Monitor Reputation and Transparency*

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Abstract

We study the disclosure policy of a regulator who oversees a monitor with reputation concerns. The monitor faces a strategic agent, who chooses how much to manipulate in response to the monitor's reputation. Manipulation increases the arrival rate of a "bad news" signal, but the agent manipulates less for higher reputations. This leads to a unique "Shirk-Work-Shirk" equilibrium in which the monitor only exerts effort for intermediate reputations.

Instead of providing transparency, the regulator's disclosure keeps the monitor's reputation intermediate. This requires releasing information which damages reputation. The regulator reveals delayed bad news for low reputations, delayed good news for intermediate reputations, but nothing for high reputations. Her policy hence becomes more lenient over time.

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1 Introduction

Monitoring is arguably the main task performed by intermediaries such as auditors, credit rating agencies (CRAs) and banks (Diamond (1991); Hansen and Torregrosa (1992)). In practice, a monitor's effort is unobservable, which potentially undermines the monitor's incentive to work diligently. The economics literature has nonetheless observed that reputation *per se* can provide the monitor a strong incentive to work (Chemmanur and Fulghieri (1994); Carter et al. (1998); Mathis et al. (2009)). When a monitor fails to detect a firm's problems—the argument goes—these problems eventually become apparent, thereby damaging the monitor's reputation and, in some cases, even driving the monitor out of the market (see, e.g., "From Sunbeam to Enron, Andersen's Reputation Suffers", NYT, 2001).

Despite the compelling logic of this mechanism, recent events (notably, the financial crisis of 2007-2008) have led the public to believe that reputational incentives are insufficient, generating a demand for regulation and a call to "monitor the monitors." (The creation of the Public Company Accounting Oversight Board (PCAOB) created by the Sarbanes—Oxley Act of 2002 to oversee the audits of public companies is a good example of this trend.)

In many industries, as part of their regulation effort, regulators collect information about monitor quality. When the information is negative, the regulator faces a dilemma: should she disclose the information or conceal it from the public? Scholars and market pundits often argue that regulators should be transparent and disclose any information they learn about the monitor's quality, including negative information, because otherwise the monitor—anticipating the regulator's opacity—would underprovide quality.[1] However, disclosing negative information about the monitor's quality may damage the monitor's reputation, affecting its incentive to provide quality in the future. The regulator might thus consider withholding negative information that if disclosed, would compromise the monitor's reputation and, in some cases, threaten his survival.

For example, in the audit market, the PCAOB conducts regular inspections to assess an auditor's quality control system. The outcome of these inspections remains private unless the monitor fails to address the defects within one year, in which case the outcome of the inspection is disclosed (for example, Sarbanes-Oxley Act Section 104 prescribes "no portions of the inspection report that deal with defects in the quality control systems of the firm under inspection shall be made public if those defects are addressed by the firm not later than 12 months after the date of the inspection report.").

In this paper, we study why a commitment to delay or even conceal information about

[1] Indeed, many government programs implement this transparency principle. For example, the Los Angeles county restaurant hygiene program monitors restaurant hygiene randomly and requires the restaurants to display the outcome of the inspection immediately on their windows. See Jin and Leslie (2002).

firm quality can be desirable from a (benevolent) regulator's perspective. We begin by studying a reputation game that features three players: a monitor, a (client) firm, and the client firm's manager. The monitor is a long-run player with reputation concerns whose quality is unknown. The manager of the client firm is a short-run player who may engage in "manipulation" but is subject to the monitor's scrutiny. Manipulation is unobservable, but the monitor can detect it. If the monitor does not detect the manipulation, the manager obtains a private benefit, but his manipulation may randomly cause a negative shock to the firm's value (e.g., a restatement, default, etc.). This shock arrives at a random time, and its intensity is proportional to the magnitude of manipulation. The monitor exerts hidden effort to detect the manager's manipulation and thus prevent the subsequent negative shock. The monitor may either be good or strategic. A good monitor always detects the manipulation. By contrast, the probability that a strategic monitor detects the manipulation depends on the effort he exerts. In each period, the firm hires the monitor and pays him a fee that is set competitively based on his relative ability and his incentives to detect the manipulation to prevent negative shocks.

This game features a unique Markov perfect equilibrium in which the monitor's reputation and his behavior evolve over time based on the history of shocks (or lack thereof.) The structure of the equilibrium depends on the severity of the moral hazard issue facing the monitor, as captured by the monitor's cost of effort. When the moral hazard issue is severe (high cost), the monitor shirks at any reputation level, and the manager engages in intense manipulation. As a consequence, negative shocks are frequent and monitor fees are low. When the moral hazard issue is moderate (moderate cost), the monitor shirks when his reputation is below a threshold—because prospects are low—but also shirks when his reputation is above a threshold—because manipulation and shocks are less prevalent. Extreme reputations, whether high or low, weaken the monitor's incentives. Finally, when the moral hazard issue is mild (low cost), the monitor works even when his reputation is close to zero. In this case, the monitor's behavior is relatively insensitive to reputation: that is, the monitor exerts relatively high effort regardless of his reputation, as long as his reputation is not too high. However, as in the moderate cost case, the monitor's effort vanishes when his reputation reaches a very high level.

The Shirk-Work-Shirk structure of the equilibrium is a novel aspect of our analysis. If manipulation intensity were independent of the monitor reputation, the equilibrium would feature the Shirk-Work structure arising in reputation settings under bad news (see e.g. Board and Meyer-terVehn (2013)). But because manipulation weakens as the monitor's reputation improves, monitoring incentives also go down, explaining the existence of the upper shirking region.

Then, we study whether and how a regulator should disclose information about the monitor's quality over time to provide incentives to the monitor and mitigate the manager's manipulation. First, we show that if the moral hazard issue is severe, such that the monitor is expected to always shirk, any disclosure policy is superfluous: positive disclosures about monitor quality mitigate manipulation, but negative disclosures exacerbate manipulation. On average, these effects cancel out.

When the moral hazard issue is mild introducing disclosure can be valuable. In general, a higher reputation benefits the regulator because it preempts manipulation, since the manager believes detection is more likely and vice-versa. This might suggest that the optimal disclosure policy is to withhold negative information about monitor quality and disclose positive information. However, this policy is never optimal, being actually dominated by a non-disclosure commitment (full opacity).

Disclosing monitor quality in a deterministic fashion is never optimal because it destroys the incentive power of reputation, leading to the same regulator value as that arising when the monitor always shirks (severe moral hazard case). For disclosure to be valuable, disclosures should arrive randomly. Furthermore, in general, the rate of good and bad news is different; for some reputations, good news are concealed and for other reputations the opposite happens.

The regulator's disclosure policy must be carefully designed. In choosing a disclosure policy, the regulator faces the following trade-off. Disclosing information may weaken monitoring incentives, since, from the monitor's perspective, disclosure is likely to erode his reputation by eventually revealing his type. In turn, by weakening monitoring incentives, disclosure exacerbates the agent's manipulation since the agent know he is less likely to be detected. However, random disclosure allows the regulator to influence the evolution of monitor reputation and keep it in the work region, where monitoring incentives are strong.

To better understand this effect, recall that the monitor starts shirking when his reputation falls below an (endogenous) threshold. At that point, a negative reputation shock increases manipulation. By contrast, a positive reputation shock decreases manipulation. However, the latter effect is stronger because, close to the threshold, an increase in reputation stimulates monitoring effort which –since this is anticipated by the agent– further mitigates manipulation. In choosing his disclosure policy, the regulator thus faces a conflict between two goals: providing monitoring incentives and maximizing the time within the work region.

We show that the regulator may benefit from a policy that discloses bad news, randomly, for relatively low reputations, and discloses good news, randomly, for relatively high reputations. Reputational incentives are strongest for intermediate reputations. The value of information for the regulator is hence to bring reputations into the region where reputational incentives "do their job". For low values, this means disclosing bad news after a delay. This does not hurt the good type and allows the regulator to induce more effort from the strategic type if there is no disclosure. For high values, this means disclosing good news after a delay. This lowers reputations for the strategic type because for him, the good news never materializes, which is beneficial for the regulator since it keeps the reputation inside the region where there is effort for a longer period of time. Of course, this is anticipated and may destroy incentives, but for high reputations, the incentive effect is second order.

The optimal disclosure policy prescribes no disclosure for top reputations. This no-disclosure-at-the-top acts as an incentive device that rewards the monitor for his past performance. This result illustrates a general principle, namely that the regulator is more willing to interfere and disclose information for lower reputations because of the relatively weaker impact on incentives (relative to higher reputations).

Literature The monitoring role of banks was first studied by Diamond (1991) and Hansen and Torregrosa (1992). The reputational incentive of monitors is considered by Bar-Isaac (2003) for sellers, Biglaiser (1993) for middlemen, Mathis et al. (2009) for credit rating agencies, Chemmanur and Fulghieri (1994) for investment banks, and Carter et al. (1998) for underwriters. We expand these models by introducing an agent who optimally responds to the monitor's reputation and conjectured monitoring effort.

Our model features perfect bad news as in Board and Meyer-terVehn (2013), which leads to a shirk-work equilibrium in their paper. Our result is different because we explicitly model the agent's behavior, which changes the dynamics of reputation.[2] Also, our monitor's type is determined ex-ante and does not change.

Varas et al. (2017) study the optimal monitoring mechanism when the agent has reputation concerns and the principal derives utility from learning the agent's type, but inspections are costly.[3] Monitoring plays a dual role: learning and incentive provision. In our setting, disclosure provides incentives to two agents at the same time: the monitor and the monitored. Also, there is no role for information acquisition –the principal learns the agent's type for free– and can implement disclosure policies that depend on the agent's type.

Several papers study the problem of a designer who decides how to reveal the *actions* of a player with reputation concerns, or, equivalently, how much noise to introduce in the market's observations: Hauser (2016), Di Pei (2016), and Lillethun (2017). Our paper differs

[2]See also Dilmé (2014) and Dilmé and Garrett (2015) for related work. Dilmé and Garrett (2015) features an inspector with switching costs and the reputation is about the inspector's state.

[3]See also Rahman (2012) who studies how to incentivize a monitor whose inspections are both unobservable and costly.

in two respects. First, it is about monitoring and hence features an agent who optimally responds to the monitor's reputation and his equilibrium effort. This aspect is absent in the papers above. Second, in our model disclosure is about the monitor's *type*, whereas in the above papers, it is about the monitor's actions. We believe our disclosure setting better fits applications such as regulatory disclosure and stress testing. Our model also leads to qualitatively different predictions. The shirk-work-shirk equilibrium is not present in the above papers and our disclosure policy affects reputation directly and aims to keep the monitor's reputation at an intermediate level.[4]

Holmstrom (1999) is the first paper to illustrate why neither perfect information nor no information are optimal disclosure systems for incentive provision purposes. Our model shares a similar flavor, because the monitor knows his type, anticipates future disclosures, and reacts to them optimally. Horner and Lambert (2016) considers a continuous time version of Holmstrom (1999) and examines the types of information systems that stimulate agent effort. Our setting is difference since it is not a career concerns model and since disclosure is about the monitor's type. Che and Mierendorff (2016) study optimal information acquisition by a decision maker who has limited attention, and can either acquire good news evidence or bad news but in his setting information does not play an incentive role. Che and Hörner (2017) consider the optimal rating system to incentivize users to learn about a product collaboratively. While they are interested in disclosures that incentivize social learning, we study disclosures that discourage manipulations.

Our paper is related to the recent Bayesian persuasion literature (see Goex and Wagenhofer (2009); Kamenica and Gentzkow (2011); Bertomeu and Cheynel (2015)), which for the most part is static,[5] in that our regulator can commit to a dynamic disclosure policy. Our model proves that depending on the regulator's priors, the optimal disclosure system may disclose bad news and conceal good news. This is related to the literature on the optimality of conservatism in firm's disclosures (see e.g., Gigler and Hemmer (2001); Gigler et al. (2009); Caskey and Hughes (2011); Bertomeu et al. (2017)).

Finally, a recent literature studies stress tests for banks (e.g. Goldstein and Leitner (2015), Shapiro and Skeie (2015), and Orlov et al. (2017)). Our paper adds to this literature by considering the effects of stress tests on the bank's incentive to build a reputation. This channel has so far been absent.

[4]Ekmekci (2011) studies Perfect Bayesian Equilibria in a repeated game with a "rating system" which maps a long-run players actions into random signals. He finds conditions so that there exists a rating system and an equilibrium which yields approximately the commitment payoff for the long-run player.

[5]See Ely (2017), Ely and Szydlowski (2017), and Orlov et al. (2018) for exceptions. These papers to not feature reputation concerns.

2 Model

In this section, we study a dynamic game between a long-run monitor and a sequence of firm managers who may opportunistically engage in manipulation. The analysis of the regulator's optimal disclosure policy is deferred until Section 5.

Time is continuous. There are three players who, for concreteness, we label monitor, firm, and manager. The firm hires the monitor to scrutinize the firm manager's behavior and, in particular, to prevent the manager from engaging in manipulation.

The monitor is a long-run player of two types: good and strategic. The monitor knows his own type, but the market is uncertain about it. The good type always detects the manager's manipulation.

In each period, a new manager comes in. The manager chooses how much to manipulate to maximize his own private gain based on his perception of the monitor's quality. However, the monitor may detect the manipulation before it leads to a public shock to firm value. Specifically, if hired by the firm, the (strategic) monitor can choose how much effort to expend scrutinizing the manager's behavior. The strategic monitor chooses $a \in [0, 1]$ at cost ca and detects the manipulation of the manager with probability a. Because the good monitor is a passive actor, hereafter by "monitor", we mean the strategic type.

The manager's choice of manipulation is denoted $m_t \in [0, 1]$. It is subject to a quadratic cost $\frac{1}{2}m_t^2$. If the manager is not detected, he receives a private benefit m_t, so his payoff is $m_t - \frac{1}{2}m_t^2$. If detected, he receives no private benefit but bears the cost. Thus, the manager manipulation solves

$$m_t = \arg\max \left(1 - x_t\right)\left(1 - \hat{a}_t\right) m - \frac{m^2}{2}$$

where $\hat{a}_t$ is the manager's conjecture of the monitor's effort a_t, and x_t is the manager's belief that the monitor is good. The first term is the likelihood the manager is not detected, which happens if the monitor is strategic (probability $1 - x_t$) and the monitoring fails (probability $1 - \hat{a}_t$). This leads to the optimal manipulation strategy

$$m\left(x_t, \hat{a}_t\right) = \left(1 - x_t\right)\left(1 - \hat{a}_t\right). \tag{1}$$

This is intuitive: if the manager expects the monitor to monitor more, he manipulates less; if he believes the monitor is good, hence likely to detect manipulation, he manipulates less as well.

Detecting the manager's manipulation benefits the firm because when undetected, the manipulation leads randomly to a negative public signal that reduces firm value. Specifically, if the manager is not detected, a bad signal arrives with Poisson intensity λm_t. Henceforth, we refer to this arrival as a "loss."

Since the good monitor always detects the manager's manipulation and prevents a loss, this is a model of perfect bad news: a loss fully reveals that the monitor is not good. By contrast, when no loss is observed, the market can't distinguish whether the manager was detected or, for random reasons, the manipulation did not trigger a loss.

The monitor is a monopolist who profits from his ability to detect manipulation. If a firm does not hire the monitor, the probability of a bad signal is λ. In other words, without monitoring (i.e., when $a = 0$ for both types) we have $m = 1$, and bad news arrive with likelihood λ each instant.

If the monitor is hired when his reputation is x_t, the perceived probability of a loss is

$$\lambda \left(1 - x_t\right) \left(1 - \hat{a}_t\right) m_t \left(x_t, \hat{a}_t\right) = \lambda m \left(x_t, \hat{a}_t\right)^2 .$$

Thus, hiring the monitor reduces the chances of a loss for two reasons: first, the presence of the monitor mitigates the manager's manipulation ($m < 1$), and second, the monitor detects the manipulation before it triggers a loss.

The firm is willing to hire the monitor because the monitor can reduce the prevalence of losses, which are costly to the firm. Specifically, a loss generates a negative cash flow to the firm equal to $-\frac{\alpha}{\lambda}$. The monitor's fee p_t is equal to the perceived value of the monitor's service:

$$p_t = \alpha \left(1 - m \left(x_t, \hat{a}_t\right) \left(1 - x_t\right) \left(1 - \hat{a}\right)\right) . \tag{2}$$

In essence, this means that the monitor, being a monopolist, extracts the entire surplus of the transaction with the firm. We thus aim to capture that some monitors in the marketplace, can charge premium fees thanks to their superior reputation.

The fee is proportional to the expected reduction in loss costs caused by the decision to hire the monitor (relative to hiring a monitor with no reputation). The reduction in the likelihood of losses is 1 minus the expected arrival rate of bad news conditional on monitor reputation, monitoring intensity, and the manager's manipulation decision. The fee is increasing in monitor reputation and monitoring effort.[6]

Using Bayes' law, the belief updating rule for the monitor's reputation, when no loss is observed, is given by[7]

$$\dot{x}_t \equiv \frac{dx_t}{dt} = \lambda x_t m \left(x_t, \hat{a}_t\right)^2 . \tag{3}$$

[6] Titman and Trueman (1986) show that firms may signal high quality by hiring more expensive but better informed monitors. Feltham et al. (1991) document empirically that riskier firms demand higher quality monitors.

[7] A sketch: For a small time $h > 0$ we have $x_{t+h} = \frac{x_t}{x_t + (1-x_t)e^{-\lambda m_t (1-\hat{a}_t)}}$. Here, $\lambda m_t(1 - \hat{a}_t)$ is the arrival rate when facing the strategic type. Taking the limit of $(x_{t+h} - x_t)/x_t$ yields Equation 3.

So, in the absence of a loss, the monitor's reputation drifts upwards: naturally, the absence of a loss is interpreted as a positive signal of monitor quality. As mentioned above, reputation x_t drops to zero when there is a loss.

As a benchmark, consider the case when the monitor has the ability to commit to exerting full effort. If the monitor could commit to full effort $a_t = 1$ forever, he would derive a value $V^{com} = \frac{\alpha - c}{r}$. We refer to V^{com} as the value of commitment and assume that $V^{com} > 0$ or $\alpha > c$.

3 Discussion

In our model, any manipulation that the monitor does not get to detect is potentially discovered later, leading to a negative shock to firm value. We can either think of this ex-post monitoring mechanism as representing the scrutiny of a regulator, such as the SEC, or as arising from the firm's own internal control mechanisms. (Alternatively, λ may capture litigation risk and whistle-blowing activities.) For the most part, we treat λ as an exogenous parameter. In Section 5, we discuss the possibility that a regulator provides additional information about the monitor.

In the model, the monitor cares about reputation. Reputation is relevant because it determines demand for the monitor and, more specifically, the fee the monitor can charge the firm. In practice, whether or not reputation drives audit fees is an open question, but the assumption that reputation has economic value in the audit market is hopefully uncontroversial.

We model the relationship between the monitor and the manager as a one-shot interaction, though in many applications, this is better described as a long-term relationship. Take the case of auditors. Conventional wisdom and empirical evidence suggest that investors interpret auditor rotation as a negative signal, presumably indicating that the firm is concealing negative information or engaging in "opinion shopping."[8] We model this relationship as a one-shot interaction to abstract away from repeated-games-like considerations and focus on our main question: namely, how a regulator can influence the evolution of monitor reputation via disclosure to mitigate manipulations.

We consider a parsimonious setting where monitor fees and manipulation evolve based on market perceptions of monitor quality. But we shut down other important considerations, such as collusion between the monitor and the manager or complex information asymmetry environments, where the manager has private information about monitor quality that the

[8] Dye (1991) shows that monitor rotation cannot happen when the monitor and the firm possess the same information and this information can be communicated to investors through financial statements.

market ignores.

Unlike Board and Meyer-terVehn (2013), we consider a setting where the monitor type is permanent. In our setting, the monitor effort does not affect its own type, but effort still has reputational consequences because it preempts the arrival of bad news. So, on some level, the monitor exerts effort to conceal his type from the public. Of course, one could think of more complex/realistic environments where the monitor type changes over time in a random fashion but is affected by effort. This assumption would make the analysis less tractable.

Our model admits several interpretations. The simplest one is between a monitor and a client firm. Alternatively, we can interpret our model as a game between the firm's board and the manager. The board monitors the manager. The manager can manipulate, and the firm sells a product. The value of the product depends on the degree of the manager's manipulation. Specifically, the product breaks at rate m_t and loses value. The board has reputation concerns. In this situation, the monitor is the firm, and the market pays a price that is proportional to the board's reputation.

Another reinterpretation is that of a bank who monitors a borrower. The borrower can put effort into his firm or not. If he does not, then the loan defaults at a Poisson rate depending on the effort. There is a competitive market for loans, and investors pay the bank based on its reputation for monitoring. A similar consideration applies to underwriters, who monitor loans before they are repackaged into, say, mortgage-backed securities.

4 Analysis

As is customary in reputation games, we focus on Markov perfect equilibria (MPE). Given the manager's manipulation strategy $m(x, \hat{a}(x))$, the monitor value solves the following HJB equation:

$$rV(x) = \max_a p(x) - ca + \dot{x}V'(x) + \lambda m(x, \hat{a}(x))(1-a)(V(0) - V(x)), \tag{4}$$

subject to the updating rule in equation 3 and the price formation rule in equation 2. The right-hand side of the above equation captures the return to the monitor. In each period, the monitor collects the fee p net of monitoring cost ca and also receives the capital gains associated with changes in his reputation. The latter come in two flavors: the positive drift in reputation arising when there is no loss, and the decrease in reputation caused by a loss, which is equal to $V(x) - V(0)$.

Observe that in any equilibrium, we must have $V(0) = 0$ and $V(1) = \frac{\alpha}{r}$. That is, when reputation is $x = 0$, the monitor has no incentive to exert effort and, consistent with this, the fee is 0 thereafter, leading to zero monitor value. This is the outcome arising when a loss hits because the loss fully reveals that the monitor is strategic. At the opposite extreme, with full reputation ($x = 1$), the manager chooses zero manipulation because he believes detection is perfect. Hence, no loss is expected. Consequently, the monitor shirks but still collects a fee α. Of course, $\frac{\alpha}{r}$ is an upper bound for the monitor value.

Consider the monitor effort strategy $a(\cdot)$. In an MPE, conjectured effort $\hat{a}(\cdot)$ depends on reputation x; hence, we can write the manipulation strategy simply as a function of reputation $m(x, \hat{a}(x)) = m(x)$. Direct inspection of equation 4 reveals that the monitor value is linear in effort a and the marginal net benefit of effort is

$$q(x) \equiv \lambda m(x) V(x) - c.$$

Thus, a marginal increase in effort reduces the probability of a capital loss due to a loss by $\lambda m(x) V(x)$, but it costs the monitor c. Hence, the monitor effort is $a(x) = 0$ when $q(x) < 0$ and $a(x) = 1$ when $q(x) > 0$. Whenever $q(x) = 0$, the monitor is indifferent between any effort level in $[0, 1]$. The next lemma establishes that detection can never be perfect in equilibrium.

Lemma 1. *Monitor effort is always strictly less than 1, or $a(x) < 1$ for all $x \in [0, 1)$.*

This is a standard property of monitoring games: if the monitor exerts full effort at any given reputation level, then the probability of detection is one. As a consequence, the manager does not want to manipulate, but then the monitor does not need to exert effort, leading to a contradiction.

Lemma 1 shows that in equilibrium, the monitor never chooses full effort; hence, there is always a positive likelihood of observing a loss ex-post. However, there are also periods in which the monitor exerts no effort at all. In fact, depending on parameter values, the monitor may shirk always, regardless of his reputation level. The next result studies the possibility of a shirking equilibrium, defined as an equilibrium in which the monitor always shirks, or $a(x) = 0$ for all x.

Proposition 1. *If $\frac{c}{\lambda} > \max_x (1-x) V_s(x)$, there is a unique equilibrium. In this equilibrium the monitor always shirks. The monitor value V_s solves the HJB equation*

$$rV_s(x) = \alpha\left(1 - (1-x)^2\right) + V_s'(x)\lambda(1-x)^2 x - \lambda(1-x) V_s(x), \tag{5}$$

with boundary condition

$$V_s(1) = \frac{\alpha}{r}.$$

The manager manipulation is

$$m(x) = 1 - x,$$

and the fee is

$$p(x) = \alpha\left(1 - (1-x)^2\right).$$

This result suggests that the ability to prevent a loss may not be sufficient incentive for the monitor to work. If c is large and the market is pessimistic about the level of effort the monitor is exerting, then the monitor may be trapped in a situation where the market believes the monitor shirks, fees are low, the manager chooses aggressive manipulation, and the monitor does not put in any effort to detect it due to the low value of his reputation. The widespread nature of manipulation reduces the monitor value, thereby weakening his incentive to exert effort.

Notice that the equilibrium is unique: under the parametric assumptions of the proposition, any equilibrium in which the monitor works would feature a discontinuity in the firm's value function, which is not consistent with the firm being forward-looking. Indeed, in any equilibrium, the value of the firm is continuous at any $x > 0$ because the monitor anticipates the future evolution of reputations and fees.

Next, we explore the possibility of an equilibrium where the monitor works with positive probability. Full shirking cannot be sustained as an equilibrium when the cost of effort c is so low that for some reputation level x, the monitor has a strict incentive to work, even when the market conjectures shirking or when $\lambda(1-x)V_s(x) > c$ for some x. Before characterizing an equilibrium with effort, we let $c \leq \max_x \lambda(1-x)V_s(x)$ and define a threshold x_h as

$$x_h \equiv \max\left\{x : \lambda(1-x)V_s(x) = c\right\}.$$

As we shall demonstrate, in any equilibrium, monitors with reputations above x_h will shirk. The next proposition characterizes the equilibrium when the cost of effort is moderate such that a shirking equilibrium is impossible.

Proposition 2. *If* $\frac{c}{\lambda} < \max_x (1-x)V_s(x)$ *and* $\frac{\lambda(\alpha-c)^{\frac{3}{2}}}{\sqrt{\alpha}}\frac{2}{3\sqrt{3}} < rc$*, there is a unique equilibrium. It takes the form of a shirk-work-shirk equilibrium. That is, there are* $0 < x_l < x_h < 1$ *such that*

$$\begin{aligned} a(x) &= 0 && if \quad x \leq x_l \\ a(x) &\in (0,1) && if \quad x \in (x_l, x_h) \\ a(x) &= 0 && if \quad x \geq x_h. \end{aligned} \tag{6}$$

On (x_l, x_h)*, the agent's manipulation* $m(x)$ *solves the ODE*

$$rc = \lambda(\alpha - c)m(x) - \alpha\lambda m(x)^3 - \lambda c x m'(x)m(x) \tag{7}$$

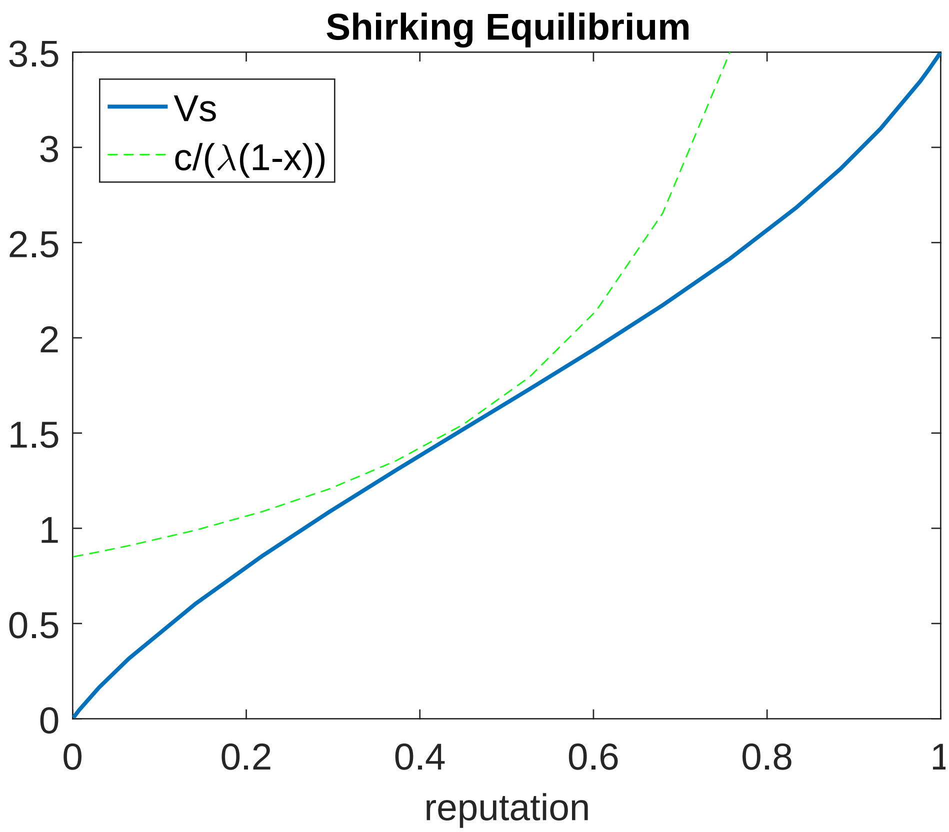

Figure 1: Parameters: $\lambda = 1, r = 0.5, \alpha = 1.75, c = 0.85$. The blue solid line represents the monitor value in a shirking equilibrium, V_s. The green dotted line captures the shirking boundary, $\frac{c}{\lambda(1-x)}$. Since this function lies above $V_s(x)$, the marginal net benefit of effort $\lambda(1-x)V_s(x) - c$ is always non-positive, which verifies the absence of incentives to deviate and exert effort.

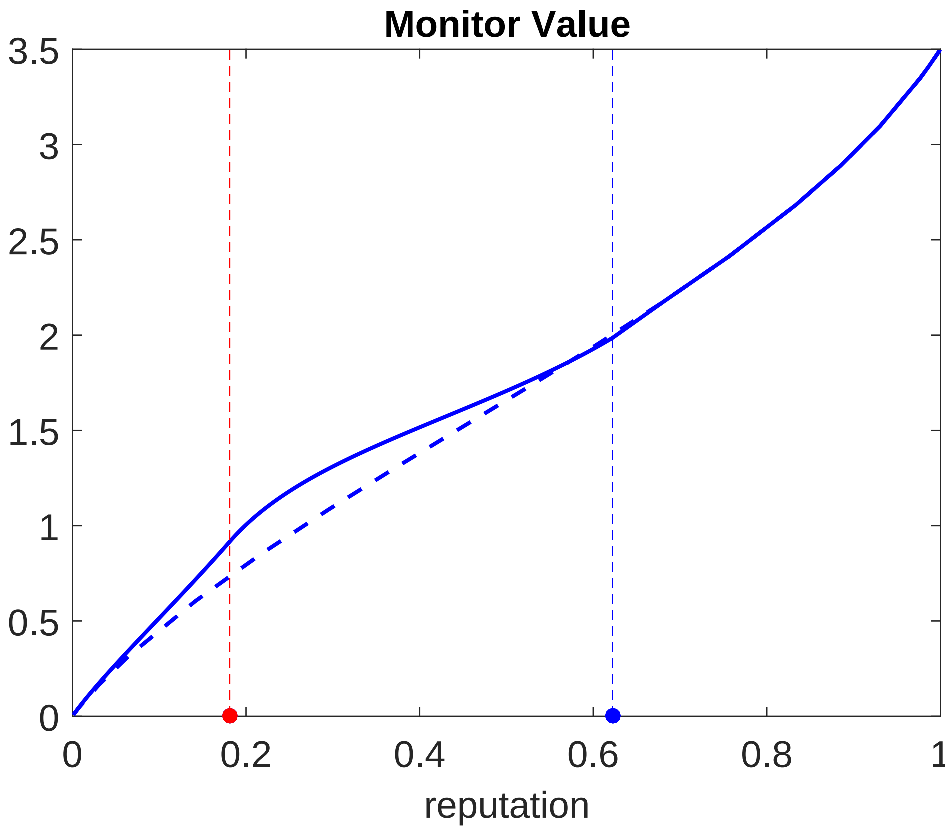

Figure 2: Parameters: $\lambda = 1, r = 0.5, \alpha = 1.75, c = 0.75$. The solid blue line represents the monitor value in the equilibrium with effort. The dotted blue line captures the monitor value in a shirking equilibrium. The interval between the two dots marks the work region.

with boundary conditions $m(x_h) = 1 - x_h$ *and* $m(x_l) = 1 - x_l$. *Outside of that interval, it is given by* $m(x) = 1 - x$. *For any* x, *manipulation is strictly decreasing in reputation.*

The monitor value $V(x)$ *is strictly increasing. It satisfies the shirking ODE* (5) *on* $[x_h, 1]$ *with boundary condition* $V(1) = \frac{\alpha}{r}$ *and on* $[0, x_l]$ *with boundary condition* $V(x_l) = \frac{c}{\lambda(1-x_l)}$. *On* (x_l, x_h), *the monitor's value satisfies the indifference condition* $V(x) = \frac{c}{\lambda m(x)}$.

When c is moderate, the monitor shirks if his reputation is at the extremes (i.e., when reputation is too low or too high) and works otherwise. Hence, very high or low reputations are equally bad for incentives, but for different reasons: while a low reputation monitor shirks because of low fees and poor prospects, the high reputation monitor shirks because losses are unlikely, so effort is not needed. The intuition is as follows. As $x \to 0$, $V(x) \to 0$, so being discovered to be a bad monitor no longer provides any incentives (monitor value is continuous at zero). Similarly, if $x \to 1$, the manager does not manipulate, and as a consequence, there is no bad news generated, but in this case, it is not worth exerting effort.

When the monitor's effort is interior, the monitor is indifferent between working and shirking, which yields the indifference condition $\lambda m(x)V(x) = c$. The monitor anticipates

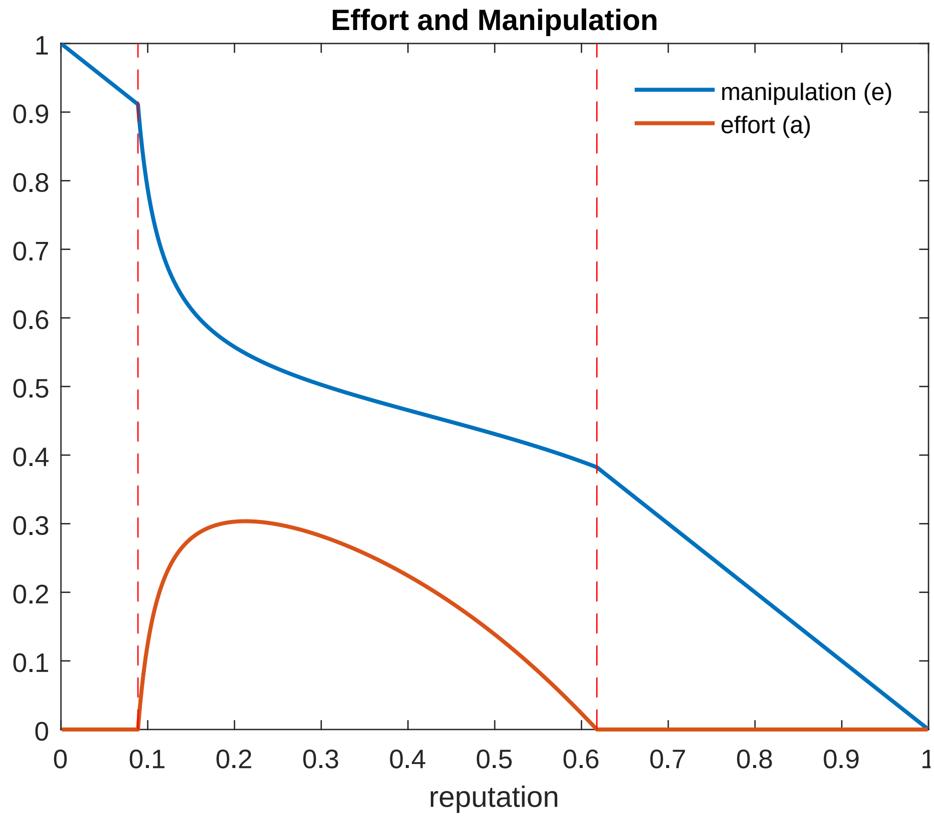

Figure 3: Parameters: $\lambda = 1, r = 0.5, \alpha = 1.75, c = 0.75$. The blue line represents the manager's manipulation strategy. Intuitively, manipulation decreases in monitor reputation, x. The red line represents the monitor effort strategy. The monitor shirks in both tails of the support of reputations and exerts effort over an intermediate range.

future effort and managerial manipulation and, given these expectations, forms a value. On any interval where the monitor works, the anticipated manipulation must be such that the monitor remains indifferent between working and shirking.

In the above shirk-work-shirk equilibrium, the monitor value is higher than that arising in a shirking equilibrium. This result is intuitive: though the monitor sometimes incurs the cost of monitoring, there is less manipulation and fewer losses, so the monitor enjoys higher fees and is less exposed to reputation shocks.

One might think there is multiplicity of equilibria in this model, but the equilibrium is actually unique. This is perhaps surprising given that the monitor's incentives to work are linked to market beliefs. This might suggest that pessimistic beliefs about the monitor effort could lead to a self-fulfilling prophecy, along the lines of the statistical discrimination phenomenon (Arrow 1972, 1973). However, contrary to the statistical discrimination setting where negative beliefs about the agent's effort weaken the agent's incentives, in our setting more pessimistic beliefs about the monitor's effort trigger more manipulation, thereby reinforcing his incentives to exert effort so as to detect the manipulation and prevent a loss. We conclude this section by examining the equilibrium when the cost of effort is low.

Proposition 3. *If* $\frac{c}{\lambda} < \max_x (1-x) V_s(x)$ *and* $\frac{\lambda(\alpha-c)^{\frac{3}{2}}}{\sqrt{\alpha}} \frac{2}{3\sqrt{3}} \geq rc$, *there is a unique equilibrium, which is a work-shirk equilibrium. That is, there is a* x_h, *such that*

$$\begin{array}{lll} a(x) \in (0,1) & if & x \in (0, x_h) \\ a(x) = 0 & if & x \geq x_h \text{ or } x = 0. \end{array} \tag{8}$$

On the work region $(0, x_h]$, *the manager's manipulation satisfies ODE 7 with boundary condition* $m(x_h) = 1 - x_h$.

This proposition shows that for a low cost of effort, the monitor works even if his reputation is close to zero, as long as it is positive. Monitor value and monitor effort are discontinuous in reputation, being zero at $x = 0$ but strictly positive for positive reputations. Technically, this is driven by $x = 0$ being an absorbing state: once the monitor losses his reputation he cannot exit that state.

In general, the monitor faces two types of incentives: i) a negative incentive arising from the risk of reputation loss when the monitor does not detect the manipulation and ii) a positive incentive arising from the benefit of improving reputation, via belief updating when he detects the manipulation. When c is low, only the negative incentive is relevant. The monitor value is high and relatively insensitive to reputation. Effort is relatively high, so belief updating is slow, because both monitor types behave similarly. However, the mere risk of losing his reputation is sufficient to provide incentives, even at low levels. To see it, notice that if effort were costless, monitor value would be independent of reputation because both types would detect the manipulation with probability one. Something similar occurs when c is low: monitor value is relatively insensitive to reputation and more similar to the value of commitment.

5 Disclosing Monitor Quality

In this section, we study the optimal disclosure policy of a regulator concerned about manipulation. A disclosure policy, in our setting, is a stochastic process that generates public verifiable signals about the monitor type based on the monitor's reputation.

At the outset, the regulator does not know the monitor's type. The regulator commits to a disclosure policy to maximize the monitor's effort and ultimately mitigate the manager's manipulation. We assume the regulator can only provide incentives to the monitor (and manager) via disclosure but cannot use monetary transfers, such as fines, to provide incentives.[9] We restrict attention to verifiable disclosures; hence the regulator can't lie about the monitor type or disclose noisy signals of the monitor's type.

Formally, the regulator chooses an information system to minimize expected manipulation. Specifically, the regulator value, in the absence of disclosure, is given by

$$W(x_0) = E^{x_0}\left[\int_0^{\infty} e^{-rs}\alpha\left(1 - m(x_s)\right) ds\right], \tag{9}$$

where x_0 is the prior probability that the monitor is good. As we shall see, the kind of disclosure policy the regulator wants to implement depends on her priors x_0.

Notice that manipulation reduces the regulator's payoff even when the manipulation is not discovered. This means the regulator internalizes the social cost of manipulation even when the manipulation does not manifest publicly. In may applications, the regulator objective would be better describe as maximizing the monitor's effort/quality. In our setting, minimizing manipulation is equivalent to maximizing the monitor's expected effort.

Benchmark Before studying the regulator's optimal disclosure policy, we consider the regulator's value in a shirking equilibrium, namely when c is so large that it's impossible to induce the monitor to exert some effort.

Anticipating no monitoring, the manager optimally chooses manipulation $m(x) = 1 - x$, and the regulator value becomes linear in reputation:

$$W_s(x_0) = E^{x_0}\left[\int_0^{\infty} e^{-rs}\alpha x_s ds\right] = \frac{\alpha x_0}{r}.$$

The linearity of W in a shirking equilibrium implies that –when the moral hazard problem is severe (large c)– disclosing information about monitor type is useless; it only increases the dispersion of manipulation –manipulation is higher when the monitor is revealed to be strategic and lower otherwise– but does not change the manipulation on average.

Proposition 4. *In a shirking equilibrium, the regulator is indifferent between any possible disclosure policy.*

The situation is very different when the cost of monitoring is low. In this case, the monitor effort varies in reputation. As a result, the regulator's value W is non-linear in monitor reputation. This suggests the regulator may benefit from influencing the evolution of monitor reputation via disclosure. In other words, the regulator may be able to exploit

[9] We shut down monetary incentives for tractability. Also, the role of monetary incentives is well known in the literature. Naturally, having the ability to use monetary incentives to discipline the monitor would likely reduce the need to implement information-based incentives.

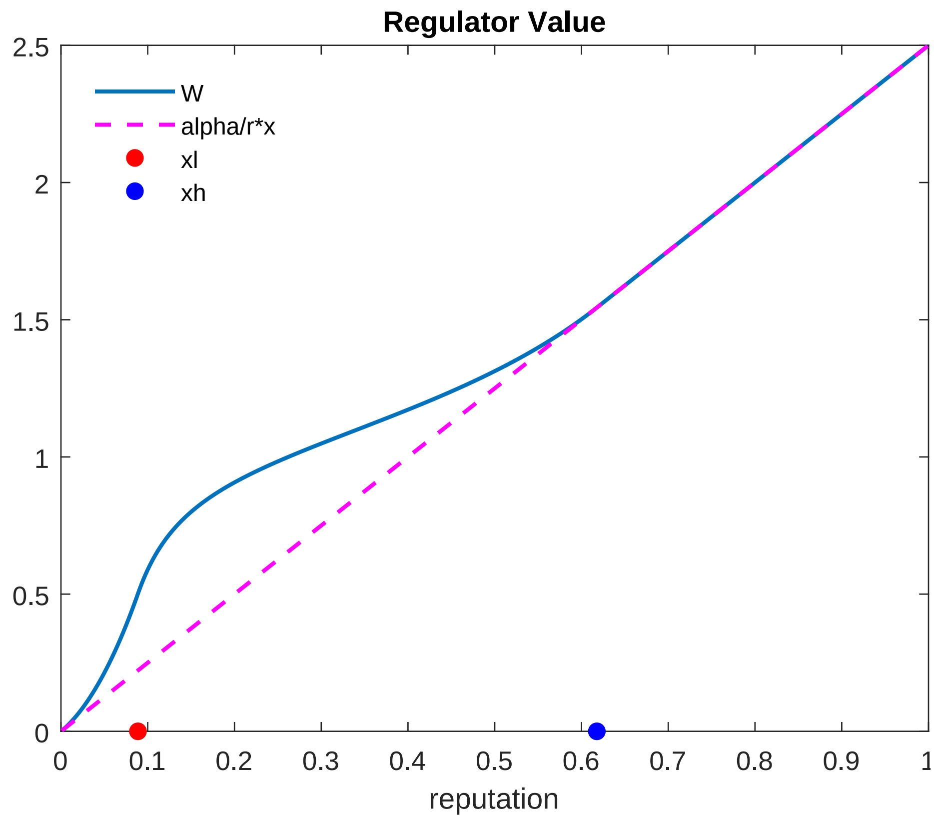

Figure 4: Parameters: $\lambda = 1.6, r = 1.4, \alpha = 3.5, c = 1$. The solid line represents the regulator value W in a shirk-work-shirk equilibrium. The dotted line captures the regulator value assuming the monitor always shirks ($\frac{\alpha}{r}x$). The monitor exerts effort when his reputation lies between the red and blue dots and shirks otherwise.

the non-linearity of his value W by resorting to disclosure. Of course, this conclusion is not direct in our setting because disclosure per se modifies the underlying monitor's behavior –in particular the monitor willingness to exert effort– ultimately affecting the shape of W.

We next study the regulator's disclosure policy. In general, the optimal disclosure policy may evolve as a function of monitor reputation, may include noisy signals, and may entail deterministic events where the monitor type is revealed with probability one. We restrict attention to verifiable disclosure policies. In the next section, we compare two forms of verifiable disclosure policies: deterministic disclosure, and disclosure with random Poisson delay. We focus on the case of moderate costs c, as this case yields the richest monitor behavior in our setting (Shirk-Work-Shirk).[10]

[10]For general dynamic disclosure policies, the monitor's HJB equation that we have characterized in Propositions 2 and 3 may not have a solution that can be characterized. Generally, it is well known that HJB-type equations can be characterized in only relatively special cases, the most common being Poisson or Brownian noise. This is why we focus on the tractable cases.

5.1 Deterministic Disclosure

As a benchmark, we study whether a deterministic disclosure policy can be valuable. A deterministic policy is a commitment to revealing the monitor's type at some known time in the future. Specifically, we consider whether a disclosure policy that reveals the monitor type perfectly when his reputation reaches an arbitrary level x can be part of an optimal disclosure policy. The following result contrasts such a policy versus a non-disclosure commitment, where the regulator stays silent.

Proposition 5. *For any initial reputation x_0, a policy of non-disclosure yields higher value to the regulator W than a commitment to disclose the monitor quality deterministically as reputation reaches x_1, for any $x_1 \in (x_0, 1)$.*

The regulator's value in a shirking equilibrium is a lower bound for the regulator's payoff in equilibrium. On the other hand, disclosing monitor quality, at any level x, leads to shirking thereafter, generating an expected value of $xW(1) + (1-x)W(0) = x\frac{\alpha}{r}$, which is exactly the regulator value in a shirking equilibrium. Such a deterministic policy also leads to zero continuation value for the monitor at $x = x_1$. Relative to no disclosure, revealing the monitor type at x_1 thus weakens the monitor's incentives to exert effort before x_1, which in turn boosts the manager's manipulation.

Therefore, it is never optimal to (commit to) disclose the monitor type in a deterministic fashion (conditional on the monitor's reputation). We can thus rule out this as part of an optimal policy, even before knowing the effect such policy has on the monitor's behavior.[11]

5.2 Random Disclosure

A deterministic policy is not valuable to the regulator. Here we explore whether the regulator may benefit from choosing a random policy where the rate of disclosure varies based on the monitor's type. For example, the regulator may choose a positive disclosure rate when the monitor is strategic (resp. good) and zero rate otherwise. We refer to this as a policy of bad (good) news.

In the sequel we analyze this option. First, we examine the bad-news case and then look at the good-news case separately. We then analyze the regulator's policy when the it can combine good and bad news.

[11] The proof of this proposition is exactly the argument we have just outlined. We hence skip the formal proof for the sake of brevity.

5.2.1 Disclosure of Bad News

We have demonstrated that disclosing monitor quality deterministically cannot be part of an optimal disclosure policy. Here, we consider whether disclosing bad news randomly can improve upon no disclosure.

Assume that, at each point, the regulator can disclose the monitor type with intensity $\gamma_t \in \{0, \gamma\}$[12] when the monitor is strategic. In other words, this policy may reveal bad news with some probability but it never discloses good news. The significance of introducing a "bias" in the disclosure rates is that no news is interpreted as favorable news, which improves the monitor's reputation in the absence of arrivals.

To understand the value of such a policy, we first consider how it affects the manager's beliefs and the monitor's incentives. With some abuse of notation, we rewrite the evolution of beliefs, without arrivals, as

$$\dot{x}_t = \lambda x_t m\left(x_t\right)^2 + \gamma x\left(1 - x\right). \tag{10}$$

Equation (10) follows from Bayes' rule and shows that when the rate of bad news is positive, the reputation drift is steeper: in the absence of arrivals the monitor reputation increases faster (relative to the case without bad news, $\gamma = 0$) because no news becomes a more favorable signal of monitor quality. Of course, as before, an arrival fully destroys the monitor's reputation, as if the monitor's license were revoked (after the disclosure arrival, the fee goes down to zero, and the monitor shirks thereafter.)

Consider how disclosure affects monitoring incentives. On average, disclosure reduces the reputation of the strategic monitor, but if lucky, it allows the monitor to increase its reputation faster. Both effects alter the monitor's incentives and his willingness to work.

Formally, denote the manager's manipulation strategy by $m(\cdot)$. The monitor value is now characterized as:

$$rV\left(x\right) = \alpha\left(1 - m\left(x\right)^2\right) + V'\left(x\right)\dot{x} - \left(\lambda m\left(x\right) + \gamma\right)V\left(x\right),$$

with boundary condition $V(1) = \frac{\alpha}{r+\gamma}$. As in Section 4, when the monitor works, he must be indifferent between working and shirking; hence, on the work region, we must have $\lambda m(x) V(x) = c$. Combining this condition along with the HJB equation for monitor value

[12] This is without loss of generality relative to $\gamma_t \in [\,0, \gamma]$ given the linearity of the regulator's problem.

$V(\cdot)$, we have that manipulation is governed by the following ODE:

$$m'(x) = \frac{\alpha\left(1 - m(x)^2\right) - \frac{(r+\gamma)c}{\lambda m(x)} - c}{cx + \frac{x(1-x)\gamma c}{\lambda m(x)^2}}.$$

This equation shows how the evolution of manipulation (and effort) is affected by disclosure, γ.

Naturally, the effect of disclosure γ depends on the monitor's reputation level x. We start by considering circumstances (i.e., reputations) under which disclosing bad news is counterproductive.

Proposition 6. *Disclosure of bad news reduces the strategic monitor's incentive to exert effort close to x_h and on $[x_h, 1]$.*

To better understand the impact of disclosure on incentives, the following contrast is useful. In our model, there are two sources of information: the first source (with intensity λ) are manipulation shocks; these shocks arise when the monitor is not able to detect the manipulation, being caused by (lack of) monitoring effort. In that sense, λ captures the scrutiny the monitor faces when he shirks. The second source of information (arriving with intensity γ) is the regulator's disclosure. Disclosure events arrive independently of the monitor effort, since disclosure is tied to monitor quality (which in our setting is constant and independent of effort).

Now, a greater disclosure intensity reduces the strategic monitor value and his incentive to exert effort because the monitor anticipates that his reputation will last less. In turn, the manager, anticipating weaker monitoring, engages in more aggressive manipulation. On the other hand, a greater intensity of manipulation shocks λ also reduces monitor value but has the opposite effect on incentives because, unlike disclosure, manipulation shocks are at least partially triggered by monitor shirking, being thus tied to the monitor's behavior rather than his type.

In summary, both sources of information, manipulation shocks and disclosure arrivals, erode the monitor's reputation and his value V but there is a fundamental difference between these two sources of information: while disclosure arrivals are independent of effort, manipulation shocks are caused by the monitor failing to detect the manager manipulation.

One additional reason why disclosure may weaken incentives in our setting is because –unlike in Bayesian persuasion settings, where the monitor ignores his type ex-ante– here the monitor knows that he is the "bad" type. Hence, with disclosure, he is particularly pessimistic about his future reputation—more so the higher the disclosure rate. In technical terms: the strategic monitor reputation is a super-martingale and disclosure only accelerates the reputation decline he expects to experience over time.[13]

Having characterized the effect of bad news on the monitor's behavior, we can now study its effect on the regulator's value W. Given γ, the regulator value follows the HJB equation:

$$rW(x) = \alpha(1 - m(x)) + \dot{x}W'(x) - \lambda m^2(x)W(x) - \gamma(1-x)W(x) \tag{11}$$

with boundary condition $W(1) = \frac{\alpha}{r}$. This equation shows that disclosure affects the regulator value in several ways. First, there is a direct effect: a disclosure arrival destroys the monitor's reputation, leading to an escalation of manipulation and a large drop in regulator value. This effect is particularly relevant when reputation is low because then an arrival is more likely, from the regulator's standpoint. Second, disclosure may reduce monitoring effort, thereby boosting the manager's manipulation, m. This has both a direct flow effect and a capital loss effect. The latter effect arises because stronger manipulation makes a manipulation shock more likely. Finally, disclosure increases the reputation drift: without arrivals, monitor reputation tends to increase faster thereby strengthening monitoring incentives.

Proposition 7. *Disclosure of bad news is valuable to the regulator when $x \leq x_d$, where $x_l \leq x_d < x_h$. It is not valuable for x close to (or above) x_h.*

Disclosure of bad news affects the regulator value in the following way. First, information lowers the expected reputation of the monitor thereby weakening monitoring incentives. Second, for some reputations, the regulator value is non-linear in reputation due to the non linearity of effort. This means that positive reputation shocks may have a stronger effect than negative shocks. For example, in the baseline problem without disclosure, the regulator value is convex below x_l. At that level, a small shock to reputation has an asymmetric effect: a negative shock increases manipulation, but a positive reputation shock reduces manipulation more strongly because it resurrects the monitor's incentive to work, further mitigating the manager's desire to manipulate. This result shows that the monitor reputation level determines whether disclosure of bad news can be valuable. Bad news is valuable only for relatively low reputations. To further understand why, notice that the effect of bad news on the regulator payoff is captured by

$$\gamma x(1-x)W'(x) - \gamma(1-x)W(x) \tag{12}$$

[13] But, disclosure also induces a countervailing effect. Indeed, we have seen that in the absence of disclosure, the monitor value $V()$ is sometimes convex in reputation (see Figure 2). This suggests that by adding volatility to reputation, disclosure could benefit the strategic monitor, despite shortening the monitor reputation's "life expectancy".

The first term captures the positive effect of bad news on the reputation drift; the second term is the loss caused by the realization of bad news. Below x_l, W' is relatively high because the regulator anticipates reaching the work region. Hence, for low reputations, disclosure of bad news has a favorable effect: the benefit of increasing the reputation drift (and hence reaching the work region faster) outweighs the risk of a negative disclosure.

In summary, when choosing a disclosure policy the regulator faces a tension between weakening monitoring incentives at any reputation level, versus increasing the time the monitor reputation stays in the work region. There are circumstances where the trade-off is trivial. Indeed, on the lower shirking region $[0, x_l]$, disclosure can be introduced without modifying the monitor's incentive. At any reputation level x, the monitor incentives are based on the values arising for higher reputations, since reputation either increases or is fully destroyed. In that sense, introducing disclosure below x_l lowers the monitor value below x_l and reinforces the monitor incentive to shirk but does not affect his incentive for higher reputations, $x \geq x_l$. Recall that monitor reputation never decreases unless an arrival is observed, in which case the monitor value drops to zero. Hence the monitor incentive given reputation x, does not depend on the disclosure policy for reputations below x.

Notice that on $[0, x_l]$ the condition for bad news to be valuable (equation 12) is related to the convexity of the regulator's payoff. Indeed, if the regulator's payoff W is convex on $[0, x]$, then $W(x) \leq xW'(x)$ since $W(0) = 0$. Thus, disclosure of bad news amplifies the dispersion of reputation, allowing the regulator to exploit the convexity of his payoff.

In summary, the regulator can hold the monitor's incentives constant and still exploit the convexity of W on $[0, x_l]$ by inducing a positive disclosure rate $\gamma > 0$ and a zero rate $\gamma = 0$ elsewhere. Of course, above x_l, bad news disclosure may deteriorate monitoring incentives; however, by continuity, we demonstrate that there is an interval to the right of x_l where bad news disclosure is valuable to the regulator.

Next we study some comparative statics focusing on the disclosure policy of the regulator. We obtain these results numerically, by solving the regulator's problem for given parameter values and then comparing the thresholds at which the regulator will disclose bad news.

The effect of enforcement Only for relatively low reputations the regulator has an incentive to disclose bad news. Indeed, the regulator implements a positive rate of bad news on the left tail of the support of reputations $[0, x_d]$. The size of this interval depends on the environment parameters. For example, the enforcement level affects the size of the disclosure region, x_d. We model enforcement as changing the agent's cost of manipulation, which now becomes $c_m \frac{1}{2} m^2$.[14]

On the surface, one would think that stronger enforcement would crowd-out disclosure, given the potentially adverse incentive effect of disclosure. Surprisingly the opposite holds. Figure 5 shows the effect of increasing the cost of manipulation to the manager on the size of the bad news region. The figure suggests that a higher cost of manipulation expands the disclosure region where the regulator is willing to interfere in the market, by disclosing bad news about the monitor type.

The intuition is as follows. Stronger enforcement leads to less manipulation by the manager. Consistent with this, the monitor has stronger incentives to shirk. Hence, the work region shrinks. Conversely the shirking region expands. As a result, the regulator finds it optimal to disclose bad news for even higher reputation levels. This suggests the frequency of bad news may go up when enforcement is stronger. In other words, a monitor may become more likely to see his reputation ruined by a regulator's disclosure in environments where manipulation is expected to be less intense.

The effect of the loss As another comparative static, consider the effect of α on the size of the bad news region. When α is higher, a manipulation shock causes larger losses to the firm. Reputation becomes more valuable (i.e., the monitor fee goes up) hence preserving a reputation becomes more important to the monitor. This triggers more monitoring effort; the work region expands. In turn, this crowds-out regulatory disclosure: the disclosure region shrinks. Empirically, this suggests that when firms are willing to pay more to the monitor, either because manipulation shocks are more costly to the firm, or because the monitor enjoys stronger monopoly power, the regulator will interfere less.

The effect of monitoring costs When the cost of monitoring goes up, moral hazard exacerbates; shirking becomes more attractive to the monitor. Accordingly, the shirking region expands. The role of disclosure grows and the bad news region expands.

5.2.2 Disclosure of Good News

In the previous section we studied the value of disclosing bad news. Sometimes the regulator may prefer to conceal bad news and, instead, disclose good news. Here we consider the circumstances under which this option is valuable.

[14] Previously, the cost was $\frac{1}{2} m^2$. By varying c_m we can capture situations where manipulation is easier or harder for the manager. From the monitor's perspective, changing c_m is equivalent to changing λ since only the manipulation in response to monitor reputation matters for the monitor's incentives. An increase in c_m is equivalent to a decrease in λ.

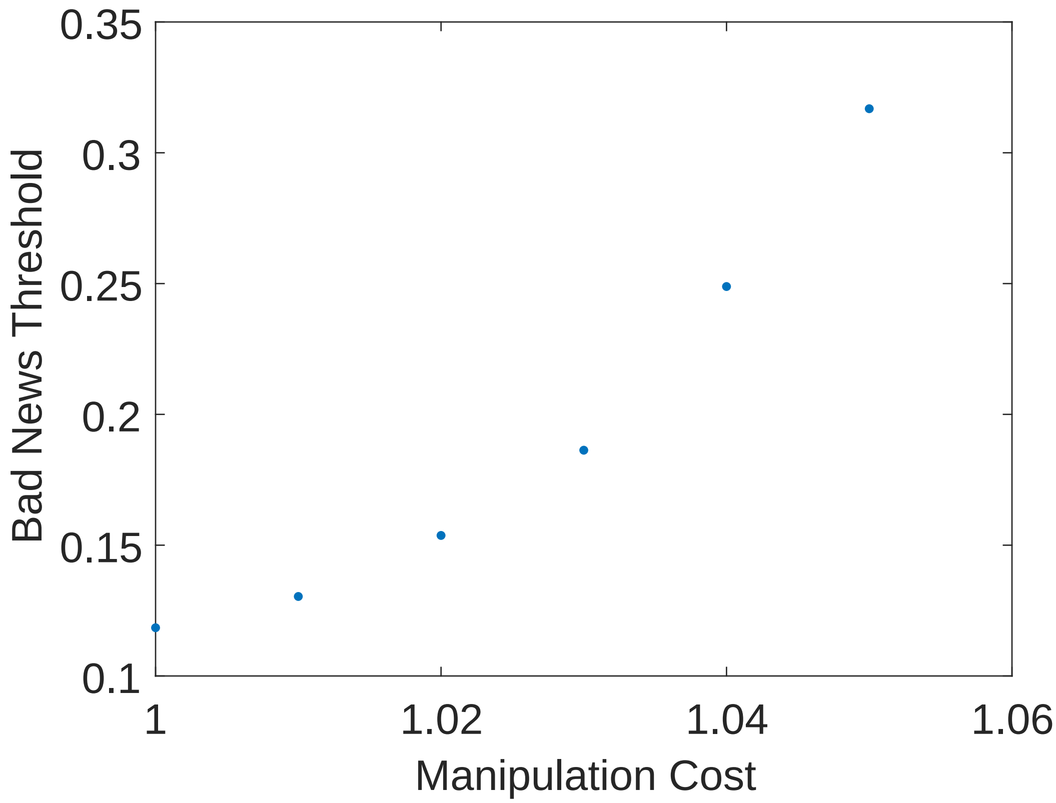

Figure 5: This figure depicts the effect of increasing the cost of manipulation to the manager on optimal size of the bad news disclosure region.

Suppose the regulator can disclose good news either at a rate γ or zero. By contrast, assume bad news can't be disclosed. When only good news are expected to arrive, then the absence of arrivals reduces the monitor's reputation relative to the case without disclosure: no news is a less favorable signal of quality compared to the case with no news, $\gamma = 0$.

Let γ represent the rate of good news. Then the change in reputation, without arrivals, is given by

$$\dot{x}_t = \lambda x_t m\left(x_t\right)^2 - \gamma x\left(1 - x\right).$$

Disclosure of good news reduces the drift. Naturally, this affects the monitor incentives. The monitor value is now characterized by

$$rV\left(x\right) = \alpha\left(1 - m\left(x\right)^2\right) + V'\left(x\right)\dot{x} - \lambda m\left(x\right)V\left(x\right).$$

Since, the strategic monitor never experiences a disclosure arrival, disclosure only affects his incentives indirectly via the reputation drift. We have the following result.

Proposition 8. *Disclosure of good news increases manipulation and decreases the value of the strategic monitor on the work region* $[x_l, x_h]$.

Good news decreases the speed at which the monitor reputation improves in the absence of arrivals. This, deteriorates the monitor's incentives and, consequently, exacerbates the manager's manipulation. Yet, such a policy may still benefit the regulator. When the good news disclosure rate is γ the regulator's value follows the HJB equation:

$$rW\left(x\right) = \alpha\left(1 - m\left(x\right)\right) + \dot{x}W'\left(x\right) - \lambda m^2\left(x\right)W\left(x\right) + \gamma x\left[W\left(1\right) - W\left(x\right)\right] \tag{13}$$

with boundary condition $W\left(1\right) = \frac{\alpha}{r}$.

Proposition 9. *For x below x_h but sufficiently close to x_h, the regulator benefits from disclosing good news. Disclosing good news is not valuable for $x \geq x_h$.*

To understand this result, notice that at any given x the value of disclosing good news is given by

$$-\gamma x\left(1 - x\right)W'\left(x\right) + \gamma x\left(\frac{\alpha}{r} - W\left(x\right)\right).$$

The first term is the negative effect of disclosure on reputation: the presence of disclosure reduces the drift, absent arrivals. The second term captures the effect of an arrival, which naturally benefits the regulator because it eliminates manager's manipulation going forward.

The good news policy is valuable when the slope of the regulator value $W'(x)$ is small, given that the effect of disclosure on the drift is negative. On the upper part of the work region (toward x_h) the slope of the regulator value $W'(x)$ is relatively low because a higher reputation means exiting the work region. This is why good news is valuable toward x_h: if no good news arrive, then the monitor reputation deteriorates, thereby delaying the process by which the reputation exits the work region. By contrast, if good news arrive, then the regulator obtains a large gain.[15]

Good and Bad News Combined We have discussed the role of bad news and good news separately. Figure 6 summarizes the main insights of the previous sections, namely: *i*) disclosing bad news is valuable for low reputations, *ii*) Good news is valuable toward x_h and *iii*) No disclosure is valuable for top reputations, in the upper shirking region.

The intuition is as follows. The value of disclosure is to bring reputations into the region where reputational incentives "do their job": for low values, this means disclosing bad news. This does not hurt the good type monitor, and allows the regulator to induce more effort from the strategic type if there is no disclosure realization. For relatively high values, this means disclosing good news, which lowers the reputation for the strategic type because for him the good news never materializes. This result is beneficial for the regulator, since it keeps the reputation inside the work region for a longer time period. Of course, this is anticipated and may destroy some incentives for monitors with lower reputations. However, close to x_h, disclosing good news is beneficial overall. Finally, the optimal policy requires that the regulator stays silent for top reputations, above x_h. This feature is key for incentives: the regulator commits to opacity once the monitor has achieved a relatively high reputation, as a reward for good past performance. Disclosure above x_h does not have the ability to improve incentives locally, but it weakens the incentives at lower reputation levels.

Because the ODE for the dynamics of manipulation is nonlinear (Equation (7)), we cannot qualitatively characterize the regulator's optimal disclosure on the interior of the work region. The full problem of the regulator is therefore not tractable. We still formulate this problem for the interested reader in Appendix D using Pontryagin's maximum principle, but it leads to a multidimensional nonlinear boundary value problem.

However, we can solve the regulator's problem numerically, by restricting attention to threshold strategies. That is, the regulator chooses one cutoff below which bad news is revealed at a constant rate and another above which good news is revealed.[16] With this restriction, we can confirm that the optimal policy takes the same form as in Figure 6.

[15] One needs to be careful here because for small γ, the adverse incentive effect of disclosure could dominate. We prove that one can always find a sufficiently low γ and an interval close to x_h such that good news disclosure is valuable.

[16] Good news stops being revealed above x_h though as we have shown this is optimal.

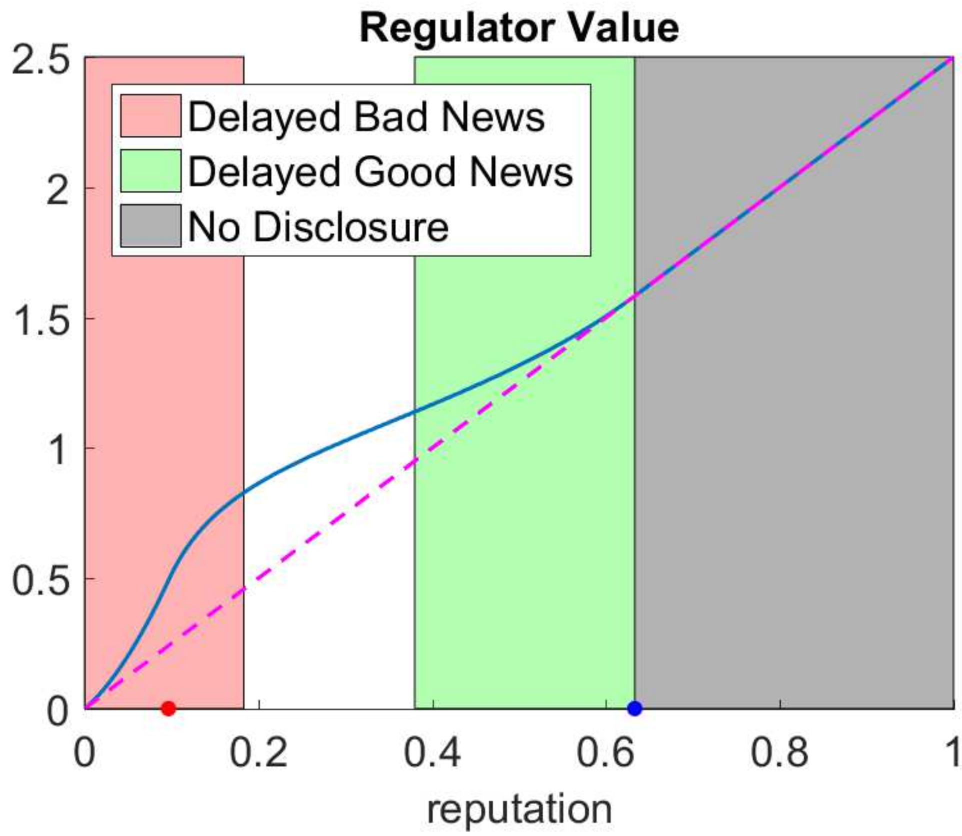

Figure 6: This figure characterizes the optimal disclosure policy of a regulator. For low reputations (pink), the regulator prefers to disclose bad news. For moderately high reputations (green), the regulator prefers to disclose good news. The regulator withholds information for top reputations, as a way to reward good past performance.

6 Conclusion

In long-run relationships, the desire to build a reputation can act as an incentive device when explicit penalties or contracting arrangements are not available. This is especially relevant for intermediaries such as banks, underwriters, rating agencies, or auditors. These intermediaries fulfill the role of monitors in the economy. Banks screen loans before they sell them off in a structured product, underwriters and venture capital firms monitor the quality of startups before their initial public offering, rating agencies monitor firms for behavior that may make default more likely, and auditors detect accounting fraud that may otherwise go unnoticed. If monitors neglect their duty, enforcement is often impractical. In the recent financial crisis, for example, mortgage underwriters have failed to properly screen applicants. Yet, it took many years before the problem became apparent, and few individual underwriters have been punished. In these situations, reputation may be the main incentive device.

In this paper, we characterize reputational incentives for monitors. In our model, the agent who is monitored is a rational player. He optimally chooses how much to misbehave in response to the monitor's reputation and the anticipated monitoring effort. This leads to a shirk-work-shirk equilibrium. When reputation is low, there is little value for the monitor to exert effort, so the monitor shirks. Likewise, when reputation is very high, the monitor shirks because the agent does not misbehave when he is faced with a high reputation monitor; in this situation, if the monitor shirks, the public is not likely to detect it. Instead, the monitor only exerts effort when reputation is in an intermediate interval. This finding has an important implication—uncertainty about the monitor is valuable.

In response to the financial crisis, regulators have started to rethink the transparency of financial intermediaries. The Sarbanes-Oxley Act (SOX) has brought with it a slew of disclosure requirements, a new regulatory authority has been formed to oversee auditors (the PCAOB), and many governments have designed stress tests for banks. If intermediaries anticipate that information about them will be revealed, how does this influence their desire to build a reputation in the first place? And how can regulators harness mandatory disclosure requirements to improve the functioning of markets for loans, equity, or auditing services?

We show that seemingly reasonable disclosures can have a detrimental effect. If the regulator provides verifiable disclosure about the monitor, any deterministic disclosure policy (i.e., a policy that reveals information with certainty at any given time) will at least partially destroy the incentive to acquire a reputation and lead the monitor to exert less effort. To improve the functioning of the underlying markets, regulators should therefore not aim to provide transparency about the monitor. Instead, they should aim to induce uncertainty about the monitor's type, since reputational incentives are strongest when reputation is in an intermediate region. This provides a rationale for disclosure policies that use delay.

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A Proofs

A.1 Shirking Equilibrium (Proposition 1)

We first show that the shirking ODE 5 with the boundary condition $V_s(1) = \frac{\alpha}{r}$ has a unique solution. This is technically involved, because the equation is singular at both $x = 0$ and $x = 1$.We express the solution as an initial value problem (IVP) at some value $x_0 \in (0, 1)$. Then, we characterize the solutions as x approaches one and show that there can be at most one solution that satisfies the boundary condition. To prove existence, we use a rescaling of $V_s(x)$ together with the Arzelà-Ascoli Theorem (see e.g. Royden (1988), Ch. 7.10, p. 167ff) and an argument similar to the shooting method (see e.g. Bailey et al. (1968)).[17] That the solution to the ODE equals the strategic type's value then follows from a standard verification argument, which we omit (See Davis (1993), Ch. 4).

To prove that this is indeed an equilibrium, we then use the firm's optimality condition for each x and show that no instantaneous deviation is optimal. Since the equilibrium is assumed to be Markovian, this is sufficient. We then establish uniqueness by showing that in any other potential equilibrium must have a discontinuous value function, which is impossible.

We start with recording some useful properties of solutions to the shirking ODE 5. The solutions can be indexed by an initial condition v_0 at a (common) initial point $x_0 \in (0, 1)$. To highlight this dependence, we denote them with $V_s(x, v_0)$. We continue writing the shirking value as $V_s(x)$.

Lemma 2. *Solutions to the initial value problem (IVP) in Equation 5 with initial condition $V_s(x_0, v_0) = v_0$ for some fixed $x_0 \in (0, 1)$ have the following properties:*

1. *For any interval $[\underline{x}, \bar{x}]$ with $0 < \underline{x} < x_0 < \bar{x} < 1$ and any v_0, the solution to the IVP exists and is unique.*
2. *$V_s(0, v_0) = 0$ for all v_0.*
3. *For any $x \in (0, 1)$, $V_s(x, v_0)$ is continuous and strictly increasing in v_0. In particular, two solutions $V_s(x, v_0')$ and $V_s(x, v_0)$ cannot cross on $(0, 1)$.*
4. *Larger solutions, i.e. $v_0' > v_0$, have larger slope: if $v_0' > v_0$, then for all $x \in (0, 1)$, $V_s'(x, v_0') > V_s'(x, v_0)$.*
5. *There exists at most one solution with $V_s(1, v_0) = \frac{\alpha}{r}$.*

[17] We defer the detailed proof to Section C, since it is purely technical.

Proof. 1. Existence and uniqueness for solutions to the IVP follows from the Picard-Lindelöf Theorem, which requires that the right hand side of

$$V_s'(x, v_0) = \frac{(r + \lambda(1-x)) V_s(x, v_0) - \alpha\left(1 - (1-x)^2\right)}{\lambda x (1-x)^2} \tag{14}$$

is Lipschitz in both V_s and x. This is true as long as the interval $[\underline{x}, \bar{x}]$ is bounded away from 0 or 1, which we have assumed.

2. To show that every solution satisfies $V_s(0, v_0) = 0$, we use the method of integrating factors (see Polyanin and Zaitsev (2002), p.4) to write

$$\begin{aligned} V_s(x, v_0) &= \exp\left(\int_{x_0}^{x} \frac{r + \lambda(1-s)}{\lambda s (1-s)^2} ds\right) \cdot \\ &\left(v_0 - \int_{x_0}^{x} \frac{\alpha\left(1-(1-s)^2\right)}{\lambda s (1-s)^2} \exp\left(-\int_{x_0}^{s} \frac{r+\lambda(1-u)}{\lambda u (1-u)^2} du\right) ds\right). \end{aligned} \tag{15}$$

This equation can be written more explicitly as

$$\begin{aligned} V_s(x, v_0) &= \left(\frac{x}{1-x}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(\frac{r}{\lambda}\frac{1}{1-x}\right) \cdot \\ &\left(\left(\frac{1-x_0}{x_0}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(-\frac{r}{\lambda}\frac{1}{1-x_0}\right) v_0 \right. \\ &\left. -\frac{\alpha}{\lambda}\int_{x_0}^{x} s^{-\left(\frac{r}{\lambda}+2\right)} (1-s)^{\frac{r}{\lambda}-1} \left(1-(1-s)^2\right) \exp\left(-\frac{r}{\lambda}\frac{1}{1-s}\right)\right) ds. \end{aligned}$$

We now show that this expression converges to zero as $x \to 0$. We can bound the value of the integral from above as follows:

$$\begin{aligned} &\int_{x_0}^{x} s^{-\left(\frac{r}{\lambda}+2\right)} (1-s)^{\frac{r}{\lambda}-1} \left(1-(1-s)^2\right) \exp\left(-\frac{r}{\lambda}\frac{1}{1-s}\right) ds \\ \leq\ & M \int_{x_0}^{x} s^{-\left(\frac{r}{\lambda}+2\right)} \left(1-(1-s)^2\right) ds \end{aligned}$$

for some $M > 0$, because all terms inside the integral except for $s^{-\left(\frac{r}{\lambda}+2\right)}$ are bounded as x converges to zero. Computing this new integral, and multiplying it by $x^{\frac{r+\lambda}{\lambda}}$, we can show that $V_s(x, v_0)$ converges to zero if and only if

$$x^{\frac{r+\lambda}{\lambda}} \left(\frac{1}{-\left(\frac{r}{\lambda}+1\right)} x^{-\frac{r}{\lambda}} + \frac{\lambda}{r} x^{-\frac{r-\lambda}{\lambda}}\right)$$

converges to zero as $x \to 0$. Inspecting the exponents, we can confirm this is the case.

3. This follows directly from Equation 15. Using that equation we can write

$$\begin{aligned} V_s\left(x, v_0'\right) - V_s\left(x, v_0\right) &= \left(\frac{x}{1-x}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(\frac{r}{\lambda}\frac{1}{1-x}\right) \cdot \\ &\quad \left(\frac{1-x_0}{x_0}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(-\frac{r}{\lambda}\frac{1}{1-x_0}\right) \cdot \left[v_0' - v_0\right] \\ &> 0 \end{aligned}$$

4. This follows from differentiating Equation 15 in x, which yields

$$V_s'\left(x, v_0'\right) - V_s'\left(x, v_0\right) = \frac{d}{dx}\left(\frac{x}{1-x}\right)^{\frac{r+\lambda}{\lambda}} \exp\left(\frac{r}{\lambda}\frac{1}{1-x}\right) \cdot \left[v_0' - v_0\right].$$

The derivative on the RHS is strictly positive for $x \in (0,1)$.

5. Suppose there exist two solutions $V_s\left(x, v_0\right)$ and $V_s\left(x, v_0'\right)$ with $V_s\left(1, v_0\right) = V_s\left(1, v_0'\right) = \frac{\alpha}{r}$. Without loss of generality, assume $v_0' > v_0$. By Point 3, for any $\varepsilon > 0$, we have $V_s\left(1-\varepsilon, v_0'\right) > V_s\left(1-\varepsilon, v_0\right)$, so there must exist a $\delta < \varepsilon$ such that $V_s'\left(1-\delta, v_0'\right) < V_s'\left(1-\delta, v_0\right)$, otherwise, the solutions cannot both hit $\frac{\alpha}{r}$ at $x = 1$. But by Point 4, such δ cannot exist.

∎

The last point of the Lemma shows that there is at most one solution to the IVP that satisfies $V_s(x) = \frac{\alpha}{r}$. In Section C, we use the Properties established in the Lemma to prove existence.

To show that shirking is indeed an equilibrium, note that shirking is optimal at x whenever

$$\lambda\left(1-x\right)V_s\left(x\right) \le c.$$

This is satisfied because of our assumption on c in the statement of Proposition 1.

We now show that the shirking equilibrium is unique. Specifically, we show that there exists no other equilibrium where $a(x) > 0$ on a set of positive measure x.[18]

To do this, we show that any equilibrium with working must feature a discontinuity in the firm's value function, which is not consistent with the firm being forward-looking.

[18] Changing effort on a set of x that has measure zero does not affect the firm's value or the evolution of the reputation. We therefore abstract from such issues. When we mean working, we mean $a(x) > 0$ on a set of positive Lebesgue measure and when we mean shirking we mean $a(x) = 0$ on such set. When we say the shirking equilibrium is unique, we mean that shirking is optimal almost everywhere.

Lemma 3. *In any equilibrium, the firm must shirk whenever* $x \geq \frac{\alpha\lambda - rc}{\alpha\lambda}$.

Proof. Shirking is optimal whenever

$$\lambda \left(1 - x\right) V\left(x\right) \leq c.$$

In any equilibrium, the value of the firm is bounded: $V_s\left(x\right) \leq \frac{\alpha}{r}$. Combining these two inequalities and rearranging yields the result. ∎

If $\alpha\lambda \leq rc$, the Lemma implies that shirking is the unique equilibrium. We thus focus on the case $\alpha\lambda > rc$. In any equilibrium that features working, there must be an interval $[x_h, 1]$ when the firm shirks. On that interval, the value of the firm is simply given by $V_s\left(x\right)$. Importantly, this value is independent of anything that happens for $x' < x$. This is because we are in a "perfect bad news" case. In any equilibrium, the value of the firm is continuous at any $x > 0$, because the future evolution of reputations and prices is anticipated.

Now, assume that $[x_h, 1]$ is the largest interval where the firm shirks. If $x_h = 0$ we are done. Thus, assume that $x_h > 0$. For any $\varepsilon > 0$, working must be optimal on $\left(x_h - \varepsilon, x_h\right)$. If this were not the case, then $[x_h, 1]$ would not be the largest interval where the firm shirks. This means that for any $\varepsilon > 0$, the value in that equilibrium $V\left(x\right)$ satisfies

$$V\left(x_h - \varepsilon\right) \geq \frac{c}{\lambda\left(1 - x_h + \varepsilon\right)} > V_s\left(x_h - \varepsilon\right).$$

By the assumption in Proposition 1,

$$\frac{c}{\lambda(1-x)} - V_s(x) \geq K \; \forall x \in [0,1]$$

for some fixed $K > 0$. Therefore,

$$V(x_h - \varepsilon) \geq V_s(x_h - \varepsilon) + K.$$

At x_h, V must satisfy the value matching condition

$$V\left(x_h\right) = V_s\left(x_h\right).$$

But this means that V is discontinuous at x_h, which is impossible. Since this argument applies for any $x_h > 0$, it must be the case that $x_h = 0$. That is, the firm shirks for all x.

Finally, we provide sharp condition in terms of the model parameters for when shirking is the unique equilibrium. To facilitate the analysis, we introduce two new functions, g and l. $g\left(x\right)$ is defined as $g\left(x\right) = \left(1 - x\right) V_s\left(x\right)$. $V_s\left(x\right)$ crosses $\frac{c}{\lambda(1-x)}$ whenever $g\left(x\right)$ crosses $\frac{c}{\lambda}$ and it is easier to study the crossing points of $g(x)$. It satisfies the ODE

$$\left(r+\lambda(1-x)^2\right)g(x)=\alpha\left((1-x)-(1-x)^3\right)+\lambda x(1-x)^2 g'(x) \tag{16}$$

with boundary conditions $g(0)=g(1)=0$. It is continuously differentiable, because $V_s(x)$ is continuously differentiable.

The slope of $g(x)$ is determined by the function $l(x,v)$ for $x\in[0,1]$ and $v\geq 0$, which is given by

$$l(x,v)=\alpha\left((1-x)-(1-x)^3\right)-\left(r+\lambda(1-x)^2\right)v. \tag{17}$$

Specifically, we can write $g(x)$ as

$$0=l(x,g(x))+\lambda x(1-x)^2 g'(x),$$

so $g'(x)$ is positive whenever $l(x,g(x))$ is negative. The function $l(x,v)$ satisfies the following properties for all $v>0$ and $x\in[0,1]$: $l(0,v)<l(1,v)<0$, $l_{xx}(x,v)<0$, $l_x(0,v)>0$ and $l_x(1,v)<0$.[19] Thus, for any fixed v, $l(x,v)$ is either always negative or hits zero exactly twice. It is also strictly decreasing in v for all x and has a unique interior maximum for all v.

Proposition 10. *Shirking is the unique equilibrium if and only if* $\max_x l\left(x,\frac{c}{\lambda}\right)\leq 0$. *The value* $\bar{c}$ *above which shirking is the unique equilibrium satisfies*

$$\max_x l\left(x,\frac{\bar{c}}{\lambda}\right)=0.$$

Equivalently, the equilibrium features working if and only if $\max_x l\left(x,\frac{c}{\lambda}\right)>0$.

Proof. Suppose that $\max_x l\left(x,\frac{c}{\lambda}\right)>0$. We show that in this case, $g(x)$ defined in Equation 16 must cross $\frac{c}{\lambda}$, which implies that the shirking equilibrium cannot exist. To show this, we denote with $\bar{g}$ the maximum of g on $[0,1]$, which is attained at $\bar{x}$, and we assume that $\bar{g}<\frac{c}{\lambda}$. The function g is continuously differentiable, so we have $g'(\bar{x})=0$,[20] and therefore $l(\bar{x},\bar{g})=0$. Since $l(x,\bar{g})$ has a unique interior maximum, we have

$$l(x,\bar{g})\leq l(\bar{x},\bar{g})=0$$

[19] Here, $l_x(x,v)$ is the partial derivative with respect to x, etc.

[20] $g(x)$ is positive and cannot be identically zero on $(0,1)$. Since $g(0)=g(1)=0$ this implies that $g(x)$ indeed has an interior maximum.

for all x. Because $l(x,v)$ is decreasing in v, we also have

$$l\left(x,\frac{c}{\lambda}\right) < l(x,\bar{g})$$

for all x. Let $\bar{x}_l$ be the maximizer of $l\left(x,\frac{c}{\lambda}\right)$. We then have

$$\max_x l\left(x,\frac{c}{\lambda}\right) = l\left(\bar{x}_l,\frac{c}{\lambda}\right) < l(\bar{x}_l,\bar{g}) \leq l(\bar{x},\bar{g}) = 0,$$

which is a contradiction. This establishes that whenever $l\left(x,\frac{c}{\lambda}\right)$ exceeds zero, $g(x)$ crosses $\frac{c}{\lambda}$ so shirking cannot be an equilibrium.

We now show that whenever $\max_x l\left(x,\frac{c}{\lambda}\right) \leq 0$, the shirking equilibrium exists. Our previous arguments will then imply uniqueness and we do not repeat them here. If $\max_x l\left(x,\frac{c}{\lambda}\right) < 0$, then $g(x) = \frac{c}{\lambda}$ implies that $g'(x) > 0$. Thus, once $g(x)$ crosses $\frac{c}{\lambda}$ from below, it must always stay above it. But this is incompatible with the boundary condition $g(1) = 0$. Thus, we must have $g(x) < \frac{c}{\lambda}$ for all x. Shirking is then an equilibrium.

Finally, we study the remaining case $\max_x l\left(x,\frac{c}{\lambda}\right) = 0$. Suppose in that case $g(x)$ exceeds $\frac{c}{\lambda}$. If this is true, then $g(x)$ crosses $\frac{c}{\lambda}$ at at least two values $x_1 < x_2$. At both values we must have $g'(x_1) = g'(x_2) = 0$. But since at most a single value of x attains $l\left(x,\frac{c}{\lambda}\right) = 0$, this is impossible. Thus, in that case, $g(x)$ is at most tangent to $\frac{c}{\lambda}$ at one point, but never crosses it. This means shirking is still an equilibrium. ∎

We have now concluded our characterization. Using the functions g and l, we record some additional properties of $V_s(x)$ and $g(x)$ below. These will be useful when analyzing the shirk-work-shirk and work-shirk cases.

Lemma 4. *Either $V_s(x) \leq \frac{c}{\lambda(1-x)}$ for all x, or $V_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ exactly twice.*

Proof. To prove the result, suppose by way of contradiction that $g(x)$ crosses $\frac{c}{\lambda}$ more than twice. Since $g(0) = g(1) = 0 < \frac{c}{\lambda}$, g must cross $\frac{c}{\lambda}$ an even number of times. Thus, there must exist three values $x_1 < x_2 < x_3$ at which $g(x)$ equals $\frac{c}{\lambda}$ for which $g'(x_1) \leq 0, g'(x_2) \geq 0$,and $g'(x_3) \leq 0$. This implies that $l\left(x_1,\frac{c}{\lambda}\right) \geq 0$, $l\left(x_2,\frac{c}{\lambda}\right) \leq 0$, and $l\left(x_3,\frac{c}{\lambda}\right) \geq 0$. But this is impossible because $l\left(x,\frac{c}{\lambda}\right)$ is strictly concave in x. If $l\left(x_2,\frac{c}{\lambda}\right)$ is non-positive, then $l\left(x_3,\frac{c}{\lambda}\right)$ must be strictly negative. Thus we have our contradiction, which establishes the result. ∎

The result in the Lemma above extends to *all* solutions of the shirking ODE 5, for which $g(1) < \frac{c}{\lambda}$, not just the one that satisfies $V_s(1) = \frac{\alpha}{r}$. Intuitively, l is independent of the particular solution we have used, all solutions satisfy $g(0) = 0$ (because $V_s(0) = 0$) and we only need $g(1) < \frac{c}{\lambda}$ to ensure that g crosses $\frac{c}{\lambda}$ an even number of times.

As we have seen in Equation 15, all solutions to the shirking ODE 5 can be indexed by an initial value v_0 at some common initial point $x_0 \in (0,1)$. We write them as $V_s(x, v_0)$ again and we denote with v_0^s the initial value that yields the shirking value. That is, $V_s(x, v_0^s) = V_s(x)$ is the solution that satisfies $V_s(1) = \frac{\alpha}{r}$.

Lemma 5. *All solutions $V_s(x, v_0)$ to Equation 5 satisfy the following properties:*

1. *The functions g never cross: Let $g(x, v_0) = (1-x)V_s(x, v_0)$. For $v_0' > v_0$, we have $g(x, v_0') > g(x, v_0)$ for all $x \in (0,1)$. Moreover, $g'(x, v_0') > g'(x, v_0)$.*
2. *Any solution that satisfies $g(1, v_s) = 0$ has a single interior maximum. It is weakly increasing to the left and decreasing at the right of the maximum.*
3. *Any solution that is larger than the shirking value, i.e. $v_0 > v_0^s$, crosses $\frac{c}{\lambda(1-x)}$ exactly once below x_h. The crossing point is left of x_l.*
4. *Any solution that is smaller than the shirking value, i.e. $v_0 < v_0^s$, crosses $\frac{c}{\lambda(1-x)}$ at most twice.*

Proof.

1. We can rewrite Equation 5 as

$$\begin{aligned}\left(r+\lambda(1-x)^2\right)V_s(x) &= \alpha\left(1-(1-x)^2\right)\\ &\quad +\lambda x(1-x)\left((1-x)V_s'(x)-V_s(x)\right).\end{aligned}$$

   We have $g'(x) = (1-x)V_s'(x) - V_s(x)$. Thus, $V_s(x, v_0') > V_s(x, v_0)$ if and only if $g'(x, v_0') > g'(x, v_0)$. But since $g(0, v_0) = g(0, v_0')$, this means that $g(x, v_0') > g(x, v_0)$ for all $x \in (0,1)$.

2. Suppose that there is a local maximum $\tilde{g}$ at point $\tilde{x}$ and a global one at $\bar{g}$. Then, $l(\tilde{x}, \tilde{g}) = 0$. Since $\tilde{g} < \bar{g}$, $g(x)$ must cross $\tilde{g}$ for at least two values $x_1 < x_2$ right of $\tilde{x}$. Without loss of generality, we can choose them so that $g'(x_1) \geq 0$ and $g'(x_2) \leq 0$. But this implies that $l(x_1, \tilde{g}) \leq 0$ and $l(x_2, \tilde{g}) \geq 0$. Inspecting the shape of $l(x, v)$, we see this is impossible. Even if $\tilde{x}$ is the first point where $l(x, \tilde{g})$ intersects zero, $l(x_1 \tilde{g}) \leq 0$ must imply $l(x_2, \tilde{g}) < 0$.

3. From Lemma 2, Point 3, we know that $g(x, v_0) > g(x)$. Therefore, it must cross $\frac{c}{\lambda}$. Also, any points where $g(x, v_0)$ crosses $\frac{c}{\lambda}$ must lie left of x_l. For $x < x_l$, we have $g'(x) > 0$, which follows from the previous point. But if $g(x, v_0)$ crosses $\frac{c}{\lambda}$ multiple times on that region there must exist a value of x where $g'\left(x, \frac{c}{\lambda}\right) < 0$. However, Lemma 2, Point 4 implies that $g'(x, v_0) > g'(x)$, so this is impossible.

4. Any solution with $v_0 < v_0^s$ must have $g(x, v^0) \leq g(x)$ for $x \in [0,1]$. Thus, $g(1, v_0) \leq 0$. We can now use exactly the argument we used to prove Lemma 4.

∎

The following result, which characterizes the derivative of $V_s(x)$ when the work region is nonempty, will be useful in further developments. We record it here to avoid repeating similar arguments later.

Corollary 11. *Whenever $V_s(x)$ crosses $\frac{c}{\lambda(1-x)}$, at two points $x_1 < x_2$, we must have*

$$V_s'(x_2) < \frac{d}{dx_2}\frac{c}{\lambda(1-x_2)}.$$

Proof. From Proposition 10 we know that $V_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ whenever $\max_x l\left(x, \frac{c}{\lambda}\right) > 0$. Its derivative is strictly below the derivative of $\frac{c}{\lambda(1-x)}$ whenever $g'(x)$ is negative. Since $g(x)$ is continuously differentiable, we have $g'(x_2) \leq 0$, so we only have to show that the inequality is strict. Suppose that $g'(x_2) = 0$.

To show the result, we study $g''(x)$, which we can express as

$$\lambda x (1-x)^2 g''(x) = 2\lambda(1-x) g(x) + 3\alpha(1-x)^3 - \alpha - g'(x)(r + 2\lambda x(1-x))$$

by differentiating Equation 16. We know that $g(x)$ attains its maximum $\bar{g}$ at some $\bar{x}$ between x_1 and x_2. At that value, we must have

$$g''(\bar{x}) \leq 0.$$

This implies that

$$2\lambda(1-\bar{x})\bar{g} + 3\alpha(1-\bar{x})^3 - \alpha \leq 0,$$

because $g'(\bar{x}) = 0$. Now, the expression above is strictly decreasing in x and strictly increasing in g. Therefore, it must be negative at $x_2 > \bar{x}$ and $g(x_2) = \frac{c}{\lambda} < \bar{g}$. Thus, if $g'(x_2) = 0$, then $g''(x_2) < 0$. But this is impossible. If these two conditions hold then x_2 is a local maximum, whereas we constructed x_2 so that g crosses $\frac{c}{\lambda}$ from above. ∎

The proof above does not rely on any particular boundary conditions for g. We can thus extend it to any arbitrary solution to the shirking ODE 5. That is, when a solution crosses $\frac{c}{\lambda(1-x)}$, its slope at the larger crossing point x_h must be strictly below the slope of $\frac{c}{\lambda(1-x)}$. This observation will be useful in the shirk-work-shirk case.

A.2 Shirk-Work-Shirk Equilibrium (Proposition 2)

We construct the equilibrium using a method similar to backward induction. For sufficiently high reputation, the firm will shirk because there is no manipulation and hence no news.[21] We construct this upper shirking interval by finding the point where $V_s(x)$ hits the function $\frac{c}{\lambda(1-x)}$. This point exists because of the parametric assumptions in Proposition 2. We call it x_h. The shirking value on $[x_h, 1]$ does not depend on the equilibrium played at $x < x_h$, so we can compute it independently.

When the strategic type's effort is interior, he must be indifferent between working and shirking. We combine this indifference condition with the ODE that describes the value function for arbitrary effort (Equation 4) and use them to derive an ODE for the effort itself. In equilibrium, effort and manipulation are equivalent[22] and we express the ODE in terms of manipulation since it is easier to study. This is Equation 7.

We then solve the initial value problem (IVP) for this ODE with the initial condition $m(x_h) = 1 - x_h$.[23] This condition is equivalent to $a(x_h) = 0$, which is consistent with the firm shirking for $x \geq x_h$. The solution to the IVP must cross the function $1-x$ exactly once below x_h, which we show. We label this crossing point x_l. The interval $[x_l, x_h]$ is then our working interval.

Finally, we construct the lower shirking interval $[0, x_l]$. To do this, we solve the shirking ODE 5 on $[0, x_l]$ with the value matching condition $V_s(x_l) = \frac{c}{\lambda(1-x_l)}$. At x_l, the firm must be indifferent between working and shirking, which motivates this condition. Then, it only remains to verify that shirking is indeed optimal on $[0, x_l]$. That is, $V_s(x)$ cannot cross $\frac{c}{\lambda(1-x)}$ on the interior of that interval.

We start with a the point $x_h \in (0,1)$ at which the shirking value satisfies

$$V_s(x_h) = \frac{c}{\lambda(1-x_n)}.$$

As we have just argued, such x_h exists. We conjecture that on an interval left of x_h, the firm exerts effort $a(x) \in (0,1)$.[24] The strategic type's value 4 is linear in effort. Whenever

[21] See Lemma 3.

[22] This is because we have $m(x) = (1-x)(1-a(x))$ in any equilibrium.

[23] To be precise, we are solving Equation 7 backwards in x. In this sense we can still understand $m(x_h) = 1 - x_h$ as an initial condition. Equivalently, we can think of it as a boundary condition which has to be satisfied by the right solution to the ODE, with the initial condition being taken at some $x_0 \in (0, x_h)$.

[24] Recall that there can never be an equilibrium where the firm exerts an effort of one. This is because exerting effort is only optimal when

$$\lambda(1-x)(1-\hat{a}(x))V(x) \geq c.$$

But if equilibrium effort is one, i.e. $\hat{a}(x) = 1$, this condition cannot hold.

$a(x) \in (0,1)$, he must therefore be indifferent between working and shirking. Equivalently, if $m(x)$ is the equilibrium manipulation, we must have

$$\lambda m(x) V(x) = c. \tag{18}$$

Differentiating this expression yields

$$m'(x) V(x) + V'(x) m(x) = 0.$$

In equilibrium, $m(x)$ must be such that the strategic type's value solves Equation 4 and the indifference condition, since he anticipates future effort and manipulation. Plugging in the two conditions above in the Equation 4, we can derive an expression for the equilibrium level of manipulation that satisfies this. After some algebra, we then arrive at Equation 7.

We pin down the working interval by showing that there exists a unique point $x_l \in (0, x_h)$ for which $m(x_l) = 1 - x_l$. To do this, we first characterize the solution to the ODE 7 with boundary condition $m(x_h) = 1 - x_h$ in the Lemmma below.

Lemma 6. *$m(x)$ has the following properties:*

1. *$m'(x) < 0$ and $m(x) > 0$ for $x \in [0, x_h]$.*
2. *$\lim_{x \to 0} m(x) = \infty$.*
3. *$m'(x_h) > -1$.*
4. *$m(x)$ crosses $1 - x$ once for $x < x_h$.*

Proof. 1. To prove the first point, note that whenever $m'(x) \geq 0$, Equation 7 implies that

$$rc < \lambda(\alpha - c) m(x) - \alpha\lambda m(x)^3.$$

Suppose that $m(x) \geq 0$. The right hand side of this inequality is strictly concave and it reaches its unique maximum at the point $m^* = \sqrt{\frac{\alpha - c}{3\alpha}}$. Its maximum value is

$$\lambda\left((\alpha - c)\sqrt{\frac{\alpha - c}{3\alpha}} - \alpha\left(\frac{\alpha - c}{3\alpha}\right)^{\frac{3}{2}}\right) = \frac{\lambda(\alpha - c)^{\frac{3}{2}}}{\sqrt{\alpha}}\frac{2}{3\sqrt{3}}.$$

By our assumption in Proposition 2, this value is below rc. Therefore, $m'(x)$ cannot be positive whenever $m(x)$ is positive. Since we have started with the condition $m(x_h) = 1 - x_h \in (0,1)$, this implies that at x_h, $m(x)$ is decreasing. But then $m(x)$ must remain positive for all $x \leq x_h$ and therefore its derivative must remain negative.

2. To show that $m(x)$ must go to infinity as x approaches zero, we rewrite Equation 7 as

$$\int_{m(x_h)}^{m(x)} \frac{\lambda cm}{\lambda\alpha\left(m-m^3\right)-\lambda cm-rc}dm = \log(x) - \log(x_h).$$

We obtain this expression by exploiting the fact that Equation 7 is a separable first-order ODE.[25] As $x \to 0$ the RHS diverges to minus infinity. The derivative of the LHS is

$$\frac{\lambda cm}{\lambda\alpha\left(m-m^3\right)-\lambda cm-rc},$$

which is negative for all $m \geq 0$. Thus, we must have $m(x) \to \infty$. Otherwise, the RHS cannot match the LHS.

3. We compare the ODE characterizing V with the one characterizing V_s. The first equation is

$$(r+\lambda m(x))V(x) = \alpha\left(1-m(x)^2\right) + \lambda x m(x)^2 V'(x)$$

while the second is the shirking ODE 5. At x_h, we have $m(x_h) = 1 - x_h$. Thus, the coefficients of both equations are the same at x_h. Since we also have $V(x_h) = V_s(x_h)$, it must be that $V'(x_h) = V_s'(x_h)$. From Corollary 11, we know that $V_s'(x_h) < \frac{c}{\lambda(1-x_h)^2}$. On the work region, we have $V'(x) = -m'(x)\frac{c}{\lambda m(x)^2}$. Combining these expressions and plugging in $m(x_h) = 1 - x_h$ then yields $m'(x_h) > 1$.

4. The proof proceeds similarly to the proof of Lemma 4. Whenever $m(x) = 1 - x$, Equation 7, which is the ODE characterizing $m(x)$, becomes

$$0 = \lambda(\alpha - c)(1-x) - \lambda\alpha(1-x)^3 - \lambda cx(1-x)m'(x) - rc.$$

We can rewrite it as

$$0 = \lambda l\left(x, \frac{c}{\lambda}\right) - \lambda cx(1-x)\left(m'(x)+1\right),$$

where $l(x,v)$ is the function we have defined previously in Equation 17. Note that $m'(x) > -1$ if and only if $l\left(x,\frac{c}{\lambda}\right) > 0$. Likewise, $m'(x) < -1$ if and only if $l\left(x,\frac{c}{\lambda}\right) < 0$. Since we are in the case with a nonempty working region, we have $\max_x l\left(x,\frac{c}{\lambda}\right) > 0$, according to Proposition 10. Recall that $l\left(x,\frac{c}{\lambda}\right)$ is hump-shaped. That is, it is negative for x small, positive for intermediate x, and negative again for large x.Also, by Point 3 we know that $m'(x_h) > -1$ and that $l\left(x_h,\frac{c}{\lambda}\right) > 0$. Since $m(x)$ diverges to infinity as

[25]See e.g. Polyanin and Zaitsev (2002), p. 3.

x becomes small, we already know that it must cross $1-x$ at least once below x_h. It must also cross $1-x$ an odd number of times left of x_h.[26] We thus only have to prove that it does not cross more than once. By way of contradiction, suppose there are three points $x_1 < x_2 < x_3 < x_h$ at which $m(x)$ crosses $1-x$. We must have $m'(x_1) \leq -1$, $m'(x_2) \geq -1$, and $m'(x_3) \leq -1$ and thus $l\left(x_1, \frac{c}{\lambda}\right) \leq 0$, $l\left(x_2, \frac{c}{\lambda}\right) \geq 0$ and $l\left(x_3, \frac{c}{\lambda}\right) \leq 0$. Since l is hump-shaped and strictly concave, the last condition is incompatible with $l\left(x_h, \frac{c}{\lambda}\right) > 0$. Once l becomes negative after being positive, it must stay negative. This is our contradiction.

∎

The first point of the Lemma guarantees that the IVP for $m(x)$ with initial condition $m(x_h) = 1 - x_h$ has a *unique* solution on any interval $[\varepsilon, x_h]$ for some small $\varepsilon > 0$. This is because $m(x) \geq 1 - x_h > 0$, so we can apply the Picard-Lindelöf Theorem to Equation 7. The last point of the Lemma characterizes the working region.

We now conclude the proof by showing that shirking is indeed optimal on $[0, x_l]$. For any point (v, x) such that $x \in (0, x_h)$ and $v = \frac{c}{\lambda(1-x)}$, there exists a solution to the shirking ODE 5 that hits that point. This is because any solution to Equation 5 is strictly increasing and continuous in its initial condition.[27] Take a solution which hits $\left(x_l, \frac{c}{\lambda(1-x_l)}\right)$, which we denote with $\tilde{V}_s(x)$.[28] For the shirking equilibrium to exist on $[0, x_l]$, we need that $\tilde{V}_s(x)$ stays below $\frac{c}{\lambda(1-x)}$ on that interval. Otherwise, the firm would strictly prefer to exert effort. At x_l, we have $m(x_l) = 1 - x_l$ and $V(x_l) = \tilde{V}_s(x_l)$. Comparing the ODEs for V and the ODEs for $\tilde{V}_s$, just as we did in the proof of Lemma 6 above, we see that $V'(x_l) = \tilde{V}_s'(x_l)$. That is, V and $\tilde{V}_s$ satisfy smooth pasting. Since at x_l, we have $m'(x_l) \leq -1$, we know that $\tilde{V}_s'(x_l) \geq \frac{c}{\lambda(1-x_l)^2}$.

Recall that solutions to the shirking ODE are ordered, so that one solution is always larger than the other on $(0,1)$.[29] If $\tilde{V}_s(x) < V_s(x)$, then Corollary 11 applies. $\tilde{V}_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ at two points $\tilde{x}_l$ and $\tilde{x}_h$, first from below and then from above. Thus either $x_l = \tilde{x}_l$ or $x_l = \tilde{x}_h$. At $\tilde{x}_h$, we have $\tilde{V}_s'(\tilde{x}_h) < \frac{c}{\lambda(1-\tilde{x}_h)^2}$, which is incompatible with the smooth pasting condition. Therefore, $x_l = \tilde{x}_l$, i.e. x_l is the first time $\tilde{V}_s(x)$ crosses $\frac{c}{\lambda(1-x)}$ and it cannot cross left of x_l. Shirking on the region $[0, x_l]$ is thus optimal. If $\tilde{V}_s(x) > V_s(x)$,by Lemma

[26]If $m(x)$ would cross $1-x$ an even number of times for $x < x_h$, then after it crosses the last time (at the lowest x), it must stay below $1-x$. This can be seen graphically. But this is impossible, because we have shown that $m(x)$ goes to infinity as x goes to zero.

[27]This can be seen by inspecting Equation 15 in the proof of Lemma 2.

[28]We label this equation differently from V_s to avoid confusion. Both equations satisfy the Equation 5, but they are *different solutions*. In particular, $\tilde{V}_s$ cannot satisfy the boundary condition $\tilde{V}_s(1) = \frac{\alpha}{r}$ unless it is identical to V_s.

[29]We have shown this in Lemma 2, Point 3.

5, Point 2, it crosses $\frac{c}{\lambda(1-x)}$ at exactly one point left of x_h. By construction, that point is x_l. Therefore, $\tilde{V}_s(x)$ for $x < x_l$.

We have established that a shirk-work-shirk equilibrium exists. We now show that it is unique. First, any equilibrium with working must have an upper shirking region. This region must equal $[x_h, 1]$. We have shown this when we proved uniqueness for the shirking equilibrium in Proposition 1 and the same argument applies here. Thus, if there exists a working region, there must be one that has x_h as its upper bound. On any nonempty working region, $m(x)$ must satisfy Equation 7, otherwise, it is not consistent with the indifference condition of the strategic type 18 or the value function. Lemma 6 guarantees that Equation 7 has a *unique* solution that satisfies $m(x_h) = 1 - x_h$. Thus, in any equilibrium with a working region bordering x_h, that working region must be $[x_l, x_h]$. We now show that there cannot be any other working region left of x_l. Let $\hat{x}_h < x_l$ be the right boundary of such a region. We must have $m(\hat{x}_h) = 1 - \hat{x}_h$ and $V(\hat{x}_h) = \frac{c}{\lambda(1-\hat{x}_h)}$. For $x \in (\hat{x}_h, x_l)$, we must have $V(x) = \tilde{V}_s(x)$. Here, $\tilde{V}_s(x)$ is the solution to the shirking ODE with boundary condition $\tilde{V}_s(x_l) = \frac{c}{\lambda(1-x_l)}$ which we have defined above.[30] But since $\tilde{V}_s(x) < \frac{c}{\lambda(1-x)}$ for $x < x_l$, we must have a discontinuity at $\hat{x}_h$. This is a contradiction, since the value V in any equilibrium must be continuous for $x > 0$. Therefore, there can be no equilibrium with a working region left of x_l. This proves that our equilibrium is unique.

A.3 Work-Shirk Equilibrium (Proposition 3)

The proof proceeds similarly to the proof of Proposition 2. We define x_h as the right-most point where $V_s(x)$ hits $\frac{c}{\lambda(1-x)}$, which given our assumptions is guaranteed to exist. Then, we solve for the ODE for $m(x)$. Unlike in Proposition 2, the solution does not necessarily hit $1 - x$. Instead, it can converge to a finite value between zero and one as x approaches zero. This will constitute a work-shirk equilibrium. The strategic type's value is discontinuous at $x = 0$. [31]

We start with recording some properties of $m(x)$. This Lemma is analogous to Lemma 6.[32]

Lemma 7. *$m(x)$ has the following properties.*

[30]Again, it is important to note here that any solution to the shirking ODE is "forward-looking". It only depends on higher values of x and is independent of the equilibrium played for lower values.

[31]That is, $V(0) = 0$ but $V(x) \geq \varepsilon > 0$ for all $x > 0$ and some $\varepsilon > 0$. Previously, we have argued that in any equilibrium, $V(x)$ needs to be continuous for $x > 0$. We have *not* argued that it needs to be continuous at $x = 0$. A possible discontinuity at zero is not surprising. The state $x = 0$ is absorbing, while for any $x > 0$ all other states $x' > x$ can be reached with positive probability.

[32]The behavior of $m(x)$ is now different because of the changed parameter assumptions, even though $m(x)$ still satisfies Equation 7 and the same boundary condition as before, albeit at a different x_h.

1. *There exist two values* $0 < \underline{m} \leq \bar{m} < 1$ *such that* $m'(x) > 0$ *if* $m(x) \in (\underline{m}, \bar{m})$, $m'(x) < 0$ *if* $m(x) > \bar{m}$ *or* $m(x) < \underline{m}$, *and* $m'(x) = 0$ *if* $x \in \{\underline{m}, \bar{m}\}$.

2. $m(x)$ *never crosses* $\underline{m}$ *or* $\bar{m}$.

3. $m(x) > 0$.

4. *If* $1 - x_h < \bar{m}$, *then* $\lim_{x \to 0} m(x) = \underline{m}$.

Proof. 1. We can write Equation 7 as

$$0 = \lambda(\alpha - c) m(x) - \lambda \alpha m(x)^3 - rc - \lambda c x m'(x) m(x). \tag{19}$$

We are interested in the properties of the function

$$h(m) = \lambda(\alpha - c) m - \lambda \alpha m^3 - rc.$$

This function determines whether $m'(x)$ is positive or negative, because

$$0 = h(m(x)) - \lambda c x m'(x) m(x).$$

Under the condition $\frac{\lambda(\alpha-c)^{\frac{3}{2}}}{\sqrt{\alpha}} \frac{2}{3\sqrt{3}} \geq rc$, $h(m)$ has two roots $0 < \underline{m} \leq \bar{m} < 1$. Inspecting its shape, we have $h(m) < 0$ for $m < \underline{m}$ and for $m > \bar{m}$ and $h(m) > 0$ for $m \in (\underline{m}, \bar{m})$. If $m(x) < \underline{m}$ or $m(x) > \bar{m}$, we must have $m'(x) < 0$. If $m(x) \in (\underline{m}, \bar{m})$ we must have $m'(x) > 0$.

2. Whenever $m(x)$ equals $\underline{m}$ or $\bar{m}$, $m'(x)$ and all higher derivatives are zero. We can show this by differentiating Equation 7 and plugging in values. Specifically, if $m = \bar{m}$ we have $m'(x) = 0$. The second derivative satisfies

$$\lambda x m(x) m''(x) = m'(x) \left(\lambda(\alpha - c) - 3\lambda\alpha m(x)^2 - \lambda c m(x) - \lambda c x |m'(x)|\right),$$

which is zero. Successively differentiating this equation and plugging in lower order terms yields the result.

3. If $1 - x_h < \underline{m}$, then $m(x)$ is decreasing, which implies that $m(x)$ is bounded above zero on $[0, x_h]$. If $1 - x_h \geq \underline{m}$ then $m(x)$ can never cross $\underline{m}$, so it must be positive.

4. We use the integral representation for Equation 7,

$$\int_{m(x_h)}^{m(x)} \frac{\lambda c m}{h(m)} dm = \log(x) - \log(x_h),$$

which we have used previously in the proof of Lemma 6. If $1 - x_h < \underline{m}$, we have $m(x) \leq \underline{m}$, so the derivative of the LHS must be negative at the solution $m(x)$. If $m(0) < \underline{m}$, then as $x \to 0$, the LHS is bounded whereas the RHS diverges to $-\infty$. Thus, we must have $m(0) = \underline{m}$. If $1 - x_h \in (\underline{m}, \bar{m})$, $m(x)$ is increasing and $h(m(x))$ is positive. Then, we can rewrite the equation as

$$-\int_{m(x)}^{m(x_h)} \frac{\lambda c m}{h(m)} dm = \log(x) - \log(x_h).$$

The LHS goes to zero as $x \to 0$ whenever $m(0) = \underline{m}$.

∎

A work-shirk equilibrium exists whenever $1 - x_h < \bar{m}$. In that case, $m(x)$ remains below $1 - x$ for all $0 < x < x_h$ and it converges to $\underline{m}$ as $x \to 0$. We now show this is the case given our assumptions in Proposition 3. To prove this result, we exploit the functions $l\left(x, \frac{c}{\lambda}\right)$ in Equation 17 and $h(m)$ in Equation 19. We construct a sequence of inequalities which will imply that $1 - x_h < \bar{m}$. We are interested in $h(m)$ only when $m = 1 - x$, so with slight abuse of notation we write it as

$$h(x) = \lambda(\alpha - c)(1 - x) - \lambda\alpha(1 - x)^3 - rc.$$

$\bar{m}$ is the largest root of $h(m)$, so $1 - \bar{m}$ is the smallest root of $h(x)$. We are trying to prove that $1 - \bar{m} < x_h$, i.e. that x_h lies above the smallest root of $h(x)$.[33] The condition $c < \alpha$ will imply that the maximum of $h(x)$ must be left of the maximum of $l\left(x, \frac{c}{\lambda}\right)$ and we will argue that x_h must lie to the right. Since h is hump-shaped its maximum is right of its smallest root, so this implies $1 - x_h < \bar{m}$.

Consider the value c_0 at which the shirking value $V_s(x)$ is tangent to $\frac{c_0}{\lambda(1-x)}$ at a single point, x_0. At that point, we must have $l\left(x_0, \frac{c_0}{\lambda}\right) = 0$ and x_0 must achieve the maximum of $l\left(x, \frac{c_0}{\lambda}\right)$ For any $c < c_0$, we have $x_l < x_0 < x_h$.[34] Inspecting $l\left(x, \frac{c}{\lambda}\right)$, we can see that $\frac{\partial^2 l\left(x, \frac{c}{\lambda}\right)}{\partial x \partial c} > 0$, i.e. the value at which l attains its maximum is increasing in c. We denote this value with x_l^* and omit the dependence on c. For $c < c_0$, we have $x_l^* < x_0$. We can rewrite $h(x)$ as

$$h(x) = \lambda l\left(x, \frac{c}{\lambda}\right) - \lambda c x(1 - x).$$

[33] Recall that whenever $h(x)$ exceeds zero on $[0, 1]$ it has exactly two roots on that interval.

[34] Note that the shirking value $V_s(x)$ is independent of c, so changing c does not affect it. For lower c, $\frac{c}{\lambda(1-x)}$ is lower while $V_s(x)$ is the same, so the intersection points must move to the left and right of x_0, respectively.

This implies

$$h'\left(x_l^*\right) = \lambda c\left(2x_l^* - 1\right).$$

If $x_l^* < \frac{1}{2}$, then h is decreasing at x_l^*. But then x_h^*, the value at which h attains its maximum, must lie left of x_l^*. $1 - \bar{m}$ is the first root of $h(x)$ and it must lie left of x_h^*. Taken together, our arguments yield the following chain of inequalities:

$$\text{if } x_l^* < \frac{1}{2}, \text{ then } 1 - \bar{m} \leq x_h^* < x_l^* < x_0 < x_h.$$

We now only have to show that $x_l^* < \frac{1}{2}$ We can compute $\frac{\partial}{\partial x} l\left(x, \frac{c}{\lambda}\right) = 0$ and find the maximizer that lies in $[0,1]$. It is given by

$$x_l^* = 1 + \frac{1}{3}\frac{c}{\alpha} - \sqrt{\frac{1}{9}\left(\frac{c}{\alpha}\right)^2 + \frac{1}{3}}.$$

Then, the result follows from the fact that we assumed $c < \alpha$ and simple algebra.

Whenever the work-shirk equilibrium exists, it is unique. The argument for this is analogous to the one in our proof of Proposition 2. As we have already seen, the shirking region $[x_h, 1]$ must be the same for any potential equilibrium. Suppose that there exists another equilibrium where on some region $[\hat{x}_l, \hat{x}_h] \subset [0, x_h]$, the agent shirks. Then, we need that $m(\hat{x}_h) = 1 - \hat{x}_h$, otherwise, the value function would not be continuous.[35] However, the ODE 7 with boundary condition $m(x_h) = 1 - x_h$ has a *unique* solution and in a work-shirk equilibrium, that solution never hits $1 - x$ for $x < x_h$. Thus, we cannot have such a region. This establishes uniqueness.

B Disclosure

B.1 Shirking Case

The proof of Proposition 4 is immediate because the regulator's value is linear: $W_s(x) = \frac{\alpha}{r}x$.

We now show that disclosure reduces the value of the strategic type in the shirking equilibrium.

Proposition 12. *In the shirking equilibrium, the strategic type prefers no disclosure to delayed or immediate disclosure.*

[35]Specifically, if $m(\hat{x}_h) < 1 - \hat{x}_h$, then $V(\hat{x}_h) = \frac{c}{\lambda m(\hat{x}_h)} > \frac{c}{\lambda(1-\hat{x}_h)}$. But since the firm is supposed to shirk left of $\hat{x}_h$, we must also have $V(x) \leq \frac{c}{\lambda(1-x)}$ for all $x \in (\tilde{x}_n - \varepsilon, \hat{x}_h)$ for some small $\varepsilon > 0$. This means the value is discontinuous, which cannot be true.

Proof. Immediate disclosure at x sends the strategic type's value to zero instantly, which is clearly detrimental. The value of delayed bad news is negative whenever $(1-x)\,xV_s'(x) - V_s(x)$ is negative. Using Equation 5, we can see that this is equivalent to $rV_s(x) < \alpha\left(1-(1-x)^2\right)$. We show this by characterizing the function $\varphi(x) = rV_s(x) - \alpha\left(1-(1-x)^2\right)$. Its derivative is $\varphi'(x) = rV_s'(x) - 2\alpha(1-x)$ and it satisfies the ODE

$$(r+\lambda(1-x))\,\varphi(x) = \lambda\alpha\left(2x(1-x)^3 - (1-x) + (1-x)^3\right) + \lambda x(1-x)^2\varphi'(x).$$

Because $V_s(0) = 0$ and $V_s(1) = \frac{\alpha}{r}$, the relevant boundary conditions are $\varphi(0) = \varphi(1) = 0$. The term multiplying $\lambda\alpha$ is negative for all $x \in (0,1)$ and zero for $x = 0$ and $x = 1$. If $\varphi(x) \geq 0$ then $\varphi'(x) > 0$ and therefore $\varphi(x') > 0$ for all $x' > x$. But this is inconsistent with $\varphi(1) = 0$. Thus, we must have $\varphi(x) < 0$ for all $x \in (0,1)$. Finally, the value of delayed good news is negative whenever $V_s'(x)$ is positive, which is true. ∎

B.2 Shirk-Work-Shirk Case

B.2.1 Delayed Bad News

The value function of the regulator in Equation 9 is constructed similarly to the function of the firm. On the upper shirking region $[x_s, 1]$, the regulator's value is her shirking value. It solves Equation 9 with $m(x) = 1-x$, i.e.

$$\left(r+\lambda(1-x)^2\right)W_s(x) = \alpha x + \lambda x(1-x)^2 W_s'(x), \tag{20}$$

with boundary condition $W(1) = \frac{\alpha}{r}$. The shirking value is has the closed form solution $W_s(x) = \frac{\alpha}{r}x$. On the working region $[x_l, x_h]$, the regulator's value solves Equation 9 and $m(x)$ is the manipulation obtanined in Equation 7 of Proposition 2. Since the regualtor's value is continuous, it satisfies the boundary condition $W(x_h) = \frac{\alpha}{r}x_h$. On the lower shirking region $[0, x_l]$, the regulator's value is again the shirking value in Equation 20, but it has boundary condition $W_s(x_l) = W(x_l)$, where W is the solution obtained on the working region.[36]

Lemma 8. *On the working region, we have $W(x) > W_s(x)$.*

Proof. On the working region, the regulator's value satisfies Equation 9 with boundary condition $W(x_h) = W_s(x_h) = \frac{\alpha}{r}x_h$. If $W(x) \leq \frac{\alpha}{r}x$ for some $x \in (x_l, x_h)$,

$$\lambda x m(x)^2\left(W'(x) - \frac{\alpha}{r}\right) \leq \alpha\,(x - (1-m(x))).$$

[36]Because of this boundary condition, $W_s(x)$ does not equal $\frac{\alpha}{r}x$ on $[0, x_l]$.

The last term is negative, because $m(x) < 1 - x$. Thus, $W'(x) < \frac{\alpha}{r}$ whenever $W(x) \leq \frac{\alpha}{r}x$. Once $W(x)$ touches $\frac{\alpha}{r}x$ on the working region, it must stay strictly below to the right. But this is incompatible with the boundary condition at x_h. Thus, the solution to $W(x)$ must satisfy $W(x) > \frac{\alpha}{r}x$ on (x_l, x_h). ∎

We are now ready to prove Proposition 7, which we restate below for convenience.

Proposition 7. *Disclosure of bad news is valuable to the regulator when $x \leq x_d$, where $x_l \leq x_d < x_h$. It is not valuable for x close to (or above) x_h.*

Proof. Delayed disclosure is valuable to the regulator whenever $W'(x)x - W(x)$ is strictly positive. This is equivalent to

$$\phi(x) := rW(x) - \alpha(1 - m(x)) > 0,$$

which follows from rearranging Equation 9. From Lemma 8, we know that $W(x_l) > \frac{\alpha}{r}x_l = \frac{\alpha}{r}(1 - m(x_l))$. Thus, $\phi(x_l) > 0$. By continuity $\phi(x)$ remains positive on a neighborhood to the right of x_l. On $[0, x_l]$, the value of information is positive whenever[37]

$$\phi(x) = rW_s(x) - \alpha x > 0,$$

where $W_s(x)$ is the shirking value in Equation 20 with boundary condition $W_s(x_l) = W(x_l) > \frac{\alpha}{r}x_l$. Thus, we have to show that the shirking value exceeds $\frac{\alpha}{r}x$. This is true because two solutions to Equation 20 cannot cross. This follows from an argument analogous to the one in Lemma 2. We can express $W_s(x)$ using the method of integrating factors and then show that each solution can be indexed by the initial value at a (wlog) common initial point. A solution with a larger initial value must then always lie above a solution with a lower one. $\frac{\alpha}{r}x$, which is the solution that satisfies $W_s(x_l) = \frac{\alpha}{r}x_l$ must therefore lie below the solution that satisfies $W_s(x_l) = W(x_l)$. This implies the value of disclosure is positive on $(0, x_l)$.

To show that disclosure is detrimental when reputation becomes sufficiently large, we plug in $\phi(x)$ and $\phi'(x) = rW'(x) + \alpha m'(x)$ into Equation 9, to obtain an ODE for $\phi(x)$, which is

$$\left(r + \lambda m(x)^2\right)\phi(x) = \lambda m(x)^2\left(x\phi'(x) - \alpha x m'(x) - \alpha(1 - m(x))\right).$$

$\phi(x)$ satisfies the boundary condition $\phi(x_h) = 0$, because $W(x_h) = \frac{\alpha}{r}x_h = \frac{\alpha}{r}(1 - m(x_h))$.

[37] Here we have just substituted $m(x) = 1 - x$ and the shirking value $W_s(x)$.

As x approaches x_h, we have

$$\phi'(x) \to \alpha(m'(x_h)+1),$$

which is strictly positive because $m'(x_h) < -1$. But if $\phi'(x) > 0$ for x close to x_h and $\phi(x_h) = 0$, then $\phi(x)$ must be negative close to x_h. ∎

Finally, we prove Proposition 6, which is restated below.

Proposition 6. *Disclosure of bad news reduces the strategic monitor's incentive to exert effort close to x_h and on $[x_h, 1]$.*

Proof. To show the result for x left of x_h, note that since $V(x) = \frac{c}{\lambda m(x)}$, it is enough to show that delayed disclosure reduces the value $V(x)$. This is true whenever $x(1-x)V'(x) - V(x)$ is negative. From Proposition 12, we know that the value of delayed disclosure strictly negative at x_h. The result then follows from continuity of $V(x)$, $V'(x)$, and $m(x)$ and the fact that $m(x_h) = 1 - x_h$. On $[x_h, 1]$, the result is immediate. Disclosure does not affect effort on that region, since the monitor shirks anyway, but it lowers the value, because is changes his HJB equation by a factor of $-\gamma V_s(x)$. This in turn lowers the monitor's value for all lower reputations and since on the work region $V(x) = \frac{c}{\lambda m(x)}$ it implies more manipulation in equilibrium. ∎

B.2.2 Delayed Good News

We first prove Proposition 8.

Proposition 8. *Disclosure of good news increases manipulation and decreases the value of the strategic monitor on the work region $[x_l, x_h]$.*

Proof. The strategic type's value for any positive γ is

$$(r + \lambda m(x))V(x) = \alpha\left(1 - m(x)^2\right) + \lambda x m(x)^2 V'(x) - \gamma x(1-x)V'(x).$$

Using the indifference condition in Equation 18 yields the analog of Equation 7, which describes $m(x)$ for any given γ.

$$rc = \lambda\alpha\left(m(x) - m(x)^3\right) - \lambda c m(x) - \lambda c x m(x) m'(x) + \gamma c m'(x)\frac{x(1-x)}{m(x)}.$$

The result follows from applying Grönwall's Lemma to this equation. We have to take additional care because the initial condition for that equation is at the right boundary of the work region, so the eqation is effectively solved backwards. We use the identity $m(x) = m(x_h - y)$ where $y = x_h - x$ to define $\tilde{e}(y) = m(x_h - y)$. Substituting this and the derivative $\tilde{e}'(y)$ into Equation 7 yields

$$\tilde{e}'(y) = \frac{-h(\tilde{e}(y))}{\lambda c(x_h - y)\tilde{e}(y) - \gamma c\frac{(x_h-y)(1-x_h+y)}{\tilde{e}(y)}},$$

which for all y and $\tilde{e}(y)$ is increasing in γ, because $h(\tilde{e}(y))$, which is the function we have defined in Equation 19, is negative. Grönwall's Lemma then implies that $\tilde{e}(y, \gamma') \geq \tilde{e}'(y, \gamma)$ for all y and $\gamma' \geq \gamma$, or, equivalently, $m(x, \gamma') \geq m(x, \gamma)$. Higher γ decreases the value because $V(x) = \frac{c}{\lambda m(x)}$ and m increases with γ. ∎

Finally, we prove Proposition 9.

Proposition 9. *For x below x_h but sufficiently close to x_h, the regulator benefits from disclosing good news. Disclosing good news is not valuable for $x \geq x_h$.*

Proof. The regulator's value for a given γ is

$$\left(r + \lambda m(x)^2\right) W(x) = \alpha(1 - m(x)) + \left(\lambda x m(x)^2 - \gamma x(1-x)\right) W'(x) + \gamma x\left(\frac{\alpha}{r} - W(x)\right).$$

Taking derivatives, we can derive the following representation describing the sensitivity or the regulator's value with respect to γ

$$\begin{aligned}\frac{\partial W(x)}{\partial \gamma} &= E_x\left[\int_0^{\tau_0 \wedge \tau_h} m^{-rt}\left(\frac{dm(x_t)}{d\gamma}(-\alpha - \lambda W(x_t) + \lambda x_t m(x_t) W'(x_t))\right.\right.\\ &\quad \left.\left. + \frac{\alpha}{r} - W(x_t) - (1 - x_t) W'(x_t)\right) dt\right].\end{aligned}$$

Delayed good news is beneficial at x if the term in the inner brackets is positive for all $x' \in [x, x_h]$. For x sufficiently close to x_h, the term in the second line is positive, because $W(x)$ approaches $\frac{\alpha}{r}x$ and $W'(x_h)$ is below $\frac{\alpha}{r}$. $m(x)$ is decreasing in γ, by Proposition 8. The term multiplying it is negative sufficiently close to x_h. Thus, the integral is positive. ∎

C Existence of Shirking Equilibrium

We now show that a solution to ODE 5 with boundary condition $V_s(1) = \frac{\alpha}{r}$ exists, using an approximation argument. The proof uses a "bounding box" which has finite upper and lower boundaries and whose right boundary is fixed below one. For any point on the boundary of this box, we can find an initial value so that the unique solution to the IVP hits this point. We then construct a sequence of boxes so that the right boundary approaches 1 and a corresponding sequence of solutions so that the value at the right boundary of the box converges to $\frac{\alpha}{r}$. To show that the limit actually satisfies $V_s(1) = \frac{\alpha}{r}$, we need to show that the sequence of solutions converges uniformly. For this we use the Arzelà-Ascoli Theorem, which we apply to a rescaled version of $V_s(x)$ that has a finite derivative.

The bounding box is for all $n \in \mathbb{N}$ given by

$$\begin{aligned} B_n &= \left\{(x,v) \in \mathbb{R}^2 | x \in [x_0, x_n], v \in \{-M, M\} \text{ if } x \in (x_0, x_n) \right. \\ &\quad \left. \text{and } v \in [-M, M] \text{ if } x \in \{x_0, x_n\}\right\} \end{aligned}$$

for some finite $M > \frac{\alpha}{r}$. Here, x_n is the right boundary of the box. We assume $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence with $x_n \in (x_0, 1)$ for all n which converges to one as $n \to \infty$. Point 3 of Lemma 2 then implies that each point on B_n can be reached by some solution to the IVP, which we show below.

Corollary 13. *For each $(\hat{x}, \hat{v}) \in B_n$, there exists a v_{0n} such that the solution to the IVP with initial condition v_{0n} satisfies $V_s(\hat{x}, v_{0n}) = \hat{v}$.*

Proof. Picking $v_{0n} = -M$ ensures that $V_s(x_0) = -M$ and picking $v_{0n} = M$ ensures that $V_s(x_0) = M$. For any $v_{0n} \in (-M, M)$, either hits the upper or lower bounds or it hits the right boundary at x_n. Since $V_s(x)$ is continuous and monotone in v_{0n} by Point 3 of Lemma 2, the continuous mapping theorem implies that for any point $(\hat{x}, \hat{v}) \in B_n$, there exists an initial condition v_{0n} such that $V_s(\hat{x}) = \hat{v}$. ∎

We use this result to construct a sequence of solutions which satisfy a boundary condition at x_n. That condition will converge to $\frac{\alpha}{r}$. Since we are only interested in the properties of these solutions as x becomes large, we omit any dependence on the initial condition v_{0n} to save notation. We denote with $V_{sn}(x)$ the solution to Equation 5 which satisfies the boundary condition

$$V_{sn}(x_n) = \frac{\alpha}{r} - \kappa(1 - x_n) \tag{21}$$

for some fixed $\kappa > 0$. As $n \to \infty$, the derivative $V'_{sn}(x_n)$ becomes potentially unbounded, because x_n approaches one and the shirking ODE 5 has a singularity at $x = 1$. Therefore, we cannot use the Arzelà-Ascoli Theorem on V_{sn} directly. Instead, we study the transformation

$$g_n(x) = V_{sn}(x)(1 - x),$$

which we extend to the entire interval $[x_0, 1]$ as follows:

$$\bar{g}_n(x) = \begin{cases} V_{sn}(x)(1-x) & \text{if} \quad x_0 \leq x \leq x_n \\ \frac{\alpha}{r}(1-x_n) - \kappa(1-x_n)^2 & \text{if} \quad x_n < x \leq 1. \end{cases}$$

Lemma 9. *For all $n \in \mathbb{N}$, $\bar{g}_n(x)$ is uniformly bounded. It is also differentiable at all $x \in [x_0, 1]$ except at x_n and has a uniformly bounded derivative.*

Proof. $\bar{g}_n(x)$ is uniformly bounded because we have constructed the sequence $V_{sn}(x)$ so that for all $x \in [x_0, x_n]$, $V_{sn}(x)$ is inside the "bounding box", i.e. $V_{sn}(x) \in [-M, M]$. Since $g_n(x) = V_{sn}(x)(1-x)$, we must also have $g_n(x) \in [-M, M]$. From the definition of $\bar{g}_n(x)$ we can also see that it is uniformly bounded on $[x_n, 1]$ for all n.

To show the derivative is uniformly bounded whenever it exists, we only have to consider the derivatives on the intervals $[x_0, x_n]$.[38] We can substitute $g_n(x) = V_{sn}(x)(1-x)$ and $g_n'(x) = V_{sn}'(x)(1-x) - V_{sn}(x)$ into Equation 5 to obtain an ODE for $g_n(x)$. This ODE is

$$\left(r + \lambda(1-x)^2\right) g_n(x) = \alpha\left((1-x) - (1-x)^3\right) + \lambda x(1-x)^2 g_n'(x). \tag{22}$$

For any n, the derivative at x_n is bounded. To see this, we first solve for $V_{sn}'(x_n)$, using Equation 14 and the condition in Equation 21. This yields

$$V_{sn}'(x_n) = \frac{1}{x_n}\left(\frac{\alpha}{\lambda} - \kappa + \left(\frac{\alpha}{r} - \frac{r\kappa}{\lambda}\right)\frac{1}{1-x_n}\right).$$

Therefore we have

$$g_n'(x_n) = \frac{1-x_n}{x_n}\left(\frac{\alpha}{\lambda} - \kappa + \left(\frac{\alpha}{r} - \frac{r\kappa}{\lambda}\right)\frac{1}{1-x_n}\right) - \frac{\alpha}{r} - \kappa(1-x_n). \tag{23}$$

As $n \to \infty$, this expression converges to $-\frac{r}{\lambda}\kappa$. This means that there exists a $\bar{K} > 0$ so that for all n, $|g_n'(x_n)| \leq \bar{K}$. To see that $g_n'(x)$ must be bounded uniformly for all n and $x \leq x_n$, we differentiate Equation 22 to obtain

$$\begin{aligned} 0 &= 2\lambda(1-x) g_n(x) + \alpha\left(3(1-x)^2 - 1\right) \\ &\quad + \lambda x(1-x)^2 g_n''(x) - \left(r + \lambda(1-x)x\right) g_n'(x). \end{aligned}$$

Suppose there exists an n and an $x_0 \leq x < x_n$ so that $|g_n'(x)| > K$. We choose K sufficiently large and larger than $\bar{K}$. Then, if $g_n'(x) > K$, the equation above immediately implies that $g_n''(x) > 0$, since $g_n(x)$ is uniformly bounded. But this means that $g_n'(x') > K$ for all $x' \geq x$.

[38] On $[x_n, 1]$, the result follows from inspecting the definition of $\bar{g}_n(x)$ above.

This is a contradiction, since we have just shown that $g_n'(x_n)$ is bounded by $\bar{K}$ for all n. Similarly, if $g_n'(x) < -K$, then $g_n''(x) < 0$, which again implies that $g_n'(x_n) < -\bar{K}$. ∎

We can now apply the Arzelà-Ascoli Theorem to the sequence of functions $\bar{g}_n(x)$. It establishes that there is a subsequence that converges to a continuous function $g^*(x)$. As we show below, we can take $g^*(x)$ to be continuously differentiable on $[x_0, 1]$ and to satisfy the ODE 22 on that interval without loss of generality.

Lemma 10. *There exists a subsequence of $\bar{g}_n(x)$ which converges uniformly to a function $g^*(x)$ which is continuously differentiable and satisfies Equation 22 on $[x_0, 1]$.*

Proof. From the previous Lemma and the Arzelà-Ascoli Theorem we know there exists a subsequence which converges to a continuous function $g^*(x)$. We now use a diagonalization procedure to show that there exists a subsequence such that $g^*(x)$ is continuously differentiable on $[x_0, 1)$. For a given n, the derivative $g_n'(x)$ satisfies

$$g_n'(x) = \frac{\left(r + \lambda(1-x)^2\right) g_n(x) - \alpha\left((1-x) - (1-x)^3\right)}{\lambda x (1-x)^2}$$

on some interval $[x_0, \bar{x}_1]$ for $\bar{x}_1 < x_n < 1$. Since the sequence g_n is equicontinuous on that interval and the right hand side of the above equation is continuous in both x and $g_n(x)$, $g_n'(x)$ is equicontinuous on that interval as well.[39] Thus, there exists a subsequence of g_n which converges to a limit that is continuously differentiable on $[x_0, \bar{x}_1]$. Proceeding iteratively, we then take a sequence of boundaries $\bar{x}_k$ which converges to one as $k \to \infty$. For each such k we can find a subsequence of g_n that converges to a continuously differentiable function. Thus, we can take the limit g^* to be continuously differentiable on $[x_0, 1)$ without loss of generality. Because of this, it also satisfies the ODE 22 on $[x_0, 1)$.

It remains to establish that g^* is continuously differentiable at $x = 1$. This follows from Equation 23 in the proof of the previous Lemma. We have

$$\lim_{n\to\infty} g^{*\prime}(x_n) = \lim_{n\to\infty} g_n'(x_n)$$

and Equation 23 shows that $\lim_{n\to\infty} g_n'(x_n) = -\frac{r\kappa}{\lambda}$. Thus, $g^{*\prime}(1)$ is finite. ∎

We now use the function g^* to show that our initial sequence of solutions $V_{sn}(x)$ converges to a limit that is continuous, solves the shirking ODE 5, and satisfies the boundary condition

[39] Note we are holding $\bar{x}_1$ fixed here.

$V_s(1) = \frac{\alpha}{r}$. To do this, we define the following function on the interval $[x_0, 1]$

$$V^*(x) = \frac{g^*(x)}{1-x}.$$

This function is continuously differentiable except perhaps at $x = 1$ and it satisfies the ODE 5, which can be seen by substituting it into Equation 22. We thus only have to show it satisfies the boundary condition at $x = 1$. If we let n_k denote the subsequence of n for which g_n converges to g^*, we have

$$V^*(x) = \lim_{k\to\infty} \frac{g_{n_k}(x)}{(1-x)} = \lim_{k\to\infty} V_{nk}(x).$$

Since for any n, $V_n(1) = \frac{\alpha}{r}$, we have

$$\begin{aligned} \lim_{x\to1} V^*(x) &= \lim_{x\to1}\lim_{k\to\infty} V_{nk}(x) \\ &= \lim_{k\to\infty} V_{nk}(1) \\ &= \frac{\alpha}{r}. \end{aligned}$$

This concludes our proof. We have shown that there exists a solution to the shirking ODE 5 on the interval $[x_0, 1]$ which satisfies the boundary condition $V_s(1) = \frac{\alpha}{r}$. Since any solution to the equation must satisfy $V(0) = 0$ (by Lemma 2, Point 1), we can extend this solution to the entire interval $[0, 1]$.

D Full Problem of the Regulator

In Section 5, we recovered several important properties of the regulator's disclosure policy. Because of the nonlinearity in the ODE for manipulation (7), we cannot completely characterize the regulator's problem analytically. In this section, we formulate the general problem.

The full problem of the regulator is to solve

$$W_0 = \max_{\{\gamma_{Gt},\gamma_{Bt}\}_{t\geq0}} E\left[\int_0^\tau \alpha(1-m_t)e^{-rt}dt + e^{-r\tau_G}\frac{\alpha}{r}1_{\{\tau=\tau_G\}} + e^{-r\tau_h}\frac{\alpha}{r}x_h 1_{\{\tau=\tau_h\}}\right] \tag{24}$$

subject to the laws of motion

$$\frac{dx_t}{dt} = \lambda m_t^2 x_t + (\gamma_{Bt} - \gamma_{Gt})\, x_t(1-x_t) \tag{25}$$

for reputation and

$$\frac{dm_t}{dt} = \frac{\lambda\alpha}{c}\left(m_t^2 - m_t^4\right) - rm_t - \left(\lambda m_t + \gamma_{Bt}\right) m_t \tag{26}$$

for manipulation.

Here, τ_h is the first time x_t hits x_h. We know that no information disclosure for $x_t \geq x_h$ is optimal, so the only continuation equilibrium once x_t hits x_h is one where the monitor shirks and the regulator provides no information. The value for the regulator is then her value in the shirking equilibrium: $W_s(x_h) = \frac{\alpha}{r}x_h$. The stopping times τ_B and τ_G are the first realizations of either good or bad news. Finally, Equation 26 acts as a promise keeping condition. The monitor anticipates future disclosures, which affect his current value and therefore his incentives to exert effort. On the work region, he still must be indifferent between working and shirking, and the evolution of manipulation must be consistent with his current value. This is captured in Equation 26.[40]

The regulator's problem in Equation 24 can be written as

$$\begin{aligned} W(x_0) &= \max_{\gamma_{Gt},\gamma_{Bt}} \int_0^{\tau_h} \left(F_t \left(\lambda m_t^2 + \gamma_{Bt}(1-x_t) + \gamma_{Gt} x_t\right) + \gamma_{Gt} x_t e^{-rt}\frac{\alpha}{r}\right)\Lambda_t dt \\ &\quad + \Lambda_{\tau_h}\left(F_{\tau_h} + e^{-r\tau_h}\frac{\alpha}{r}x_h\right) \end{aligned} \tag{27}$$

where the auxiliary state variable

$$\Lambda_t = \exp\left(-\int_0^t \left(\lambda m_s^2 + \gamma_{Bs}(1-x_s) + \gamma_{Gs}x_s\right) ds\right)$$

satisfies

$$\frac{d\Lambda_t}{dt} = -\left(\lambda m_t^2 + \gamma_{Bt}(1-x_t) + \gamma_{Gt}x_t\right)\Lambda_t$$

with initial condition $\Lambda_0 = 1$ and the auxiliary state variable

$$F_t = \int_0^t e^{-rs}\alpha(1-m_s)\,ds$$

satisfies

$$\frac{dF_t}{dt} = e^{-rt}\alpha(1-m_t)$$

with initial condition $F_0 = 0$. Here, τ_h is the time where x_t hits x_h. Using the ODE for

[40] Equation 26 is derived the same way as Equation 7, except we have to take into account γ_{Gt} and γ_{Bt}.

manipulation in equation 7, we can derive the law of motion

$$\frac{dm_t}{dt} = \frac{\lambda\alpha}{c}\left(m_t^2 - m_t^4\right) - rm_t - \left(\lambda m_t + \gamma_{Bt}\right) m_t$$

and use m_t as a state variable.

Equation 27 follows form the superposition and marking theorems for Poisson processes (see e.g. Kingman (1993)). Specifically, τ_B is the first arrival time of an inhomogeneous poisson process with arrival rate $\lambda_{Bt} = \lambda m_t^2 + \gamma_{Bt}(1 - x_t)$ and τ_G is the first arrival rate of an independent poisson process with arrival rate $\lambda_{Gt} = \gamma_{Gt} x_t$. By the superposition theorem, τ is the first arrival time of a poisson process with arrival rate $\lambda_t = \lambda_{Bt} + \lambda_{Gt}$. $1 - \Lambda_t$ is the probability that $\tau < t$. By the marking theorem, we have the following result. If $\tau = t$, then the conditional probability that $\tau = \tau_G$ is $\frac{\lambda_{Gt}}{\lambda_{Gt}+\lambda_{Bt}}$. Using these facts to explicitly compute the expectation in the regulator's value in Equation 24 then yields equation 27.

In equation 27, we also exploit the fact that for $x_t > x_h$, the only continuation equilibrium is the shirking equilibrium, so that no information provision is optimal when $x_t > x_h$. This yields the terminal value $e^{-r\tau_h}\frac{\alpha}{r}x_h$, which is the discounted shirking value of the regulator at x_h. At $x_t = x_h$, the continuation values of the regulator and monitor depend only the future. x_h therefore remains constant for any information policy $(\gamma_{Gt}, \gamma_{Bt})_{t \leq \tau_h}$. We can therefore treat it as a constant in the regulator's optimization problem.

To save notation, we write the state as a vector $y_t = (x_t, m_t, F_t, \Lambda_t)$ and the control as $\gamma_t = (\gamma_{Gt}, \gamma_{Bt})$. We assume that the final time τ_h is free and we impose the constraint that $x_{\tau_h} = x_h$. We also impose the state constraint

$$m_t \leq 1 - x_t.$$

The Hamiltonian is given by

$$H\left(y_t, \gamma_t, p_t, t\right) = \left(F_t\left(\lambda m_t^2 + \gamma_{Bt}\left(1 - x_t\right) + \gamma_{Gt} x_t\right) + \gamma_{Gt} x_t e^{-rt}\frac{\alpha}{r}\right)\Lambda_t + p_t \frac{dy_t}{dt}$$

and the terminal value function is

$$h\left(y_{\tau_h}, \tau_h\right) = \Lambda_{\tau_h}\left(e^{-r\tau_h}\frac{\alpha}{r}x_h + F_{\tau_h}\right).$$

The adjoint $p_t = \left(p_t^x, p_t^m, p_t^F, p_t^\Lambda\right)$ satisfies the law of motion

$$\frac{dp_t}{dt} = -\nabla_y H\left(y_t, \gamma_t, p_t, t\right)$$

with boundary condition

$$p(\tau_h) = \nabla_y h(y_{\tau_h}, \tau_h).$$

Additionally, since the time τ_h is free, we have[41]

$$H\left(y_{\tau_h}, \gamma_{\tau_h}, p_{\tau_h}, \tau_h\right) + \frac{\partial}{\partial t} h(y_t, t)\,|_{\tau_h} = p(\tau_h)^T \frac{dy_t}{dt}|_{\tau_h}.$$

By the definition of the Hamiltonian, the above equation simplifies to

$$\left(F_{\tau_h}\left(\lambda m_{\tau_h}^2 + \gamma_{B\tau_h}(1 - x_{\tau_h}) + \gamma_{G\tau_h} x_{\tau_h}\right) + \gamma_{G\tau_h} x_{\tau_t} e^{-r\tau_h}\frac{\alpha}{r}\right)\Lambda_{\tau_h} = r\Lambda_{\tau_h} e^{-r\tau_h}\frac{\alpha}{r} x_h.$$

To enforce the state constraint $m_t \leq 1 - x_t$,we introduce a penalty function

$$-M \max\{m_t - (1 - x_t), 0\}^2$$

and to deal with the constraint $x_{\tau_h} = x_h$, we introduce the penalty $-e^{-rt}M(x_t - x_h)^2$ into the terminal value. We can then rewrite the Hamiltonian as

$$\begin{aligned} H(y_t, \gamma_t, p_t, t) &= \left(F_t\left(\lambda m_t^2 + \gamma_{Bt}(1 - x_t) + \gamma_{Gt} x_t\right) + \gamma_{Gt} x_t e^{-rt}\frac{\alpha}{r}\right)\Lambda_t \\ &\quad -M \max\{m_t - (1 - x_t), 0\}^2 + p_t \frac{dy_t}{dt} \end{aligned}$$

and the final payoff as

$$h(y_t, t) = \Lambda_t\left(e^{-rt}\frac{\alpha}{r} x_h + F_t\right) - e^{-rt}M(x_t - x_h)^2.$$

We can enforce $m_{\tau_h} = 1 - x_h$ via the appropriate boundary condition. See e.g. Seierstad and Sydsaeter (1986) for how to determine the appropriate boundary conditions for a Hamiltonian.

[41] Here $p(\tau_h)^T$ denotes the transpose.

Thus, we solve the following system of ODEs

$$\begin{aligned}
\frac{dx_t}{dt} &= \left(\lambda m_t^2 x_t + (g_{Bt} - g_{Gt})\, x_t\,(1 - x_t)\right) \\
\frac{dm_t}{dt} &= \frac{\lambda\alpha}{c}\left(m_t^2 - m_t^4\right) - r m_t - (\lambda m_t + \gamma_{Bt})\, m_t \\
\frac{dF_t}{dt} &= \alpha e^{-rt}(1 - m_t) \\
\frac{d\Lambda_t}{dt} &= -\left(\lambda m_t^2 + \gamma_{Bt}(1 - x_t) + \gamma_{Gt} x_t\right)\Lambda_t \\
-\frac{dp_t^x}{dt} &= \left(F_t(\gamma_{Gt} - \gamma_{Bt}) + \gamma_{Gt} e^{-rt}\frac{\alpha}{r}\right)\Lambda_t - 2M\,(m_t - (1 - x_t))\,1\{m_t > 1 - x_t\} \\
&\quad + p_t^x\left(\lambda m_t^2 + (g_{Bt} - g_{Gt})(1 - 2x_t)\right) - p_t^\Lambda(\gamma_{Gt} - \gamma_{Bt})\,\Lambda_t \\
-\frac{dp_t^m}{dt} &= 2F_t\Lambda_t\lambda m_t - 2M\,(m_t - (1 - x_t))\,1\{m_t > 1 - x_t\} + p_t^x 2\lambda m_t x_t \\
&\quad + p_t^m\left(\frac{\lambda\alpha}{c}\left(2m_t - 4m_t^3\right) - r - 2\lambda m_t - \gamma_{Bt}\right) - p_t^F \alpha e^{-rt} - p_t^\Lambda 2\lambda m_t \\
-\frac{dp_t^F}{dt} &= \left(\lambda m_t^2 + \gamma_{Bt}(1 - x_t) + \gamma_{Gt} x_t\right)\Lambda_t \\
-\frac{dp_t^\Lambda}{dt} &= F_t\left(\lambda m_t^2 + \gamma_{Bt}(1 - x_t) + \gamma_{Gt} x_t\right) + \gamma_{Gt} x_t e^{-rt}\frac{\alpha}{r} \\
&\quad - p_t^\Lambda\left(\lambda m_t^2 + \gamma_{Bt}(1 - x_t) + \gamma_{Gt} x_t\right)
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
x_0 &= x_l \\
m_{\tau_h} &= 1 - x_h \\
F_0 &= 0 \\
\Lambda_0 &= 1 \\
p_{\tau_h}^x &= e^{-r\tau_h}\frac{\alpha}{r}\Lambda_{\tau_h} - 2e^{-r\tau_h} M\,(x_{\tau_h} - x_h) \\
p_{\tau_h}^m &= 0 \\
p_{\tau_h}^F &= \Lambda_{\tau_h} \\
p_{\tau_h}^\Lambda &= e^{-r\tau_h}\frac{\alpha}{r} x_h + F_{\tau_h} \\
0 &= -re^{-r\tau_h}\left(\frac{\alpha}{r}\Lambda_{\tau_h} x_h + M\,(x_{\tau_h} - x_t)^2\right) \\
&\quad + \left(F_{\tau_h}\left(\lambda m_{\tau_h}^2 + \gamma_{B\tau_h}(1 - x_{\tau_h}) + \gamma_{G\tau_h} x_{\tau_h}\right) + \gamma_{G\tau_h} x_{\tau_h} e^{-r\tau_h}\frac{\alpha}{r}\right)\Lambda_{\tau_h} \\
&\quad - M \max\left\{m_{\tau_h} - (1 - x_{\tau_h}), 0\right\}^2.
\end{aligned}$$

The solution of the Hamiltonian system determines the optimal disclosure policy, which is bang-bang and follows

$$\begin{aligned}\gamma_{Gt} &> 0 \text{ iff } F_t \Lambda_t x_t + e^{-rt} \Lambda_t \frac{\alpha}{r} x_t - p_t^x x_t (1 - x_t) - p_t^\Lambda x_t \Lambda_t > 0, \\ \gamma_{Bt} &> 0 \text{ iff } F_t \Lambda_t (1 - x_t) + p_t^x x_t (1 - x_t) - p_t^m m_t - p_t^\Lambda (1 - x_t) \Lambda_t > 0.\end{aligned}$$