = 1.264$ and $i^{FB} = 0.132$. The solid lines depict the inalienability case: the firm always under-invests, and $i(s)$ increases with s (and equivalently $\mathfrak{i}(w)$ increases with w.)

Under-investment becomes more severe either (a.) when liquidity is lower, liabilities are higher (lower S), or (b.) when the entrepreneur's outside value is higher as a result of a larger capital stock K. Our liquidity and risk management problem reveals how debt (when $S < 0$) affects investment and compensation. In our model, there is a debt over-hang effect even though debt is risk-free. The reason is that debt reduces valuable financial slack thus crowding out future investments.

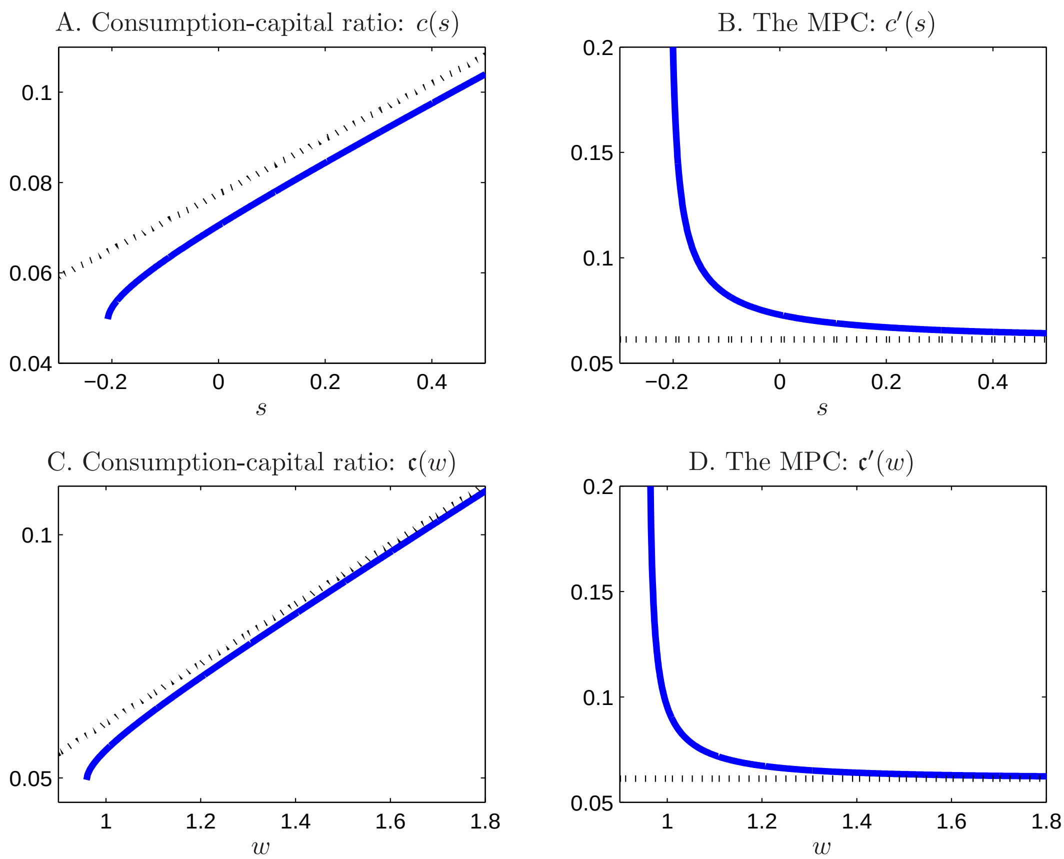

Figure 5: **Consumption-capital ratio and the MPC.** The dotted lines depict the first-best results: $c^{FB}(s) = \chi m(s) = \chi w = \mathfrak{c}^{FB}(w)$. The solid lines depict the inalienability case: the entrepreneur always under-consumes, and $c(s)$ is increasing and concave in s (and equivalently $\mathfrak{c}(w)$ is increasing and concave with w.)

Consumption and the MPC. The entrepreneur's FOC for consumption is the standard condition: $\zeta U'(C) = J_S(K,S)$. Panels A and B of Figure 5 plot $c(s)$ and the MPC $c'(s)$. The dotted lines in Panels A and B describe Merton's linear consumption rule under the first-best: $c^{FB}(s) = \chi(s+q^{FB})$, where the constant MPC is $\chi = 6.13\%$ and $q^{FB} = 1.264$. Under inalienability the entrepreneur under-consumes: c_t is lower than $c^{FB}(s)$ under all circumstances. But, the higher the financial slack s the higher is the entrepreneur's consumption. Again, this result can be derived analytically by differentiating $c(s)$ given in (23) and using the concavity of $m(s)$:

$$c'(s) = \chi \left[m'(s)^{1-\frac{1}{\gamma}} - \frac{1}{\gamma} m''(s) m'(s)^{-\left(1+\frac{1}{\gamma}\right)} m(s) \right] > 0. \tag{80}$$

It is striking that financially constrained entrepreneurs with s close to $\underline{s} = -0.208$ have substantially larger MPCs than suggested by Friedman's permanent-income hypothesis. For example, when $s = -0.2$, the MPC is $c'(-0.2) = 19.6\%$, which is much higher than the MPC $\chi = 6.13\%$ given by the standard permanent-income hypothesis. The prediction that MPCs for severely financially constrained consumers can be very high is consistent with empirical evidence in Parker (1999) and

Souleles (1999).

The dual contracting problem conveys the same insights as the entrepreneur's liquidity and risk management problem. Panels C and D of Figure 5 show that $\mathfrak{c}(w)$ is lower than the first-best consumption rule due to the inalienability constraint, and $\mathfrak{c}(w)$ is increasing and concave in w.

5.5 Which Outside Option: Recontracting or Autarky?

When limited commitment is due to the inalienability of human capital it is natural to assume that the entrepreneur's outside option is employment at another firm, which involves *recontracting*.[32] The point is that the mere decision to quit does not mean the entrepreneur has to hide and can no longer engage in any contracts. In contrast, when limited commitment takes the form of absconsion it is more natural to assume that the entrepreneur has to continue in autarky. To avoid being caught she has to hide and therefore cannot engage in any new contracts. The absconsion/autarky perspective is more common in the literature.

Why does it matter whether the outside option is autarky or recontracting? We address this question in this section and show that even for reasonable coefficients of relative risk aversion, autarky is such an unappealing and costly option for the entrepreneur that the first-best allocation can be supported, which means that it loses its bite in generating plausible economic predictions.

Autarky means that the entrepreneur is shut out of capital markets and therefore has to divide operating revenues AK_t into consumption C_t and investment I_t (including adjustment costs), so that $AK_t = C_t + I_t + G_t$. As we show, autarky is a severe punishment even for an entrepreneur with moderate risk aversion, as she is then fully exposed to the firm's operating risk and cannot diversify it away. Hence, *ex ante* limited commitment may not result in much or any distortion in investment and consumption. We illustrate this key insight in Panels A and B of Figure 6 by plotting $m(s)$ and $m'(s)$ for both $\gamma = 2$ and $\gamma = 5$, when the outside option is autarky.

As risk aversion γ increases from 2 to 5, $\underline{s}$ changes from -0.756 to $-q^{FB} = -1.264$. Panel B shows furthermore that when $\gamma = 2$ the marginal value of liquidity $m'(\underline{s})$ decreases from 1.544 to unity as s increases from $\underline{s} = -0.756$ to ∞. In contrast, when $\gamma = 5$, the marginal value of liquidity equals unity ($m'(s) = 1$) for all s (see the dashed line in Panel B), achieving the first-best. That is, the first-best is attainable with $\gamma = 5$ under autarky because the punishment is so severe. The limited commitment constraint never binds in equilibrium under autarky when $\gamma = 5$. This reduces the empirical relevance of the limited commitment model with autarky.

In contrast, under our recontracting formulation the first-best is far from attainable. The reason

[32]Unless, of course, the entrepreneur is prevented from working by a non-compete clause, which we have ruled out. However, in general, non-compete clauses are of finite duration and hence in theory, the employee still has options to re-contract in the future.

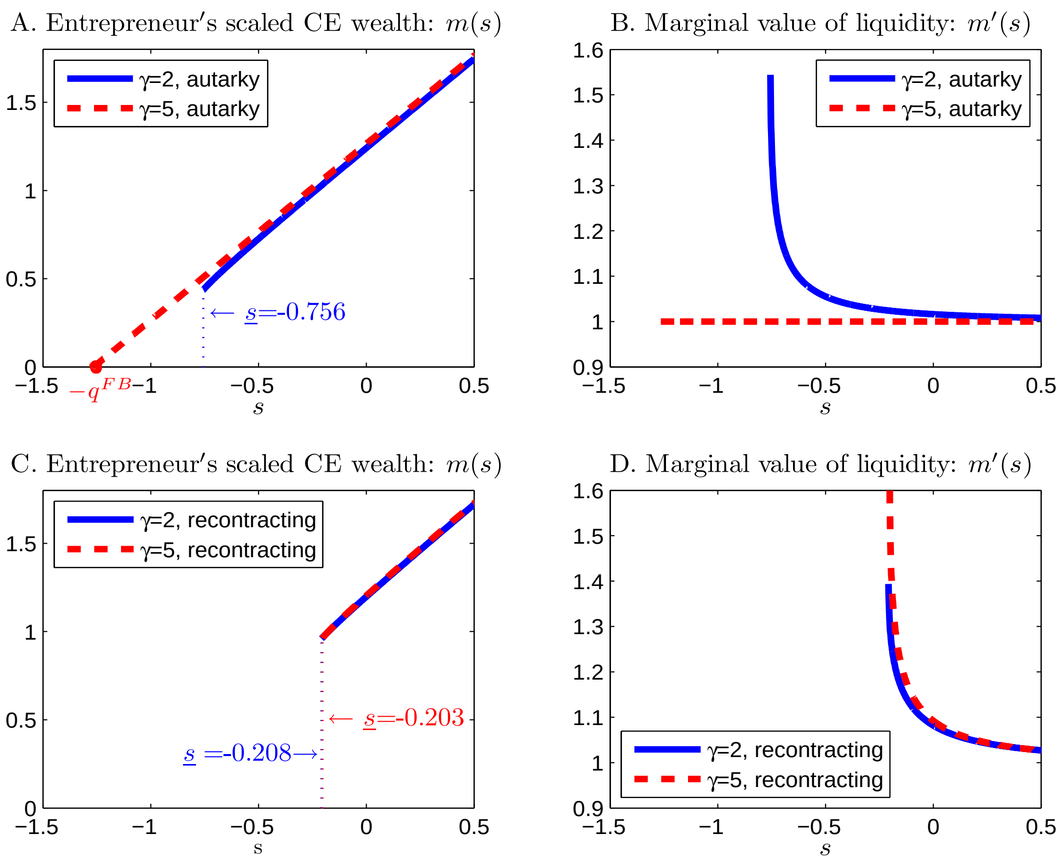

Figure 6: **Recontracting versus autarky.** Panels A and B plot the autarky case, and Panels C and D plot the recontracting case for $\gamma = 2$ and $\gamma = 5$. Under recontracting, the solutions for $\gamma = 2$ and $\gamma = 5$ are similar. For example, $\underline{s} = -0.208$ for $\gamma = 2$ and $\underline{s} = -0.203$ for $\gamma = 5$. However, under autarky, when $\gamma = 5$, the solution features first-best and hence $\underline{s} = -q^{FB}$, but when $\gamma = 2$, $\underline{s} = -0.756$.

is that the entrepreneur's risk aversion has comparable quantitative effects on her value function and her outside option value. Panels C and D of Figure 6 report $m(s)$ and $m'(s)$ with $\gamma = 2$ and $\gamma = 5$ for our recontracting formulation. We find that changes in risk aversion have almost no impact on debt capacity: $\underline{s}$ barely changes from -0.208 to -0.203 as we increase γ from 2 to 5. Finally, observe that inalienability imposes a much tighter debt limit than autarky. For example, even when $\gamma = 2$, the debt capacity under recontracting is 0.208, which is less than one-third of the debt capacity under autarky, 0.756.

Comparisons with Ai and Li (2015). The reformulation of our model with autarky as the outside option is closely related to the contracting problem analyzed by Ai and Li (2015). They consider a contracting problem between an infinitely-lived risk-neutral principal and a risk-averse agent with CRRA preferences, who is subject to a limited-commitment constraint with autarky as the outside option. The contracting formulation of our model differs from Ai and Li (2015) in several respects. First, in our model both the principal and the entrepreneur are risk-averse and are exposed to both aggregate and idiosyncratic shocks. Given that the principal is risk-neutral in Ai and Li (2015), the distinction between aggregate and idiosyncratic shocks is not meaningful in their setup.[33] As we have shown, aggregate and idiosyncratic shocks have very different implications for consumption, investment, portfolio choice, and risk management. Second, the *state variable* that we choose to work with in our contracting problem is the entrepreneur's promised certainty-equivalent wealth, while in Ai and Li (2015) it is the agent's promised utility. In other words, our units are *dollars* while Ai and Li's units are the agent's *utils*. It is only by expressing the entrepreneur's compensation in dollars that we can interpret the entrepreneur's future promised compensation as a liquidity buffer and measure how financially constrained the firm is with the investor's marginal value of liquidity $p'(w)$.

Third, the entrepreneur's consumption in our problem is stochastic, while in Ai and Li the agent's consumption is deterministic as long as the constraint is not binding. More precisely, when the constraint does not bind between $(t, t+s)$, the entrepreneur's consumption satisfies $C_{t+s} = C_t e^{-(\zeta - r)s/\gamma} \exp\left[\frac{1}{\gamma}\left(\frac{\eta^2 s}{2} + \eta(\mathcal{Z}_{m,t+s} - \mathcal{Z}_{m,t})\right)\right]$, whereas in Ai and Li (2015), consumption is deterministic, $C_{t+s} = C_t e^{-(\zeta - r)s/\gamma}$, as $\eta = 0$ in their model.[34]

6 Persistent Productivity Shocks

We now extend the model by introducing persistent productivity shocks. The firm faces two conflicting forces in the presence of such shocks. First, as Froot, Scharfstein, and Stein (1993) have emphasized, the firm will want to have sufficient funding capacity to take maximal advantage of the investment opportunities that become available when productivity is high. To do so, the firm may want to take hedging positions that allow it to transfer funds from the low to the high productivity state. Second, the firm also wants to smooth the entrepreneur's compensation across productivity states, allowing the entrepreneur to consume a higher share of earnings in the low than in the high productivity state. To do so, the firm will need to ensure that it has sufficient liquidity and funding capacity in the low productivity state. This may require taking hedging positions such that funds are transferred from the high to the low productivity state.

Which of these two forces dominates? We show that even for extreme parameter values for the productivity shocks the consumption smoothing effect dominates. One reason is that, when productivity is high, the firm's endogenous credit limit is also high, so that transferring funds from the low to the high productivity state is less important. In contrast, the consumption smoothing

[33]In our model, the principal uses the SDF $\mathbb{M}_t = e^{-rt} \exp\left(-\frac{\eta^2}{2}t - \eta\mathcal{Z}_{m,t}\right)$, while in Ai and Li (2015), the principal uses $\mathbb{M}_t = e^{-rt}$. That is, the market price of risk is $\eta > 0$ in ours and $\eta = 0$ in theirs.

[34]With the additional assumption that $\zeta = r$, consumption between t and $t+s$ is a sub-martingale in our model, while it is constant in Ai and Li (2015).

benefits of transferring funds from the high to the low productivity state are significant.

We model persistent productivity shocks $\{A_t;\, t \geq 0\}$ as a two-state Markov switching process, $A_t \in \left\{A^L, A^H\right\}$ with $0 < A^L < A^H$. We denote by $\lambda_t \in \left\{\lambda^L, \lambda^H\right\}$ the transition intensity from one state to the other, with λ^L denoting the intensity from state L to H, and λ^H the intensity from state H to L. The counting process $\{\mathcal{N}_t;\, t \geq 0\}$ (starting with $\mathcal{N}_0 = 0$) keeps track of the number of times the firm has switched productivity $\{A_s : s \leq t\}$ up to time t; it increases by one whenever the state switches from either H to L or from L to H: $d\mathcal{N}_t = \mathcal{N}_t - \mathcal{N}_{t-} = 1$ if and only if $A_t \neq A_{t-}$, and $d\mathcal{N}_t = 0$ otherwise.

In the presence of such shocks the entrepreneur will want to purchase or sell insurance against stochastic changes in productivity. We characterize the optimal insurance policy against such shocks and how investment, consumption, risk management, and credit limit vary with productivity. For brevity, we only consider the case where productivity shocks are purely idiosyncratic.[35]

Productivity Insurance Contract. Consider the following insurance contract offered at current time $t-$. Over the time interval $dt = (t-, t)$, the entrepreneur pays the unit insurance premium $\xi_{t-}dt$ to the insurance counterparty in exchange for a unit payment at time t if and only if $A_t \neq A_{t-}$ (i.e., $d\mathcal{N}_t = 1$). That is, the underlying event for this insurance contract is the change in productivity. Under our assumptions of perfectly competitive financial markets and idiosyncratic productivity shocks, the actuarially fair insurance premium is given by the intensity of the change in productivity state: $\xi_{t-} = \lambda_{t-}$.

Let Π_{t-} denote the number of units of insurance purchased by the entrepreneur at time $t-$. We refer to Π_{t-} as the insurance demand. If $\Pi_{t-} < 0$, the firm sells insurance and collects insurance premia at the rate of $\lambda_{t-}\Pi_{t-}$. Then, S_t evolves as follows:

$$dS_t = (rS_t + Y_t - C_t + \Phi_{m,t}(\mu_m - r) - \lambda_{t-}\Pi_{t-})\, dt + \Phi_{h,t}\epsilon_K d\mathcal{Z}_{h,t} + \Phi_{m,t}\sigma_m d\mathcal{Z}_{m,t} + \Pi_{t-}d\mathcal{N}_t\,. \quad (81)$$

Note that the only differences between (81) and (10) are the insurance premium payment $\lambda_{t-}\Pi_{t-}$ and the contingent liability coverage $\Pi_{t-}d\mathcal{N}_t$.

The solution for the firm's value is a pair of state-contingent value functions $J(K, S; A^L) \equiv J^L(K, S)$ and $J(K, S; A^H) \equiv J^H(K, S)$, which solve two inter-linked HJB equations, one for each

[35] We have analyzed more general situations that incorporate systematic productivity shocks. Generalizing our model to allow for a systematic risk premium requires an application of the standard change of measure technique by choosing different transition intensities under the physical measure and the risk-neutral measure. See for example, Bolton, Chen, and Wang (2013). As one may expect, the generalized liquidity and risk management problem in this section also has an equivalent optimal contracting formulation.

state.[36] The HJB equation in state L is:[37]

$$\begin{aligned}
\zeta J^L(K,S) \quad = \quad \max_{C,I,\Phi_h,\Phi_m,\Pi^L} \quad & \zeta U(C) + (I-\delta_K K)J_K^L + \frac{\sigma_K^2 K^2}{2} J_{KK}^L \\
& + \left(rS + \Phi_m(\mu_m - r) + A^L K - I - G(I,K) - C - \lambda^L \Pi^L\right) J_S^L \\
& + \left(\epsilon_K^2 \Phi_h + \rho\sigma_K\sigma_m\Phi_m\right) K J_{KS}^L + \frac{(\epsilon_K\Phi_h)^2 + (\sigma_m\Phi_m)^2}{2} J_{SS}^L \\
& + \lambda^L [J^H(K, S+\Pi^L) - J^L(K,S)] \,. \qquad (82)
\end{aligned}$$

Two important features differentiate (82) from the HJB equation (14). First, the drift term involving the marginal utility of liquidity J_S^L now includes the insurance payment $-\lambda^L\Pi^L$. Second, the last term in (82) captures the adjustment of S by the amount Π^L and the corresponding change in the value function following a productivity change from A^L to A^H.

The inalienability constraint must hold at all times t in both productivity states, so that

$$S_t \geq \underline{S}(K_t; A_t)\,, \qquad (83)$$

or equivalently,

$$s_t \geq \underline{s}(A_t)\,. \qquad (84)$$

Naturally, the firm's time-t credit limit $|\underline{s}(A_t)|$ depends on its productivity A_t. We use $\underline{s}^H$ and $\underline{s}^L$ to denote $\underline{s}(A_t)$ when $A_t = A^H$ and $A_t = A^L$, respectively.

The entrepreneur determines her optimal insurance demand Π^L in state L by differentiating (82) with respect to Π^L and setting Π^L to satisfy the FOC:

$$J_S^L(K,S) = J_S^H(K, S+\Pi^L)\,, \qquad (85)$$

provided that the solution Π^L to the above FOC satisfies the (state-contingent) condition:

$$S + \Pi^L \geq \underline{S}^H. \qquad (86)$$

Otherwise, the entrepreneur sets the insurance demand so that $\Pi^L = \underline{S}^H - S$, in which case the firm will be at its maximum debt level $\underline{S}^H$ when productivity switches from A^L to A^H.[38]

Quantitative Analysis. We consider two sets of parameter values. The first set is such that $A^H = 0.25$, $A^L = 0.14$, and $\lambda^L = \lambda^H = 0.2$, with all other parameter values as in Table 3. The

[36] For contracting models involving jumps and/or regime switching, see Biais, Mariotti, Rochet, and Villeneuve (2010), Piskorski and Tchistyi (2010), and DeMarzo, Fishman, He and Wang (2012), among others.

[37] For brevity, we omit the coupled equivalent HJB equation for $J(K,S;A^H) \equiv J^H(K,S)$ in state H.

[38] There is an equivalent set of conditions characterizing Π^H, which we refer readers to the Appendix.

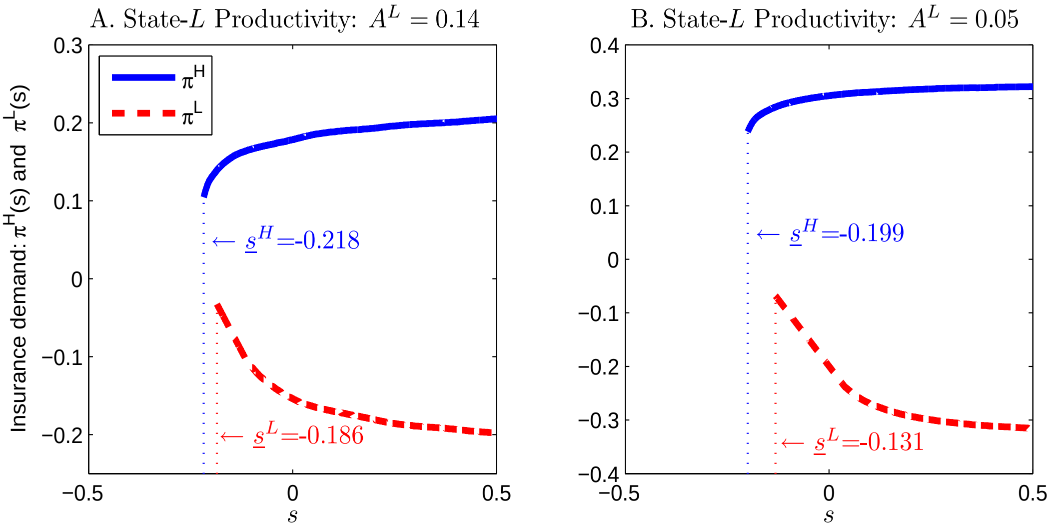

Figure 7: **Insurance demand: $\pi^H(s)$ and $\pi^L(s)$.** State-H productivity is $A^H = 0.25$ in both panels. In Panel A, State-L productivity is $A^L = 0.14$ and $\pi^L(s) = \underline{s}^H - s$ when $-0.186 < s < -0.114$. In Panel B, State-L productivity is $A^L = 0.05$ and $\pi^L(s) = \underline{s}^H - s$ when $-0.131 < s < 0.039$.

transition intensities $(\lambda^H, \lambda^L) = (0.2, 0.2)$ imply that the expected duration of each state is five years. The second set of parameter values is identical to the first, except that $A^L = 0.05$. That is, productivity in the low state, A^L, is much lower (0.05 instead of 0.14).

Figure 7 plots the entrepreneur's insurance demand $\pi^H(s)$ as the solid line, and $\pi^L(s)$ as the dashed line. Panel A plots the insurance demand in both states when productivity differences are $(A^H - A^L)/A^H = (0.25 - 0.14)/0.25 = 44\%$, while Panel B plots the insurance demand when productivity differences are very large, $(A^H - A^L)/A^H = (0.25 - 0.05)/0.25 = 80\%$. Remarkably, under both sets of parameter values the firm optimally buys insurance in state H, $\pi^H(s) > 0$, and sells insurance in state L, $\pi^L(s) < 0$. This result is not obvious *a priori*, for when productivity differences are large the benefit from transferring liquidity from state L to H and thereby taking better advantage of investment opportunities when they arise, could well be the dominant consideration for the firm's risk management. But that turns out not to be the case. Even when productivity differences are as large as 80%, the dominant consideration is still to smooth the entrepreneur's consumption. Moreover, a comparison of Panels A and B reveals that for the larger productivity differences, the insurance demand is also larger, with $\pi^H(s)$ exceeding 0.2 everywhere in Panel B, but remaining well below 0.2 in Panel A, and $\pi^L(s)$ attaining values lower than -0.25 in Panel B (when $s + \pi^L \geq \underline{s}^H$ is not binding), while $\pi^L(s)$ is always larger than -0.2 in Panel A.[39]

[39] These results are robust and hold for other more extreme parameter values, which for brevity we do not report.

7 Deterministic Formulation *à* la Hart and Moore (1994)

Our contracting problem is also closely related to Hart and Moore's contracting problem under inalienability. Hart and Moore (1994) consider a special case with a single deterministic project and linear preferences for both the investor and the entrepreneur. They emphasize the idea that debt financing is optimal when the entrepreneur's human capital is inalienable. Our more general framework reveals that the optimality of debt financing is not a robust result. Instead, the robust ideas are that inalienability gives rise to: i) an endogenous *financing capacity*; and ii) an optimal corporate liquidity and risk management problem.

To highlight the critical role of liquidity management, it is instructive to consider the special case of our model where there are no shocks, so that $\sigma_K = 0$ and $\eta = 0$, as in Hart and Moore (1994). Although output and capital accumulation are then deterministic, this special case of our model is still more general than Hart and Moore (1994) in two respects: 1) the entrepreneur has a strictly concave utility function and therefore a strict preference for smoothing consumption; 2) the firm's operations are not fixed by a one-time lump-sum investment, but can be adjusted over time through capital accumulation (or decumulation). That is, our model can be viewed as a convex version of Hart and Moore (1994), as the additional controls in our deterministic formulation are consumption and investment, both of which are convex and characterized by FOCs.

With $\sigma_K = 0$ and $\eta = 0$, $\delta = \delta_K$ and the liquidity ratio s_t evolves as follows:

$$\mu^s(s_t) \equiv \frac{ds_t}{dt} = (r + \delta - i_t)s_t + A - i_t - g(i_t) - c_t \,, \tag{87}$$

given a contract $\{c_t, i_t; t \geq 0\}$. To ensure that the entrepreneur stays with the firm, and the debt capacity is maximized, $\mu^s(\underline{s}) = 0$ has to hold. The ODE given in (34) can be simplified to:

$$0 = \frac{m(s)}{1-\gamma}\left[\gamma\chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta\right] + \left[rs + A - i(s) - g(i(s))\right] m'(s) + (i(s) - \delta)(m(s) - sm'(s))\,, \tag{88}$$

where $\chi = r + \gamma^{-1}(\zeta - r)$ and $\lim_{s\to\infty} m(s) = q^{FB} + s$.

Under the first-best, with $i_t = i^{FB}$ and $c_t = c^{FB}$, the drift of s, $\mu^s_{FB}(s_t)$, is then:

$$\mu^s_{FB}(s_t) = \left(r + \delta - i^{FB}\right)\left(s_t + q^{FB}\right) - c^{FB} = -\left(i^{FB} - \delta + \gamma^{-1}(\zeta - r)\right) m^{FB}(s_t), \tag{89}$$

where the first equality uses (41), the second uses (36) and (37). It immediately follows that the first-best drift is negative, $\mu^s_{FB}(s_t) \leq 0$, if and only if the following condition holds:

$$i^{FB} \geq \delta - \gamma^{-1}(\zeta - r)\,. \tag{90}$$

When does condition (90) hold? Under the auxiliary assumption that the entrepreneur's discount rate ζ equals the interest rate r, (90) holds if and only if the firm's first-best net investment policy is positive: $i^{FB} \geq \delta$. In other words, condition (90) requires the firm to grow under the first-best policy, which is the natural case to focus on. The alternative case is when (90) is not satisfied. Then the firm's size is decreasing over time even under the first-best policy. In this latter case, the inalienability of human capital constraint is irrelevant and the first-best outcome (optimal downsizing) is attained.[40] We summarize this discussion in the proposition below.

Proposition 1 *When (90) is satisfied the drift of s equals zero at the endogenous debt limit $\underline{s}$: $\mu^s(\underline{s}) = 0$. When (90) is not satisfied, the first-best outcome is obtained.*

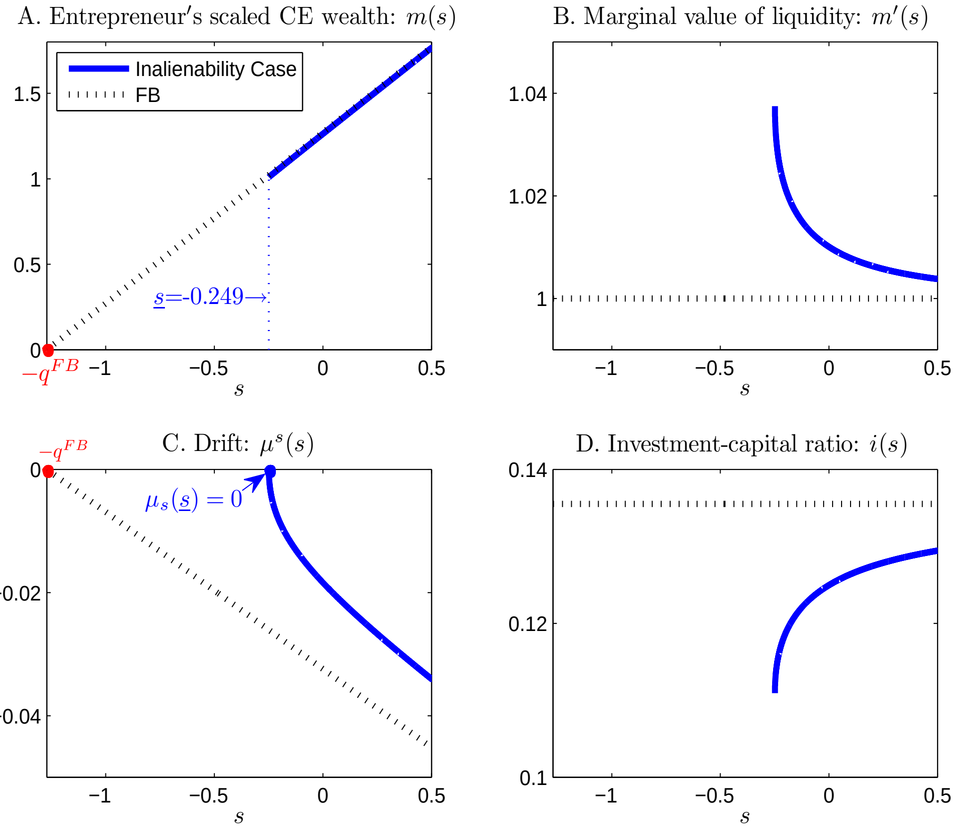

Figure 8: **The deterministic case ($\sigma_K = 0$ and $\eta = 0$) where the firm is financially constrained.** Productivity $A = 0.185$ and other parameter values are given in Table 3.

Figure 8 plots the solution when $A = 0.185$. Note that $i^{FB} = 0.136$, which is greater than $\delta = \delta_K = 0.11$. Hence, (90) is satisfied and the first-best is not attainable. The firm underinvests and under-compensates the entrepreneur, since the marginal value of liquidity is greater than one,

[40]For example, when productivity $A = 0.18$ (together with $\sigma_K = 0$ & $\eta = 0$), $q^{FB} = 1.17$ and $i^{FB} = 0.0852$. Because $\delta = 11\%$ and $r = \zeta = 5\%$, it is immediate to see that (90) is violated and hence $\mu^s_{FB}(s_t) > 0$. That is, s_t increases over time even under first-best and thus her limited-commitment constraint never binds. Of course, the net worth $s + q^{FB}$ is positive which implies $s \geq \underline{s}$ where $\underline{s} = -q^{FB} = -1.17$ in this case.

$m'(s) > 1$. However, the gap relative to the first-best decreases with s. Liquidity s decreases over time and reaches the maximal debt limit $\underline{s}$ and permanently stays there. In our example, $\underline{s} = -0.249$. Starting at $s_0 = 0$, it takes the firm $t = 25.77$ years to reach the absorbing state where the borrowing constraint binds permanently at $s_t = \underline{s} = -0.249$. Over time, the entrepreneur reduces investment i smoothly as is shown in Panel D.[41]

8 Two-Sided Limited Commitment

In our baseline model, the firm's optimal policy requires that investors incur losses with positive probability. As Figure 1 illustrates, investors make losses, $p(w) < 0$, when $w > 1.18$. But investors' *ex ante* commitment to continue compensating the entrepreneur *ex post* even when this means incurring large losses may not be credible. What if investors cannot commit to such loss-making promises to the entrepreneur *ex post*? We next explore this issue and characterize the solution when neither the entrepreneur nor investors are able to commit.

Suppose that investors can only commit to making losses *ex post* up to a fixed fraction ℓ of the total capital stock, so that $p(w_t) \geq -\ell$ at all t. For expositional simplicity we set $\ell = 0$. Then, the main change relative to the one-sided commitment problem analyzed so far is that there is also an upper boundary $\overline{s} = -p(\overline{w}) = 0$, with the following new conditions at $\overline{s} = 0$:

$$\sigma_h^s(0) = \sigma_m^s(0) = 0\,. \tag{91}$$

Using the same argument as for (32), we may express (91) as $m''(0) = -\infty$, and we verify that $\mu^s(s)$ given in (30) is weakly negative at $\overline{s} = 0$, so that $s \leq \overline{s} = 0$ with probability one.

Panel A of Figure 9 shows that investors' lack of commitment significantly destroys value. For example, at $s = 0$, $m(0) = 1.198$ under full commitment by investors, which is 42% higher than $m(0) = 0.843$ under limited commitment. Value destruction arises from the direct effect of the entrepreneur's inability to hold liquid savings ($s \leq 0$), and also the indirect effect of distorted investment and consumption decisions. Panel B shows that $i(s)$ under two-sided limited-commitment fundamentally differs from that under one-sided limited-commitment. For example, at $s = 0$, $i(0) = 0.331$ under one-sided commitment, which is six times higher than $i(0) = 0.053$ under two-sided limited commitment.

Compared with the first-best, the firm under-invests when $s < -0.13$, but over-invests when $-0.13 < s \leq 0$. Whether the firm under-invests or over-invests depends on the net effects of the entrepreneur's and investors' limited-commitment constraints. For sufficiently low values of s

[41] She also reduces consumption smoothly. We do not plot this result for brevity.

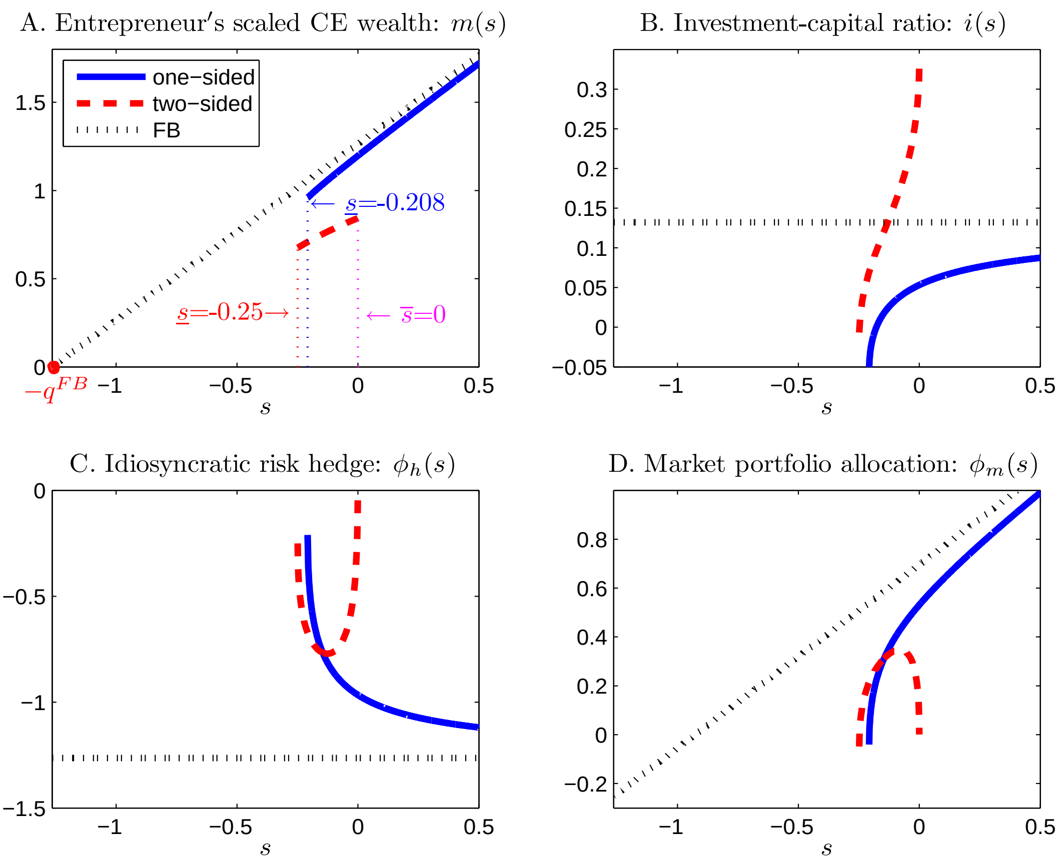

Figure 9: **Two-sided limited commitment.** The endogenous upper boundary $\overline{s} = 0$, which means the firm cannot save in this example. Compared with the first-best, s lies in the range $(\underline{s}, \overline{s}) = (-0.25, 0)$ under two-sided limited commitment. The firm under-invests when s is close to $\underline{s} = -0.25$ and over-invests when s is close to $\overline{s} = 0$. The credit limit under the two-sided limited commitment $|\underline{s}| = 0.25$ is larger than the credit limit $|\underline{s}| = 0.208$ under the one-sided limited commitment.

(when the entrepreneur is deep in debt) the entrepreneur's constraint matters more and hence the firm under-invests. When s is sufficiently close to zero, investors' limited-liability constraint has a stronger influence on investment. To ensure that $s \leq 0$ the entrepreneur needs to transform liquid assets into illiquid capital. This causes the firm to over-invest relative to the first-best.

Phrased in terms of the equivalent contracting problem, the intuition is as follows. Given that the entrepreneur cares about her total compensation $W = w \cdot K$ and given that investors are constrained by their ability to promise the entrepreneur w beyond an upper bound $\overline{w}$, (in this case, $\overline{w} = m(0) = 0.843$), investors reward the entrepreneur along the extensive margin, firm size K, which allows the entrepreneur to build more human capital through over-investment.

Figure 9 also shows that in the two-sided limited-commitment case, s lies between $\underline{s} = -0.25$ and $\overline{s} = 0$, so that the entrepreneur has a larger credit limit of $|\underline{s}| = 0.25$ instead of $|\underline{s}| = 0.208$, the limit under one-sided limited commitment. However, a firm with a larger debt capacity is not necessarily less financially constrained, since investors' limited-liability constraint limits the entrepreneur's self-insurance capacity.

Panels C and D of Figure 9 plot the idiosyncratic risk hedge $\phi_h(s)$ and the market portfolio allocation $\phi_m(s)$. Neither $\phi_h(s)$ nor $\phi_m(s)$ is monotonic in s under two-sided limited-commitment. The reason is that the volatilities $\sigma_h^s(s)$ and $\sigma_m^s(s)$ for s must be turned off at both $\underline{s} = -0.25$ and $\overline{s} = 0$ boundaries in order to prevent separation by not only the entrepreneur but also investors. This is achieved by setting $\phi_h(\underline{s}) = \underline{s} = -0.25$, $\phi_m(\underline{s}) = \beta^{FB}\underline{s} = -0.05$, and $\phi_h(0) = \phi_m(0) = 0$, as implied by the volatility boundary conditions for $\sigma_h^s(s)$ and $\sigma_m^s(s)$ at both boundaries.

9 Conclusion

We have shown that the 'key-man' risk management problem for the firm boils down to a compromise between $(a.)$ the maximization of the entrepreneur's net worth, which requires full insurance against idiosyncratic risk and Merton's mean-variance allocation of her net worth to the stock market, and $(b.)$ the maximization of the firm's financing capacity $|\underline{s}|$, which involves reducing the volatility of scaled liquidity. When the firm is close to exhausting its line of credit, the priority is to survive. From a liquidity and risk management perspective, this means that the firm cuts back all expenditures, generates new sources of liquidity, e.g. by selling insurance on persistent productivity shocks, and takes hedging positions to ensure that the volatility of its liquidity s is minimal. In contrast, when liquidity is plentiful the firm adapts its risk management policies towards maximizing the entrepreneur's net worth.

Our theory is particularly relevant for industries where human capital is an essential input. The survival of human-capital intensive companies rests on their ability to retain talent. It is optimal in general to offer a substantial fraction of promised compensation in the form of future compensation. However, the promised future compensation must be credible, which lead the firm to optimally manage corporate liquidity and risk exposures. Corporate risk management complements the firm's liquidity management by reducing unnecessary exposures of the firm's liquidity to idiosyncratic risk and also capturing risk-adjusted excess returns on behalf of key employees, thereby enhancing the firm's ability to make credible future promises.

Although our framework is already quite rich, we have imposed a number of strong assumptions, which are worth relaxing in future work. For example, one interesting direction is to allow for equilibrium separation between the entrepreneur and investors. This could arise, when after an adverse productivity shock the entrepreneur no longer offers the best use of the capital stock. Investors may then want to redeploy their capital to other more efficient uses. By allowing for equilibrium separation our model could be applied to study questions such as the life-span of entrepreneurial firms, managerial turnover, or investment in firm-specific or general human capital.

References

Abel, A.B., Eberly, J.C., 1994. A unified model of investment under uncertainty. American Economic Review, 84, 1369–1384.

Ai, H., Li, R., 2015. Investment and CEO Compensation under limited commitment. Journal of Financial Economics, 116, 452–472.

Albuquerque, R., 2003. The composition of international capital flows: risk sharing through foreign direct investment. Journal of International Economics, 61, 353-383.

Albuquerque, R., Hopenhayn, H., 2004. Optimal lending contracts and firm dynamics. Review of Economic Studies, 71, 285–315.

Alvarez, F., and Jermann, U.J., 2000. Efficiency, equilibrium, and asset pricing with risk of default. Econometrica, 68, 775–797.

Becker, G., 1975. Human Capital: A Theoretical and Empirical Analysis with Special Reference to Education, 2nd edition, National Bureau of Economic Research, Columbia University Press.

Berk, J.B., Stanton, R., Zechner, J., 2010. Human capital, bankruptcy and capital structure. Journal of Finance, 65(3), 891–926.

Ben-Porath, Y., 1967. The Production of human capital and the life cycle of earnings. The Journal of Political Economy, 75(4), 352–365.

Biais, B., Mariotti, T., Plantin, G., Rochet, J.-C., 2007. Dynamic security design: convergence to continuous time and asset pricing implications. Review of Economic Studies, 74, 345–390.

Biais, B., Mariotti, T., Rochet, J.-C., 2013. Dynamic financial contracting. Advances in Economics and Econometrics, Tenth World Congress (Daron Acemoglu, Manuel Arellano, Eddie Dekel), vol. 1, Cambridge University Press.

Biais, B., Mariotti, T., Rochet, J.-C., Villeneuve, S., 2010. Large risks, limited liability and dynamic moral hazard. Econometrica, 78, 73–118.

Bolton, P., Chen, H., Wang, N., 2011. A unified theory of Tobin's q, corporate investment, financing, and risk management. Journal of Finance, 66, 1545–1578.

Bolton, P., Chen, H., Wang, N., 2013. Market timing, investment, and risk management. Journal of Financial Economics, 109, 40–62.

Bolton, P., Dewatripont, M., 2005. Contract Theory. MIT Press.

Brown, J., Matsa, D.A., 2016. Boarding a sinking ship? An investigation of job applications to distressed firms. The Journal of Finance, 71(2), 507–550.

Campello, M., Giambona, E., Graham, J.R., Harvey, C.R., 2011. Liquidity management and corporate investment during a financial crisis. Review of Financial Studies, 24, 1944–1979.

Carroll, C.D., 1997. Buffer-stock saving and the life cycle/permanent income hypothesis, Quarterly Journal of Economics, 112(1), 1–55.

Clementi, G.L., Hopenhayn, A., 2006. A theory of financing constraints and firm dynamics. Quarterly Journal of Economics, 121, 229–265.

Cox, J.C., Ingersoll, J.E.Jr., Ross, S.A., 1985. An intertemporal general equilibrium model of asset prices. Econometrica, 53, 363–384.

Deaton, A., 1991. Saving and liquidity constraints. Econometrica, 59, 1221–1248.

DeMarzo, P. M., Fishman, M.J., 2007a. Agency and optimal investment dynamics. Review of Financial Studies 20, 151–88.

DeMarzo, P.M., Fishman, M.J., 2007b. Optimal long-term financial contracting. Review of Financial Studies, 20, 2079–2128.

DeMarzo, P.M., Fishman, M.J., He, Z., Wang, N., 2012. Dynamic agency and the q theory of investment. Journal of Finance, 67, 2295–2340.

DeMarzo, P.M., Sannikov, Y., 2006. Optimal security design and dynamic capital structure in a continuous–time agency model. Journal of Finance, 61, 2681–2724.

Di Tella, S., and Sannikov, Y., 2016. Optimal Asset Management with Hidden Savings. Working Paper, Princeton University.

Donangelo, A., Labor Mobility: Implications for Asset Pricing. Journal of Finance. *forthcoming*.

Duchin, R., Gilbert, T., Harford, J., and Hrdlicka, C., 2016. Precautionary Savings with Risky Assets: When Cash is Not Cash.

Duffie, D., 2001. Dynamic asset pricing theory. Third Edition, Princeton University Press.

Duffie, D., Huang, C., 1985. Implementing Arrow-Debreu equilibria by continuous trading of a few long-lived securities. Econometrica. 53, 1337–1356.

Eberly, J.C., Rebelo, S., Vincent, N., 2009. Investment and value: A neoclassical benchmark. Unpublished working paper. Northwestern University and HEC Montréal.

Edmans, A., Gabaix, X., 2016. Executive compensation: A modern primer. Journal of Economic Literature.

Epstein, L.G., Zin, S.E., 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. Econometrica, 57, 937–969.

Froot, K.A., Scharfstein, D.S., Stein, J.C., 1993. Risk management: coordinating corporate investment and financing policies. Journal of Finance, 48, 1629–1658.

Frydman, C., Jenter, D., 2010. CEO compensation. Annual Review of Financial Economics, Annual Reviews, 2(1), 75–102.

Gale, D., Hellwig, M., 1985. Incentive-compatible debt contracts: the one-period problem. Review of Economics Studies, 52, 647–663.

Ganor, M., 2013. Agency costs in the Era of economic crisis–The enhanced connection between CEO compensation and corporate cash holdings. Arizona Law Review, 55(1), 105–149.

Green, E. J., 1987. Lending and the smoothing of uninsurable income. In Contractual Arrangements for Intertemporal Trade. Minnesota Studies in Macroeconomics Series, Vol. 1, ed. E. C. Prescott and N. Wallace, 3–25. Minneapolis: University of Minnesota Press.

Hall, R.E., 2004. Measuring factor adjustment costs. Quarterly Journal of Economics, 119, 899–927.

Harris, M., Holmstrom, B., 1982. A theory of wage dynamics. The Review of Economic Studies, 49, 315–333.

Harrison, J., Kreps, D., 1979. Martingales and Arbitrage in Multiperiod Securities Markets, Journal of Economic Theory, 20, 381-408.

Hart, O., Moore, J., 1994. A theory of debt based on the inalienability of human capital. Quarterly Journal of Economics, 109, 841–879.

Hart, O., Moore, J., 1998. Default and renegotiation: A dynamic model of debt. Quarterly Journal of Economics, 113(1), 1–41.

Hayashi, F., 1982. Tobin's marginal q and average q: A neoclassical interpretation. Econometrica, 50, 213–224.

Holmstrom, B., 1979. Moral hazard and observability. The Bell Journal of Economics, 74–91.

Holmstrom, B., Milgrom, P., 1987. Aggregation and linearity in the provision of intertemporal incentives. Econometrica, 55(2), 303–328.

Holmstrom, B., Tirole, J., 1997. Financial intermediation, loanable funds, and the real sector. The Quarterly Journal of Economics, 112(3), 663–691.

Holmstrom, B. and Tirole, J., 2001. LAPM: A liquidity-based asset pricing model. the Journal of Finance, 56(5), 1837-1867.

Innes, R.D., 1990. Limited liability and incentive contracting with ex-ante action choices. Journal of Economic Theory, 52(1), 45–67.

Jones, L.E., Manuelli, R.E., 2005. Neoclassical models of endogenous growth: The effects of fiscal policy, innovation and fluctuations. Handbook of Economic Growth, Elsevier.

Kehoe, P.J., Perri, F., 2002. International business cycles with endogenous incomplete markets. Econometrica, 70, 907-928.

Kehoe, T.J., Levine, D.K., 1993. Debt-constrained asset markets. Review of Economic Studies, 60, 865–888.

Kiyotaki, N., Moore, J., 1997. Credit cycles. Journal of Political Economy, 105, 211–248.

Kocherlakota, N., 1996. Implications of efficient risk sharing without commitment. Review of Economic Studies, 63, 595–609.

Lambrecht, B.M., Myers, S. C., 2012. A Lintner model of payout and managerial rents. Journal of Finance, 67(5), 1761–1810.

Lintner, J., 1956. Distribution of incomes of corporations among dividends, retained earnings, and taxes, American Economic Review, 46, 97–113.

Ljungqvist, L., Sargent, T.J., 2004. Recursive macroeconomic theory. Cambridge, MA, MIT Press.

Lorenzoni, G., Walentin, K., 2007. Financial frictions, investment and Tobin's q. Unpublished working paper. MIT.

Lucas, R.E.Jr., Prescott, E.C., 1971. Investment under uncertainty. Econometrica, 39, 659–681.

Marcet, A., Marimon, R., 1992. Communication, commitment and growth. Journal of Economic Theory, 58(2), 219–249.

Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory, 3, 373–413.

Miao, J., Zhang, Y., 2015. A duality approach to continuous-time contracting problems with limited commitment. Journal of Economic Theory, 159, 929–988.

Parker, J. A., 1999. The reaction of household consumption to predictable changes in social security taxes. American Economic Review, 89, 959-973.

Piskorski, T., Tchistyi, A. 2010. Optimal mortgage design. Review of Financial Studies, 23(8), 3098–3140.

Quadrini, V., 2004. Investment and liquidation in renegotiation-proof contracts with moral hazard. Journal of Monetary Economics, 51, 713–751.

Rampini, A., Sufi, A., Viswanathan, S., 2014. Dynamic risk management. Journal of Financial Economics, 111, 271–296.

Rampini, A., Viswanathan, S., 2010. Collateral, risk management, and the distribution of debt capacity. Journal of Finance, 65, 2293–2322.

Rampini, A., Viswanathan, S., 2013. Collateral and capital structure. Journal of Financial Economics, 109, 466–492.

Riddick, L. A., Whited, T. M., 2009. The corporate propensity to save. Journal of Finance, 64, 1729–1766.

Sannikov, Y., 2008. A continuous-time version of the principal-agent problem. The Review of Economic Studies, 75(3), 957–984.

Sannikov, Y., 2013. Dynamic security design and corporate financing. Handbook of the Economics of Finance 2, 71-122.

Schaettler, H., Sung, J., 1993. The first-order approach to the continuous-time principal?agent problem with exponential utility. Journal of Economic Theory, 61, 331-371.

Smith, C. W., Stulz, R., 1985. The determinants of firms' hedging policies. Journal of Financial and Quantitative Analysis, 20, 391-405.

Souleles, N. S., 1999. The response of household consumption to income tax refunds. American Economic Review, 89, 947-958.

Stulz, R.M., 1984, Optimal hedging policies. Journal of Financial and Quantitative Analysis, 19, 127-140.

Stulz, R.M., 1996. Rethinking risk management. Journal of applied corporate finance, 9(3), 8–25.

Thomas, J., Worrall, T., 1990. Income fluctuation and asymmetric information: An example of a repeated principal-agent model. Journal of Economic Theory, 51(2), 367–390.

Townsend, R.M., 1979. Optimal contracts and competitive markets with costly state verification. Journal of Economic Theory, 21, 265–293.

Tufano, P., 1996. Who manages risk? An empirical examination of risk management practices in the gold mining industry. the Journal of Finance, 51(4), 1097–1137.

Wang, C., Wang, N., Yang, J., 2012. A unified model of entrepreneurship dynamics. Journal of Financial Economics, 106, 1–23.

Appendix

A The Entrepreneur's Optimization Problem

We conjecture that the entrepreneur's value function $J(K,S)$ takes the following form:

$$J(K,S)=\frac{(bM(K,S))^{1-\gamma}}{1-\gamma}=\frac{(bm(s)K)^{1-\gamma}}{1-\gamma}\,, \tag{A.1}$$

where b is constant and will be determined later. We then have:

$$\begin{aligned}
J_S &= b^{1-\gamma}(m(s)K)^{-\gamma}m'(s), && \text{(A.2)}\\
J_K &= b^{1-\gamma}(m(s)K)^{-\gamma}(m(s)-sm'(s)), && \text{(A.3)}\\
J_{SK} &= b^{1-\gamma}(m(s)K)^{-1-\gamma}\left(-sm(s)m''(s)-\gamma m'(s)(m(s)-sm'(s))\right), && \text{(A.4)}\\
J_{SS} &= b^{1-\gamma}(m(s)K)^{-1-\gamma}\left(m(s)m''(s)-\gamma m'(s)^2\right), && \text{(A.5)}\\
J_{KK} &= b^{1-\gamma}(m(s)K)^{-1-\gamma}\left(s^2m(s)m''(s)-\gamma(m(s)-sm'(s))^2\right). && \text{(A.6)}
\end{aligned}$$

Substituting these terms into the HJB equation (14) and simplifying, we obtain:

$$\begin{aligned}
0 \;=\; & \max_{c,i,\phi_h,\phi_m}\quad \zeta m(s)\frac{\left(\frac{c}{bm(s)}\right)^{1-\gamma}-1}{1-\gamma}+(i-\delta_K)(m(s)-sm'(s))\\
&+(rs+\phi_m(\mu_m-r)+A-i-g(i)-c)m'(s)+\frac{\sigma_K^2}{2}\left(s^2m''(s)-\frac{\gamma(m(s)-sm'(s))^2}{m(s)}\right)\\
&+\left(\epsilon_K^2\phi_h+\rho\sigma_K\sigma_m\phi_m\right)\left(-sm''(s)-\frac{\gamma m'(s)(m(s)-sm'(s))}{m(s)}\right)\\
&+\frac{(\epsilon_K\phi_h)^2+(\sigma_m\phi_m)^2}{2}\left(m''(s)-\frac{\gamma m'(s)^2}{m(s)}\right).
\end{aligned} \tag{A.7}$$

The first order conditions for consumption and investment in (15) and (16) then become:

$$\begin{aligned}
\zeta U'(c) &= b^{1-\gamma}m(s)^{-\gamma}m'(s), && \text{(A.8)}\\
1+g'(i) &= \frac{m(s)}{m'(s)}-s\,. && \text{(A.9)}
\end{aligned}$$

From the first order conditions (17) and (18), we obtain (25) and (26). And then substituting (25) and (26) into (28) and (29) respectively, we have (74) and (77).

Finally, substituting these policy functions for $c(s)$, $\phi_h(s)$and $\phi_m(s)$ into (A.7), we obtain the ODE given in (34) for $m(s)$:

$$\begin{aligned}
0 \;=\; & \frac{m(s)}{1-\gamma}\left[\gamma\chi m'(s)^{\frac{\gamma-1}{\gamma}}-\zeta\right]+\left[rs+A-i(s)-g(i(s))\right]m'(s)+(i(s)-\delta)(m(s)-sm'(s))\\
&-\left(\frac{\gamma\sigma_K^2}{2}-\rho\eta\sigma_K\right)\frac{m(s)^2m''(s)}{m(s)m''(s)-\gamma m'(s)^2}+\frac{\eta^2m'(s)^2m(s)}{2(\gamma m'(s)^2-m(s)m''(s))}\,,
\end{aligned} \tag{A.10}$$

where χ is defined by

$$\chi \equiv b^{\frac{\gamma-1}{\gamma}} \zeta^{\frac{1}{\gamma}} \,. \tag{A.11}$$

Substituting γ_e given by (22) into (A.10), we obtain the ODE given in (34).

A.1 First-Best

Under the first-best case, we have $m^{FB}(s) = s + q^{FB}$. Substituting this expression for $m^{FB}(s)$ into the ODE (A.10) we obtain:

$$\begin{aligned} 0 &= \frac{s+q^{FB}}{1-\gamma}\left[\gamma\chi-\zeta\right] + \left[rs + A - i^{FB} - g(i^{FB})\right] + (i^{FB}-\delta)q^{FB} + \frac{\eta^2(s+q^{FB})}{2\gamma} \\ &= \left(\frac{\gamma\chi-\zeta}{1-\gamma} + \frac{\eta^2}{2\gamma} + r\right)(s+q^{FB}) + \left[A - i^{FB} - g(i^{FB}) - (r+\delta-i^{FB})q^{FB}\right]. \end{aligned} \tag{A.12}$$

As (A.12) must hold for all $m^{FB}(s) = s + q^{FB}$, we must have

$$\chi = r + \frac{\eta^2}{2\gamma} + \gamma^{-1}\left(\zeta - r - \frac{\eta^2}{2\gamma}\right), \tag{A.13}$$

as given by (37), and

$$0 = A - i^{FB} - g(i^{FB}) - (r+\delta-i^{FB})q^{FB}, \tag{A.14}$$

so that (41) holds. In addition, using (A.11), we obtain the expression (20) for the coefficient b. Next, substituting $m(s) = m^{FB}(s) = s + q^{FB}$ into (A.8) and (A.9) gives the first-best consumption rule (36) and investment policy (38). To ensure that the optimization problem is well posed, we require positive consumption and positive Tobin's Q, i.e. $\chi > 0$ and $q^{FB} > 0$, which imply:

$$\textit{Condition 1}: \; r + \frac{\eta^2}{2\gamma} + \gamma^{-1}\left(\zeta - r - \frac{\eta^2}{2\gamma}\right) > 0\,, \tag{A.15}$$

$$\textit{Condition 2}: \; i^{FB} < r + \delta\,, \tag{A.16}$$

where i^{FB} is given by (41). Substituting $m(s) = m^{FB}(s) = s + q^{FB}$ into (25) and (26) respectively, we obtain the first-best idiosyncratic risk hedge $\phi_h^{FB}(s)$ given in (42) and the market portfolio allocation $\phi_m^{FB}(s)$ given in (43).

The expected return for Q_t^{FB}, μ^{FB}, satisfies the CAPM where

$$\mu^{FB} = \frac{A - i^{FB} - g(i^{FB})}{q^{FB}} + \left(i^{FB} - \delta_K\right) = r + \delta - i^{FB} + \left(i^{FB} - \delta_K\right) = r + \beta^{FB}\left(\mu_m - r\right)\,, \tag{A.17}$$

and β^{FB} is given by (40). The value of capital Q_t^{FB} follows a GBM process given by:

$$dQ_t^{FB} = Q_t^{FB}\left[\left(i^{FB} - \delta_K\right)dt + \left(\epsilon_K d\mathcal{Z}_{h,t} + \rho\sigma_K d\mathcal{Z}_{m,t}\right)\right]\,, \tag{A.18}$$

with the drift $\left(i^{FB} - \delta_K\right)$, idiosyncratic volatility ϵ_K, and systematic volatility $\rho\sigma_K$. These coefficients are identical to those for $\{K_t : t \geq 0\}$. Next, turning to the dynamics of $\{M_t^{FB}; t \geq 0\}$, we apply Ito's formula to $M_t^{FB} = S_t + Q_t^{FB} = S_t + q^{FB} K_t$ and obtain the following dynamics:

$$dM_t^{FB} = M_t^{FB}\left[\left(r - \chi + \frac{\eta^2}{\gamma}\right)dt + \frac{\eta}{\gamma} d\mathcal{Z}_{m,t}\right]. \tag{A.19}$$

A.2 Inalienability

Under inalienability, the entrepreneur's credit constraint binds at $\underline{S}(K)$, which implies:

$$J(K, \underline{S}) = J(\alpha K, 0). \tag{A.20}$$

Substituting the value function (19) into (A.20), we obtain $M(K, \underline{S}) = M(\alpha K, 0)$, which implies (45). The boundary conditions given in (32) are necessary to ensure that the entrepreneur will stay with the firm, which implies that

$$\phi_h(\underline{s}) = \underline{s}, \quad \phi_m(\underline{s}) = \underline{s}\beta^{FB}. \tag{A.21}$$

Comparing (A.21) with (25) and (26), it is straightforward to show that (32) is equivalent to $\lim_{s \to \underline{s}} m''(s) = -\infty$ as given in (46). Finally, we have $\lim_{s \to \infty} m(s) = m^{FB}(s) = s + q^{FB}$.

B Equivalent Optimal Contract

B.1 Solution of the Contracting Problem

HJB Equation for $F(K, V)$. Using Ito's formula, we have

$$d(\mathbb{M}_t F(K_t, V_t)) = \mathbb{M}_t dF(K_t, V_t) + F(K_t, V_t) d\mathbb{M}_t + < d\mathbb{M}_t, dF(K_t, V_t) >, \tag{B.1}$$

where

$$\begin{aligned}
dF(K_t, V_t) &= F_K dK_t + \frac{F_{KK}}{2} < dK_t, dK_t > + F_V dV_t + \frac{F_{VV}}{2} < dV_t, dV_t > + F_{VK} < dV_t, dK_t > \\
&= \left[(I - \delta_K K)F_K + \frac{\sigma_K^2 K^2 F_{KK}}{2} + \zeta\left(V - U(C)\right) F_V\right] dt \\
&\quad + \left[\frac{(z_h^2 + z_m^2)V^2 F_{VV}}{2} + (z_h \epsilon_K + z_m \rho \sigma_K) K V F_{VK}\right] dt \\
&\quad + V F_V (z_h d\mathcal{Z}_{h,t} + z_m d\mathcal{Z}_{m,t}) + \sigma_K K F_K \left(\sqrt{1 - \rho^2} d\mathcal{Z}_{h,t} + \rho d\mathcal{Z}_{m,t}\right).
\end{aligned} \tag{B.2}$$

Using the SDF $\mathbb{M}$ given in (8) and the following martingale representation,

$$\mathbb{E}_t[d(\mathbb{M}_t F(K_t, V_t))] + \mathbb{M}_t (Y_t - C_t) dt = 0, \tag{B.3}$$

we obtain (51), which is the HJB equation for the optimal contracting problem.

Optimal Policy Functions and ODE for $p(w)$. Applying Ito's formula to (54) and transforming (51) for $F(K,V)$ into an HJB equation for $P(K,W)$, we obtain the following:

$$\begin{aligned} rP(K,W) = \max_{C,I,x_h,x_m} \quad & \left\{ Y - C + \left(\frac{\zeta(U(bW)-U(C))}{bU'(bW)} - x_m \eta K \right) P_W + (I - \delta_K K - \rho\eta\sigma_K K) P_K \right. \\ & + \frac{\sigma_K^2 K^2}{2} P_{KK} + \frac{(x_h^2 + x_m^2)K^2}{2} \frac{P_{WW} bU'(bW) - P_W b^2 U''(bW)}{bU'(bW)} \\ & \left. + (x_h \epsilon_K + x_m \rho \sigma_K) K^2 P_{WK} \right\} . \end{aligned} \tag{B.4}$$

Using the FOCs for I, C, x_h and x_m respectively, we obtain

$$1 + G_I(I,K) = P_K(K,W) , \tag{B.5}$$

$$U'(bW) = -\frac{\zeta}{b} P_W(K,W) U'(C) , \tag{B.6}$$

$$x_h = -\frac{\epsilon_K P_{WK}}{P_{WW} - P_W bU''(bW)/U'(bW)} , \tag{B.7}$$

$$x_m = -\frac{\rho \sigma_K P_{WK}}{P_{WW} - P_W bU''(bW)/U'(bW)} + \frac{\eta P_W}{K[P_{WW} - P_W bU''(bW)/U'(bW)]} . \tag{B.8}$$

Substituting $P(K,W) = p(w)K$ into (B.5)-(B.8), we obtain the optimal consumption, investment, and risk management policies given by (56)-(59), respectively. In addition, substituting $P(K,W) = p(w)K$ and the corresponding optimal policies (56)-(59) into the PDE (B.4), we obtain the investor's value $p(w)$ satisfies ODE (64).

Dynamics of the Entrepreneur's Promised Scaled Wealth w. Using Ito's formula, we have the following dynamics for W:

$$dW_t = \frac{\partial W}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 W}{\partial V^2} < dV_t, dV_t > = \frac{dV_t}{V_W} - \frac{V_{WW}}{2V_W^3} < dV_t, dV_t > , \tag{B.9}$$

where $< dV_t, dV_t >$ denotes the quadratic variation of V, (B.9) uses $\partial W / \partial V = 1/V_W$, and

$$\frac{\partial^2 W}{\partial V^2} = \frac{\partial V_W^{-1}}{\partial V} = \frac{\partial V_W^{-1}}{\partial W} \frac{\partial W}{\partial V} = -\frac{V_{WW}}{V_W^2} \frac{1}{V_W} = -\frac{V_{WW}}{V_W^3} . \tag{B.10}$$

Substituting the dynamics of V given by (50) into (B.9) yields

$$\begin{aligned} dW_t &= \frac{1}{V_W} [\zeta(V - U(C_t))dt + z_h V d\mathcal{Z}_{h,t} + z_m V_t d\mathcal{Z}_{m,t}] - \frac{(z_h^2 + z_m^2) V^2 V_{WW}}{2V_W^3} dt \\ &= \left[\frac{\zeta(V - U(C_t))}{V_W} - \frac{(x_h^2 + x_m^2) K^2 V_{WW}}{2V_W} \right] dt + x_h K_t d\mathcal{Z}_{h,t} + x_m K_t d\mathcal{Z}_{m,t} , \end{aligned} \tag{B.11}$$

where $x_m = \frac{z_m V}{K V_W}$ and $x_h = \frac{z_h V}{K V_W}$. Using the dynamics for W and K, we can write the dynamic evolution of the certainty equivalent wealth w as follows:

$$dw_t = d\left(\frac{W_t}{K_t} \right) = \mu^w(w) dt + \sigma_h^w(w) d\mathcal{Z}_{h,t} + \sigma_m^w(w) d\mathcal{Z}_{m,t} , \tag{B.12}$$

where the drift and volatility processes, $\mu^w(\,\cdot\,)$ $\sigma^w_h(\,\cdot\,)$, and $\sigma^w_m(\,\cdot\,)$, are given by

$$\mu^w(w) = \frac{\zeta}{1-\gamma}\left(w + \frac{\mathfrak{c}(w)}{\zeta p'(w)}\right) - w(\mathfrak{i}(w) - \delta_K) + \frac{\gamma(x_h^2 + x_m^2)}{2w} - \left(\epsilon_K \sigma^w_h(w) + \rho\sigma_K\sigma^w_m(w)\right), \quad \text{(B.13)}$$

and

$$\sigma^w_h(w) = x_h(w) - w\epsilon_K\,, \quad \sigma^w_m(w) = x_m(w) - w\rho\sigma_K\,. \quad \text{(B.14)}$$

Substituting $x_h(w)$ given in (58) and $x_m(w)$ given in (59) into (B.14), we have (61) and (62).

Dynamics of Consumption C_t in the Interior Region.

Lemma 4. When the inalienability constraint does not bind between $(t, t+s)$, the entrepreneur's consumption follows $C_{t+s} = C_t e^{-(\zeta-r)s/\gamma}\exp\left[\frac{1}{\gamma}\left(\frac{\eta^2 s}{2} + \eta(\mathcal{Z}_{m,t+s} - \mathcal{Z}_{m,t})\right)\right]$.

Proof: The FOC for consumption given in (15) implies that $C = \left(\frac{J_S(K,S)}{\zeta}\right)^{-\frac{1}{\gamma}}$. Using Ito's formula, we have

$$\begin{aligned}
dC_t &= -\frac{\zeta^{\frac{1}{\gamma}}}{\gamma} J_S^{-\frac{1+\gamma}{\gamma}}\left(dJ_S - \frac{1+\gamma}{2\gamma J_S}(dJ_S)^2\right), \\
&= -\frac{\zeta^{\frac{1}{\gamma}}}{\gamma} J_S^{-\frac{1+\gamma}{\gamma}}\Big[J_{SS}dS_t + \frac{J_{SSS}(dS_t)^2}{2} + J_{SK}dK_t + \frac{J_{SKK}(dK_t)^2}{2} + J_{SSK} < dS_t, dK_t > \\
&\quad -\frac{1+\gamma}{2\gamma J_S}\left(J_{SS}^2(dS_t)^2 + 2J_{SS}J_{SK} < dS_t, dK_t > + J_{SK}^2(dK_t)^2\right)\Big] \\
&= -\frac{\zeta^{\frac{1}{\gamma}}}{\gamma} J_S^{-\frac{1+\gamma}{\gamma}}\Big[\Big((rS + \Phi_m(\mu_m - r) + AK - I - G(I,K) - C)J_{SS} + (I - \delta_K K)J_{KS} + \frac{\sigma_K^2 K^2}{2}J_{KKS} + \\
&\quad + \left(\epsilon_K^2\Phi_h + \rho\sigma_K\sigma_m\Phi_m\right)KJ_{KSS} + \frac{(\epsilon_K\Phi_h)^2 + (\sigma_m\Phi_m)^2}{2}J_{SSS}\Big)dt + (\Phi_{h,t}\epsilon_K d\mathcal{Z}_{h,t} + \Phi_{m,t}\sigma_m d\mathcal{Z}_{m,t})J_{SS} \\
&\quad + \sigma_K K_t\left(\sqrt{1-\rho^2}d\mathcal{Z}_{h,t} + \rho d\mathcal{Z}_{m,t}\right)J_{KS} - \frac{1+\gamma}{2\gamma J_S}\left(J_{SS}^2(dS_t)^2 + 2J_{SS}J_{SK} < dS_t, dK_t > + J_{SK}^2(dK_t)^2\right)\Big].
\end{aligned}$$

Differentiating the HJB equation given in (14) with respect to S, we obtain

$$\begin{aligned}
\zeta J_S(K,S) &= rJ_S + (rS + \Phi_m(\mu_m - r) + AK - I - G(I,K) - C)J_{SS} + (I - \delta_K K)J_{KS} + \frac{\sigma_K^2 K^2}{2}J_{KKS} \\
&\quad + \left(\epsilon_K^2\Phi_h + \rho\sigma_K\sigma_m\Phi_m\right)KJ_{KSS}(K,S) + \frac{(\epsilon_K\Phi_h)^2 + (\sigma_m\Phi_m)^2}{2}J_{SSS}\,.
\end{aligned}$$

Collecting terms and simplifying, we obtain

$$\begin{aligned}
\frac{dC_t}{-\frac{\zeta^{\frac{1}{\gamma}}}{\gamma}J_S^{-\frac{1+\gamma}{\gamma}}} &= (\zeta - r)J_S dt + (\Phi_{h,t}\epsilon_K d\mathcal{Z}_{h,t} + \Phi_{m,t}\sigma_m d\mathcal{Z}_{m,t})J_{SS} + \sigma_K K_t\left(\sqrt{1-\rho^2}d\mathcal{Z}_{h,t} + \rho d\mathcal{Z}_{m,t}\right)J_{KS} \\
&\quad -\frac{1+\gamma}{2\gamma J_S}\left(J_{SS}^2(dS_t)^2 + 2J_{SS}J_{SK} < dS_t, dK_t > + J_{SK}^2(dK_t)^2\right) \\
&= (\zeta - r)J_S dt + (\Phi_{h,t}J_{SS} + K_t J_{KS})\epsilon_K d\mathcal{Z}_{h,t} + (\Phi_{m,t}J_{SS} + \beta^{FB}K_t J_{KS})\sigma_m d\mathcal{Z}_{m,t} \\
&\quad -\frac{1+\gamma}{\gamma J_S}\left(\frac{(\epsilon_K\Phi_h)^2 + (\sigma_m\Phi_m)^2}{2}J_{SS}^2 + \left(\epsilon_K^2\Phi_h + \rho\sigma_K\sigma_m\Phi_m\right)KJ_{SS}J_{SK} + \frac{\sigma_K^2 K^2}{2}J_{SK}^2\right).
\end{aligned}$$

Substituting the optimal idiosyncratic risk hedging rule Φ_h given by (17) and the optimal stock-market portfolio allocation Φ_m given by (18), we obtain

$$\frac{dC_t}{-\frac{\zeta^{\frac{1}{\gamma}}}{\gamma}J_S^{-\frac{1+\gamma}{\gamma}}} = (\zeta - r)J_S dt - \eta J_S d\mathcal{Z}_{m,t} - \frac{\eta^2(1+\gamma)}{2\gamma}J_S dt\,. \tag{B.15}$$

Recall that $C = \left(\frac{J_S(K,S)}{\zeta}\right)^{-\frac{1}{\gamma}}$, so that

$$dC_t = -\frac{\zeta^{\frac{1}{\gamma}}}{\gamma}J_S^{-\frac{1}{\gamma}}\left[\left((\zeta - r) - \frac{\eta^2(1+\gamma)}{2\gamma}\right)dt - \eta d\mathcal{Z}_{m,t}\right] = \frac{C_t}{\gamma}\left[\left(\frac{\eta^2(1+\gamma)}{2\gamma} - (\zeta - r)\right)dt + \eta d\mathcal{Z}_{m,t}\right]. \tag{B.16}$$

Finally, applying Ito's formula to (B.16), we obtain

$$C_{t+s} = C_t e^{-(\zeta - r)s/\gamma}\exp\left[\frac{1}{\gamma}\left(\frac{\eta^2 s}{2} + \eta(\mathcal{Z}_{m,t+s} - \mathcal{Z}_{m,t})\right)\right]. \tag{B.17}$$

B.2 Equivalence

Having characterized the optimal contract in terms of the entrepreneur's promised certainty-equivalent wealth W, we show next how to implement the optimal contract by flipping the optimal contacting problem on its head and considering a dynamic entrepreneurial finance problem, where the entrepreneur owns the firm's productive, illiquid capital stock and chooses consumption and corporate investment by optimally managing liquidity and risk subject only to satisfying the endogenous liquidity constraint. A key observation is that the entrepreneur's inalienability constraints naturally translate to endogenous liquidity constraints in the entrepreneur's problem.

The optimization problem for the entrepreneur is equivalent to the optimal contract problem for the investor in (47) if and only if the borrowing limits, $\underline{S}(K)$, are such that:

$$\underline{S}(K) = -P(K, \underline{W})\,, \tag{B.18}$$

where $P(K, \underline{W})$ is the investor's value when the entrepreneur's inalienability constraint binds. We characterize the implementation solution by first solving the investor's problem in (51) and then imposing the constraint (B.18).

The optimal contracting problem gives rise to the investor's value function $F(K, V)$, with the promised utility to the entrepreneur V as the key state variable. The investor's value $F(K, V)$ can be expressed in terms of the entrepreneur's promised certainty-equivalent wealth W, $P(K, W)$. The optimization problem for the entrepreneur gives rise to the entrepreneur's value function $J(K, S)$, with $S = -P(K, W)$ as the key state variable. Equivalently, the entrepreneur's objective is her certainty equivalent wealth $M(K, S)$ and the relevant state variable is her savings $S = -P$.

The following relations between s and w hold:

$$s = -p(w) \text{ and } m(s) = w\,. \tag{B.19}$$

The standard chain rule implies:

$$m'(s) = -\frac{1}{p'(w)} \text{ and } m''(s) = -\frac{p''(w)}{p'(w)^3}\,. \tag{B.20}$$

Next, we demonstrate the equivalence between the two problems by showing first that when substituting $s = -p(w)$ into the ODE for $m(s)$, we obtain the ODE for $p(w)$, and vice versa. Substituting (B.19) and (B.20) into the ODE (34) for $m(s)$, we obtain the ODE (64) for $p(w)$. Substituting (B.19) and (B.20) into consumption and investment policies (23) and (24) in the primal problem, we obtain the optimal consumption (56) and investment policies (57) in the contracting problem. Substituting (B.19) and (B.20) into (45) and (46), the boundary conditions for $m(s)$, we obtain (68) and (71), the boundary conditions for $p(w)$.

B.3 Comparative Statics with respect to α

Under inalienability, two conditions pin down the endogenous borrowing limit $\underline{s}$: first, the zero-volatility condition for s at $\underline{s}$ given in (32); and second, the value-matching condition for the entrepreneur between quitting or staying with the firm, $m(\underline{s}) = \alpha m(0)$. A lower value of α means

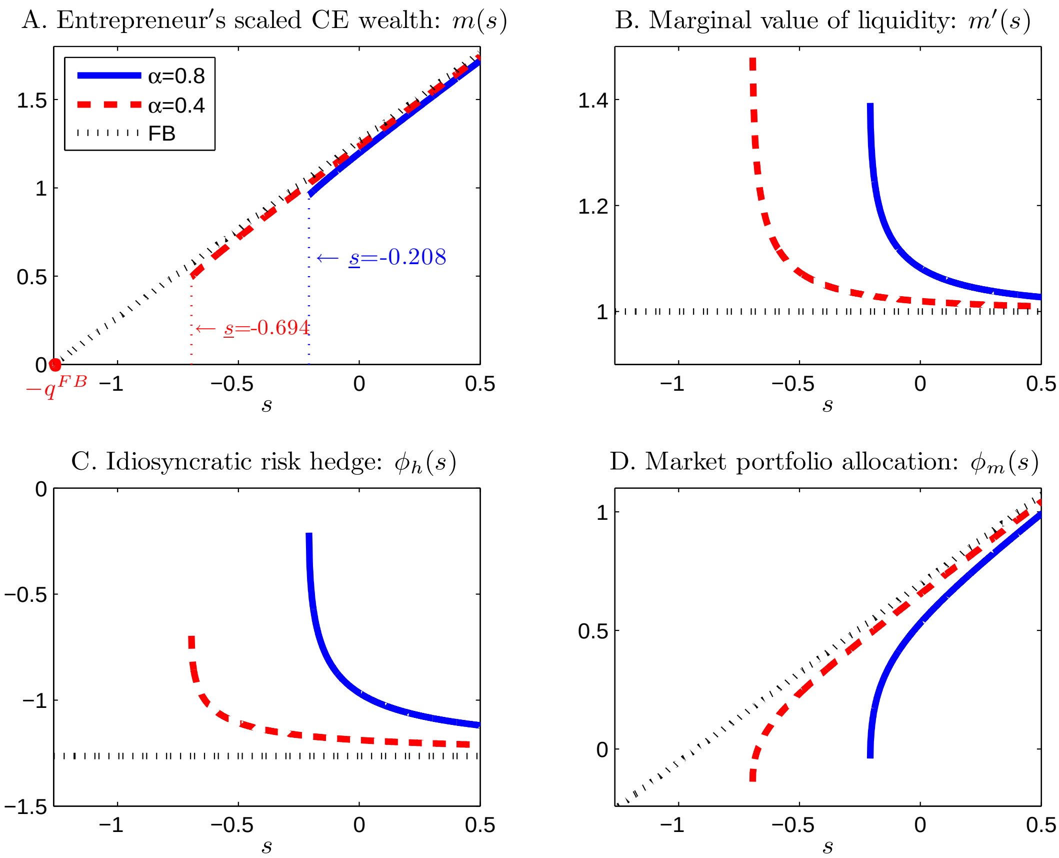

Figure 10: **Comparative Statics with respect to α.** A lower value of α means a higher debt capacity $|\underline{s}|$. For example, $|\underline{s}| = 0.208$ for $\alpha = 0.8$ and $|\underline{s}| = 0.694$ for $\alpha = 0.4$.

a lower outside option value for the entrepreneur, which makes it possible to extend a larger line of credit to the firm. Figure 10 compares the solution for the baseline parameter values where $\alpha = 0.8$ with the situation where $\alpha = 0.4$. When α drops from 0.8 to 0.4, the credit limit increases from $|\underline{s}| = 0.208$ to $|\underline{s}| = 0.694$, and the marginal value of liquidity at $s = -0.208$ decreases significantly from $m'(-0.208) = 1.394$ to $m'(-0.208) = 1.029$. The entrepreneur invests more, gets a higher compensation, hedges more idiosyncratic risk ($|\phi_h(s)|$ is higher), and takes a larger stock market portfolio position ($|\phi_m(s)|$ is higher). Stated differently, for a given level of s the entrepreneur puts more weight on long-run net-worth maximization than on short-term distress-cost minimization when she has less attractive outside options (a lower α).

B.4 Autarky as the Entrepreneur's Outside Option

Let $\widehat{J}(K_t)$ denote the entrepreneur's value function under autarky defined as follows,

$$\widehat{J}(K_t) = \max_{I} \mathbb{E}_t \left[\int_t^{\infty} \zeta e^{-\zeta(v-t)} U(C_v) dv \right] , \tag{B.21}$$

where autarky implies that the entrepreneur's consumption C_t equals output Y_t, in that

$$C_t = Y_t = A_t K_t - I_t - G(I_t, K_t) \, . \tag{B.22}$$

The following proposition summarizes the main results.

Proposition 2 *Under autarky, the entrepreneur's value function $\widehat{J}(K)$ is given by*

$$\widehat{J}(K) = \frac{(b\widehat{M}(K))^{1-\gamma}}{1-\gamma} , \tag{B.23}$$

where b is given by (20) and $\widehat{M}(K)$ is the entrepreneur's certainty equivalent wealth given by

$$\widehat{M}(K) = \widehat{m} K , \tag{B.24}$$

where

$$\widehat{m} = \frac{(\zeta(1 + g'(\widehat{i}))(A - \widehat{i} - g(\widehat{i}) \,)^{-\gamma})^{\frac{1}{1-\gamma}}}{b} , \tag{B.25}$$

and $\widehat{i}$ is the optimal investment-capital ratio solving the following implicit equation:

$$\zeta = \frac{A - \widehat{i} - g(\widehat{i})}{1 + g'(\widehat{i})} + (\widehat{i} - \delta_K)(1-\gamma) - \frac{\sigma_K^2 \gamma (1-\gamma)}{2} \, . \tag{B.26}$$

Proof of Proposition 2. The value function $\widehat{J}(K)$, satisfies the following HJB equation:

$$\zeta \widehat{J} = \max_{I} \, \zeta \frac{C^{1-\gamma}}{1-\gamma} + (I - \delta_K K) \widehat{J}_K + \frac{\sigma_K^2 K^2}{2} \widehat{J}_{KK} \, . \tag{B.27}$$

Using $\widehat{J}(K) = \frac{(b\widehat{M}(K))^{1-\gamma}}{1-\gamma}$ and $c = A - i - g(i)$, we have

$$\zeta = \max_{i} \, \zeta \left(\frac{A - i - g(i)}{\widehat{m} b} \right)^{1-\gamma} + (i - \delta_K)(1-\gamma) - \frac{\sigma_K^2 \gamma (1-\gamma)}{2} \, . \tag{B.28}$$

Using the FOC for i, we have (B.25). Substituting (B.25) into (B.28), we obtain i given by (B.26). The entrepreneur's value function $J(K, S)$ satisfies the following condition:

$$J(K_t, S_t) \geq \widehat{J}(K_t)\,, \tag{B.29}$$

which implies $M(K_t, S_t) \geq \widehat{M}(K_t)$ and $M(K_t, \underline{S}_t) = \widehat{M}(K_t)$, By using the homogeneity property, we show that the lower boundary $\underline{s}$ satisfies: $m(\underline{s}) = \widehat{m}$.

C Persistent Productivity Shocks

By using the dynamics of S_t given by (81), we obtain the HJB equation for the value function $J^L(K, S)$ in State L given by (82) and the following HJB equation for $J^H(K, S)$ in State H:

$$\begin{aligned}
\zeta J^H(K, S) \quad = \quad \max_{C, I, \Phi_h, \Phi_m, \Pi^H} \quad & \zeta U(C) + (I - \delta_K K) J_K^H + \frac{\sigma_K^2 K^2}{2} J_{KK}^H \\
& + \left(rS + \Phi_m(\mu_m - r) + A^H K - I - G(I, K) - C - \lambda^H \Pi^H\right) J_S^H \\
& + \left(\epsilon_K^2 \Phi_h + \rho\sigma_K \sigma_m \Phi_m\right) K J_{KS}^H + \frac{(\epsilon_K \Phi_h)^2 + (\sigma_m \Phi_m)^2}{2} J_{SS}^H \\
& + \lambda^H [J^L(K, S + \Pi^H) - J^H(K, S)]\,.
\end{aligned} \tag{C.1}$$

We then obtain the following main results:

Proposition 3 *In the region $s > \underline{s}^L$, $m^L(s)$ satisfies the following ODE:*

$$\begin{aligned}
0 \quad = \quad \max_{i^L, \pi^L} \quad & \frac{m^L(s)}{1-\gamma}\left[\gamma\chi m^{L\prime}(s)^{\frac{\gamma-1}{\gamma}} - \zeta\right] + \left[rs + A^L - i^L - g(i^L) - \lambda^L \pi^L(s)\right] m^{L\prime}(s) \\
& - \left(\frac{\gamma\sigma_K^2}{2} - \rho\eta\sigma_K\right) \frac{m^L(s)^2 m^{L\prime\prime}(s)}{m^L(s) m^{L\prime\prime}(s) - \gamma m^{L\prime}(s)^2} + \frac{\eta^2 m^{L\prime}(s)^2 m^L(s)}{2(\gamma m^{L\prime}(s)^2 - m^L(s) m^{L\prime\prime}(s))} \\
& + (i^L - \delta)(m^L(s) - s m^{L\prime}(s)) + \frac{\lambda^L m^L(s)}{1-\gamma}\left(\left(\frac{m^H(s+\pi^L)}{m^L(s)}\right)^{1-\gamma} - 1\right),
\end{aligned} \tag{C.2}$$

subject to the following boundary conditions:

$$\lim_{s\to\infty} m^L(s) = q_L^{FB} + s\,, \qquad m^L(\underline{s}^L) \;=\; \alpha m^L(0)\,, \qquad \text{and} \quad m^{L\prime\prime}(\underline{s}^L) = -\infty\,, \tag{C.3}$$

where q_L^{FB} is provided below in Proposition 4. The insurance demand $\pi^L(s)$ solves:

$$m^{H\prime}(s + \pi^L) = m^{L\prime}(s)\left(\frac{m^L(s)}{m^H(s+\pi^L)}\right)^{-\gamma}, \tag{C.4}$$

as long as $\pi^L(s)$ satisfies $\pi^L(s) \geq \underline{s}^H - s$. Otherwise, the entrepreneur sets $\pi^L = \underline{s}^H - s$. We have another set of analogous equations and boundary conditions for $m^H(s)$ and $\pi^H(s)$ in state H.

The following proposition summarizes the solutions for the first-best case.

Proposition 4 *Under the first-best, the firm's value $Q_n^{FB}(K)$ in state $n = \{H, L\}$ is proportional to K: $Q_n^{FB}(K) = q_n^{FB} K$, where q_H^{FB} and q_L^{FB} jointly solve:*

$$\left(r + \delta - i_L^{FB}\right) q_L^{FB} = A^L - i_L^{FB} - g(i_L^{FB}) + \lambda^L \left(q_H^{FB} - q_L^{FB}\right), \tag{C.5}$$

$$\left(r + \delta - i_H^{FB}\right) q_H^{FB} = A^H - i_H^{FB} - g(i_H^{FB}) + \lambda^H \left(q_L^{FB} - q_H^{FB}\right), \tag{C.6}$$

and where i_L^{FB} and i_H^{FB} satisfy: $q_L^{FB} = 1 + g'(i_L^{FB})$ and $q_H^{FB} = 1 + g'(i_H^{FB})$. The insurance demands in state L and H are respectively given by: $\pi^L = q_H^{FB} - q_L^{FB}$ and $\pi^H = q_L^{FB} - q_H^{FB}$.