A Theory of Liquidity in Private Equity

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Abstract

We propose a model of Private Equity (PE) investment that can rationalize several empirical findings about fundraising and returns. General partners (GPs) possess superior investment skills and raise capital from Limited Partners (LPs) to finance illiquid projects within funds. The optimal fund contract incentivizes GPs to maximize the expected payoff of fund investments by giving them a profit share in the fund, while compensating LPs for the liquidity risk they face. The size of a PE fund increases with the amount of wealth the GP co-invests in the fund. When PE investments become attractive, GPs prefer to increase the size of their fund rather than increasing their profit share. In markets with low liquidity risk for LPs, expected returns to LPs are lower, and aggregate fundraising as well as average fund sizes are larger. When LPs can trade PE fund investments in a secondary market, partnership claims trade at a discount when aggregate liquidity is scarce. LPs with higher tolerance for liquidity risk will realize higher average returns compared to other LPs, and the difference is larger when liquid capital is more scarce. The introduction of a secondary market can lead to market segmentation, where LPs facing lower liquidity risk switching to the secondary market and those with higher liquidity risk staying in the primary market.

Keywords: Private Equity, Liquidity, Secondary Markets.

1 Introduction

Private equity firms are stewards of other people's capital. In practice, they operate through investment vehicles organized as limited partnerships, in which the general partners (GPs) - employees of the private equity firm itself - raise capital from outside investors, known as limited partners (LPs). These partnerships are typically structured as closed-end funds, in which the LP initially pledges (but does not provide) capital that is then called over time by the GP who identifies investment opportunities. Once committed to a private equity fund, the LPs' capital becomes illiquid. First, limited partnership agreements between GPs and LPs often associate severe penalties with reneging on pledged commitments. Second, private equity investments take time to mature so that GPs must typically wait for several years to realize a positive return on their capital - an effect known as the J-curve. It is only natural that a secondary market for partnership claims has emerged over the past 20 years. When selling their claims, LPs can cash out on their investment before the underlying assets mature and transfer their commitment to meet future capital calls to secondary buyers. Recent empirical evidence suggests that these claims typically trade at a discount to their Net Asset Value (NAV), and this discount-to-NAV is varying over time (Nadauld et al. 2018, Albuquerque et al. 2018). Hence, investors in private equity still face liquidity risk even if the can sell their PE claims in the secondary market.

Several academic studies indeed suggest a role for (il)liquidity of the asset class in explaining returns, aggregate fundraising, and discounts in the secondary market. There is a broad academic consensus that private equity investment has outperformed similar but more liquid public equity investments over the last thirty years (Harris et al. 2014). However, it is not clear to what extent these returns reflects a premium for illiquidity, and how this liquidity premium is determined. Sensoy and Robinson (2016) show that net cash flows to LPs in private equity funds are higher in good times, further suggesting that private equity liquidity is pro-cyclical. Moreover, vintage returns to private equity funds have been shown to be negatively related to aggregate PE fundraising in the particular vintage year by Kaplan and Strömberg (2009) but the reasons for this relationship are still debated. While the studies mentioned above show that secondary claims trade at a discount, it is unclear how these discounts are related to primary market returns, if at all. Finally, there is also evidence that different LPs earn different returns on their investments in private equity funds (Lerner et al. 2007, Cagnavaro et al. 2018, Dyck and Pomorski (2016)), but the reasons for these differences, and their persistence over time is still debated (Sensoy et al. 2014).

Our main objective is to provide a theory on the determinants of the liquidity premium in private equity which is consistent with these empirical facts. To do so, we

develop a model of private equity investment where GPs raise capital from LPs for up to two long-term investments. The model has two key ingredients: GPs must be given incentives to properly manage investments and LPs are exposed to liquidity shocks that limits their willingness to commit capital. This simple framework rationalizes several institutional features of private equity partnership. The fund structure arises endogenously since it is optimal for LPs to commit capital for a series of investment rather than on a project by project basis. With the fund structure, GPs can cross-pledge income across multiple projects and reduce the agency cost. The mechanism is similar to Axelson et al. (2009) but with a moral hazard friction (rather than adverse selection) and allows PE fund size to be endogenous. We show that when investments are scalable, GPs prefer to increase the size of their funds rather than the unit fees they charge. Our model predicts that fund size, and profits for GPs, increase with the expected return of the underlying portfolio investments as well as with the amount of wealth that GPs can invest on their own.

The most novel feature of our model is to account for the liquidity risk faced by LPs. LPs with little exposure to liquidity risk require a lower liquidity premium to commit capital on a long-term basis. This lowers the cost of financing for GPs, who can then raise larger funds. Hence, the empirical finding that PE fundraising as well as average fund sizes are negatively related to subsequent returns arises naturally in our model, and all depend on the average liquidity risk that LPs face. We also consider the case when different LPs face different liquidity risk. If the return on the underlying PE investments is too low, only LPs with low liquidity risk enter the private equity market. As the profitability of private equity increases and GPs seek to raise more capital, LPs higher liquidity risk enter the market, and LPs with low liquidity exposure earn an excess return beyond their break-even compensation for liquidity risk. Hence, our model is consistent with different LPs realizing different returns on their PE investment portfolios.

In the second part of the paper, we introduce a secondary market for private equity. LPs who face liquidity shocks would like to sell their claim to new investors. This exit option is beneficial to LPs who can then cash in early on their investment. Everything else equal, this *liquidity effect* in the secondary market lowers the return required by investors to commit capital to PE funds. GPs tend to benefit from the secondary market since they now face a lower cost of capital. Interestingly, more liquid LPs are typically hurt by the introduction of a secondary market, since their rents are competed away by less liquid LPs, who benefit the most from the exit option in the secondary market.

We show that a secondary market affects fundraising in the primary market in a subtle way. In our model, liquidity provision in the secondary market is endogenous: investors initially decide whether to keep cash (dry powder) to trade in the secondary

market. If a severe liquidity shock occurs, many LPs will sell their claims, while few investors will stand ready to buy them. Secondaries then trade at a discount because buyers must be compensated for providing liquidity. This liquidity premium in the secondary market increases the value of short-term assets, such as cash, vis a vis long-term assets, such as private equity, in the primary market. This *opportunity cost effect* may cancel out the benefits the *liquidity effect* for GPs. In such cases, GPs actually lose from the introduction of a secondary market. We also show that the introduction of a secondary market can generate a significant composition shift among LPs. More liquid investors, who before would have provide capital to PE funds, may now choose to specialize in secondary market trading if claims are expected to trade a discount to net asset value.

Relation to the literature

This paper analyzes the liquidity properties of private equity investment in a model where several key institutional features are endogenized. In this respect, our rationale for LPs to finance several projects through a fund structure is similar to Axelson et al. (2009). Similar to this paper, investing through funds rather than deal-by-deal creates some "inside equity" for the GP, which reduces agency costs. In their case, the quality of projects is the GP private information. Cross-pledging or projects' cash flows prevents the GP form investing in some bad projects, if they have the hope of finding good projects in the future. Axelson et al. (2009) also analyze the role of third-party debt financing to avoid over-investment in bad projects. In contrast, we consider a GP moral hazard problem similar to Holmstrom and Tirole (1997), which enables us to endogenous fund size. Also, Axelson et al. (2009) do not consider the consequences of LP illiquidity or the role of the secondary market, which is the focus of our model.

Sorensen et al. (2014) also investigate the illiquidity cost of private equity to investors – i.e. the required excess return for investing in private rather than public equity – in a dynamic portfolio choice model. In their paper, the cost of private equity is that it exposes a risk-averse LP to additional uninsurable risk. In our paper, in contrast, LPs are risk neutral, but suffer from liquidity shocks, as in Diamond and Dybvig (1983). Our paper and thus our measure of illiquidity accounts for the presence of a secondary market. Using a proprietary data set of secondary markets bid, Albuquerque et al. (2018) find evidence that buyers earn higher expected return in response to liquidity shocks, in line the predictions of our model.

Several papers have studied the the degree of persistence in returns across private equity funds raised by the same GP, which was first documented in Kaplan and Schoar (2005). Hochberg et al. (2014) provide a model where existing LPs learn the GPs' skill over time, giving rise to informational holdup in a setting with overlapping funds.

They show that this informational holdup reduces the ability of good GPs to increase fees in their next fund, leading to performance persistence across funds, consistent with the empirical evidence in Kaplan and Schoar (2005). Korteweg and Sorensen (2017), however, argue that this performance persistence is not investable, since the ultimate performance of the previous fund is not known at the time the GP raises the next fund. Also, Harris et al. (2014) show, the performance persistence in VC is much stronger than in buyouts, and that persistence among buyout funds have more or less disappeared after 2000, a period during which the overall buyout market as well as average fund sizes have grown significantly (Döskeland and Strömberg 2018). In our paper, GPs do not compete on fees but rather on size when facing different investors, and our model does not have performance persistence for GPs. Our model also assumes that GPs can increase fund size without negative effects on the performance of the underlying investments. The model might therefore be better suited to capture buyout investment, where funds have been shown to be more scalable than in VC (Metrick and Yasuda 2010).

Apart from performance persistence among GPs, several papers have also examined whether different LPs have systematically different performance in their PE investment portfolio over time. Lerner et al. (2007) compare private equity returns different types of LPs using data on PE funds raised before 1999, and performance numbers as of 2004. They find significant differences in LP performance, with endowments exhibiting the highest returns on their PE fund portfolio, which they attribute to better fund selection ability. In a more recent study, Sensoy et al. (2014) no longer find any consistent differences in performance across LP types when they include funds raised until 2006, and performance numbers as of 2011, and they attribute the different results to the growth and maturing of the PE market. In another study including the same authors (Cagnavaro et al. 2018), they do find evidence of performance persistence for individual LPs (rather than LP types), where some LPs consistently outperform others over time. Focusing on private equity investments by public pension funds, Dyck and Pomorski (2016) find that LPs with larger PE portfolios significantly outperforms the ones with smaller portfolios, both in the 1990s and 2000s. Our model provides an explanation for why LP performance can systematically differ, based on the sensitivity of a particular LP to liquidity shocks.

While they do no derive an optimal fund structure, Lerner and Schoar (2004) argue that GPs endogenously limit trading of Limited Partnership claims to to screen for "good" LPs. Similar to our model, LPs can be hit by liquidity shocks and then wish to exit their investment through a sale of the partnership claim. In their model, LPs also acquire private information about the skills of GPs during the investment. Outside investors only see whether incumbent LPs sell their claim but do not know why. Hence,

observing a sale is a bad signal for future LPs and raises the cost of capital for the GP's follow-up fund. Preventing LPs from selling their share may thus facilitate subsequent financing for GPs. Similar to them, we also consider the effect of LP liquidity shocks and secondary markets, and predict that performance should differ across LPs depending on their tolerance for liquidity shocks. We differ in that we focus primarily on the effect of liquidity shocks and secondary markets on PE fundraising and returns.

2 Model

The model has three periods, denoted $t = 0, 1, 2$. The economy is populated with investors or LPs who have large financial resources and managers or GPs who can invest in long-term projects but have limited financial resources. GPs seek financing from LPs to leverage their investment skills. However, LPs face liquidity shocks that reduce their willingness to commit capital for long-term investments.

LPs

There is a mass M of LPs who are risk-neutral and consume in period 1 and 2. Each LP is endowed at date 0 with 1 unit of cash that is storable at the risk-free rate of $r = 0$. LPs may experience a liquidity shock in period 1 that increases their discount rate. We model this shock in the spirit of Diamond and Dybvig (1983). Formally, a LP with a probability λ of a receiving a liquidity shock has the following preferences:

$$u(c_1, c_2) = c_1 + \tilde{\delta} c_2, \qquad \text{where} \quad \tilde{\delta} = \begin{cases} 1 & (1-\lambda) \\ 0 & \lambda \end{cases} \tag{1}$$

where c_t is consumption in period t. A LP with a high probability λ of a liquidity shock has a strong preference for liquidity and discounts date 2 payoffs at a rate

$$\underline{r}(\lambda) = \frac{1}{1-\lambda} - 1 \tag{2}$$

We will first assume that LPs are ex-ante identical, that is they all face the same probability of a liquidity shock λ. We relax this assumption in Section 4.

GPs and Projects

There is a unit mass of GPs who are risk-neutral and do not discount future cash flows. GPs have an initial endowment of A units of cash at date 0. GPs can invest in projects at date 0 and at date 1. All projects mature at date 2 and pay either R in

case of success or 0 in case of failure, per unit of investment. We assume that a GP can only start one project per date but that he can manage two projects started at different dates. Project returns are independent across dates.

Moral Hazard and Project Return

A GP can increase the probability of success of a project by exerting effort. We follow the approach by Holmstrom and Tirole (1997) and assume that effort is not observable by LPs. When the GP exerts effort, a project succeeds with probability p. When shirking, the probability of success is q and the GP enjoys a private benefit B per unit of funds invested in the project. Exerting effort is efficient while shirking is inefficient since

$$pR \geq 1 \geq qR + B \tag{3}$$

The leftmost term is the expected payoff of a project per unit of investment. The rightmost term sums the monetary and non-monetary payoffs from shirking on a project.

The moral hazard problem implies that GPs must have a monetary stake in the project, since otherwise they would rather shirk to enjoy the private benefit. We assume that this stake must be large enough so as to constrain financing. This condition writes

$$p\left(R - \frac{pB}{p^2 - q^2}\right) < 1 \tag{4}$$

We will show that the left hand side is the maximum payoff the GPs can pledge per unit of investment in a fund.[1] If condition (4) did not hold, the income pledgeable to investors would cover their initial outlay and GPs would be able to raise funds with infinite leverage. Under condition (4), GPs must contribute their own resources to invest and leverage is limited.

We suppose that the resources of LPs are large compared to that of GPs, that is human capital is a scarce resource compared to physical capital. The following assumption formalizes this statement

$$M \geq \frac{A}{1 - p\left(R - \frac{pB}{p^2 - q^2}\right)} - A \tag{5}$$

[1]The reader may be more familiar with the condition

$$p\left(R - \frac{B}{p - q}\right) < 1 \tag{4b}$$

which prevents self-financing of a project in the Holmstrom-Tirole model. Condition (4) is stronger than (4b) because, as we will show, joint financing of two projects is easier than financing of a single project.

The left hand side is the total resources in the hands of LPs. As will become clear later, the right hand side is the maximum borrowing capacity of GPs. Hence, assumption (5) will imply that financing is not constrained by the lack of available capital.

Equilibrium concept and contracts

The main variable of interest will be the (net) buy and hold return on a private equity commitment, denoted by r_{PE}. For each \$1 committed to a fund run by a GP, a LP receives $\$(1 + r_{PE})$ at date 2 where r_{PE} is determined in equilibrium. As we will see, r_{PE} can be interpreted as the cost of capital for a GP to increase the size of the fund beyond its own resources A. GPs and LPs operate in a competitive environment and take r_{PE} as given.

At date 0, GPs offer contracts for investment partnerships with LPs. The contract specifies *i*) the total fund size I, *ii*) the shares x and $1-x$ of the fund capital allocated to the date 0 and the date 1 projects and *iii*) the compensation schedule of the GP. Since the technology is linear, the compensation of the GP is also linear in the fund size I. For any possible (unit) fund payoff

$$y \in \{0, R(1-x), Rx, R\} \tag{6}$$

we can then define $w(y)$ as the wage received by the GP per unit of investment. The four possible outcomes in equation (6) correspond respectively to a joint failure, a success of the second project only, a success of the first project only and a joint success.

We will require that the compensation schedule satisfies the following constraints:

$$0 \leq w(y') - w(y) \leq y' - y, \qquad y' \geq y \tag{7}$$

The first part of Condition (7) states that the GP compensation must increase with the fund payoff. This condition is stated for completeness since it will arise as an equilibrium property. The second part of Condition (7) implies that the compensation of the GP cannot increase more than one for one with the unit payoff. This assumption ensures that GPs have no incentive to manipulate the fund cash flows.[2]

At date 1, a secondary market for private equity claims may open. To streamline the exposition, we delay its introduction until Section 5. Finally, at date 2, the fund cash flows are realized and distributed according to the compensation schedule designed at date 0. Figure 1 represents the timeline of the model.

[2] Suppose that the manager compensation schedule instead verifies $y - w(y) > y' - w(y')$ while $y < y'$. Then, when the payoff is y, the manager would claim that it is in fact y', borrow $y' - y$ at the market rate of 0 and leave $y' - w(y')$ to the investors.

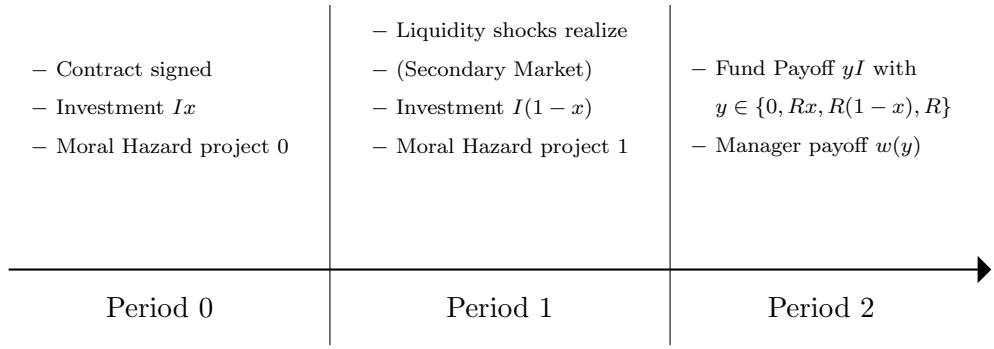

Figure 1: Timeline

3 Fund Design

In this section, we study the benchmark case where LPs cannot trade partnerhship claims in a secondary market. In the absence of an early exit option, investors must bear the full illiquidity cost of investing with GPs. Our benchmark analysis delivers three main results that capture institutional features of private equity. First, with scalable investments, GPs face a trade-off between fee and size. In our model, it is optimal to maximize fund size while keeping fees just high enough to incentivize the GP for effort. Second, GPs can raise larger funds and increase profit when LPs are less sensitive to liquidity risk. Finally, our model also rationalizes the fund structure whereby LPs commit capital for a series of investments rather than on a project by project basis.

Fee Size Trade-off

Let us consider a partnership between a GP and several LPs. Take first as given the share of capital $x \in [0, 1]$ allocated to the first project. Under assumption (3), it is optimal that a GP exerts effort on both projects. This requirement defines a set of incentive compensation schedules. For each such schedule we can define the expected compensation $W(x)$ for the GP per unit of investment - called expected fee for simplicity. Since the GP exerts effort, projects 0 and 1 have an expected return of pRx and $pR(1-x)$ respectively, per unit of investment. The total expected return of the fund is thus equal to pRI. Hence, for any expected fee $W(x)$ resulting from an incentive-compatible compensation schedule, the total payoff net of fees to LPs investing in a fund of size I is given by

$$(pR - W(x))I$$

To determine the size of the fund I, note that LPs commit $I - A$ since A is the

GP's own contribution. Observe that the pattern of capital calls $(x, 1-x)$ is immaterial here because LPs earn a zero net return when storing cash between date 0 and date 1. Hence, the rate of return GPs must offer on the total commitment by LPs is the buy and hold return r_{PE} for an investment paying at date 2. Hence, if a GP raises a fund of size $I \geq A$, the ratio of the total expected distribution to LPs, $(pR - W(x))I$ divided by their initial contribution $I - A$ should be given by

$$\frac{(pR - W(x))I}{I - A} = 1 + r_{PE} \tag{8}$$

The left hand side is the gross expected return on a capital commitment of \$1. It should be equal to the market return for an investment paying off at date 2.

The profit earned by a GP from running a fund of size I and charging expected fee $W(x)$ is given by

$$\Pi_{GP} = \max\{pRA, W(x)I\} \qquad \text{subject to} \quad (8) \tag{9}$$

where the GP takes as given the cost of capital $1+r_{PE}$. The formulation in equation (9) nests the case where GPs choose not to raise funds and only invest their own resources. Observe that the left hand side of equation (8) is decreasing both in $W(x)$ and I. Hence, a GP can either choose to run a large fund with a low expected fee or a small fund with a high expected fee. These two quantities should adjust for the return to LPs to be equal to r_{PE}. It is intuitive that a smaller fee increases the expected return to LPs. For a given fee, a smaller fund also benefits LPs since they get a share of the total investment I while their own contribution $I - A$ is comparatively lower. In our linear model, this trade-off has a simple solution.The following Lemma shows that GPs either do not raise financing from LPs or charge the minimum expected fee when they do.

Lemma 1 (Fee Size Trade-off)
GPs raise capital from LPs if and only if their liquidity risk is low, that is

$$\lambda \leq \hat{\lambda}(p, R) := \frac{pR - 1}{pR} \tag{10}$$

If condition (10) holds, the equilibrium return on a private equity commitment is

$$r^*_{PE} = \underline{r}(\lambda) \tag{11}$$

GPs then charge the minimum feasible expected fee denoted $W^*(x)$.

When $\lambda > \hat{\lambda}(p, R)$, the fund's assets are too illiquid for LPs since then

$$pR - 1 < \underline{r}(\lambda)$$

where the right hand side is the minimum rate of return required by investors. Hence, when condition (10) does not hold, LPs would not invest even if GPs could pledge all the projects cash flows. In this case, the cost of capital faced by GPs is too high and they prefer not to raise financing. Hence, external financing is only optimal when the probability of a liquidity shock is not too high.

When condition (10) holds however, the equilibrium buy and hold return is given by $r^*_{PE} = \underline{r}(\lambda)$. Since LPs have abundant resources, they must be indifferent between holding cash or investing in private equity. When facing such a cost of capital, GPs chose to minimize the expected fee per unit of capital, $W(x)$, and to maximize the amount of capital raised in the fund, I. To build intuition, it is useful to rewrite the payoff of the GP as

$$\Pi^{GP} = pRI - (1 + \underline{r}(\lambda))(I - A)$$

This payoff is equal to to the total fund cash flows minus the funding cost. When condition (10) holds, that is when $pR > 1 + \underline{r}(\lambda)$, it is strictly optimal to increase the fund size I.

To characterize the equilibrium, we must find the incentive compatible schedule that leads to the minimum expected fee and the optimal investment split between projects. Note that our preliminary analysis established that the split between projects $(x, 1-x)$ only matters to the extent that it affects the expected compensation $W(x)$.

Fund Structure

We now explain why allocating the fund capital over two projects, that is choosing $x \in (0, 1)$ allows GPs to charge a lower expected fee. This result justifies why GPs and LPs contract for a series of investments rather than on a deal-by-deal basis. Several standard motives for diversification can be discarded in our model. First, both LPs and GPs are risk-neutral so that hedging motives cannot explain this result. Second, there is no built-in complementarity between projects because their returns are independent. However, we show that diversification allows to mitigate the moral hazard problem of the GPs. Bundling investments in a fund allows GPs to cross-pledge the income of two projects. In other words, the bonus paid for the success of a project can be used to incentivize effort on the second project.[3] The following Proposition states that an equal split across projects is optimal.

[3]This insight is similar to the benefit of diversification in the context of delegated monitoring, originally due to Diamond (1984). Axelson et al. (2009) rely on a similar argument to derive the compensation structure of GPs in a model with asymmetry of information.

Proposition 1 (Diversification Benefits)
There exists $\Delta \in \left[0, \frac{1}{2}\right)$ *such that any investment split* $x \in \left[\frac{1}{2} - \Delta, \frac{1}{2} + \Delta\right]$ *is optimal. For* $x^* = \frac{1}{2}$*, the optimal compensation of the GP per unit of expected capital is given by* $w^*(0) = 0$ *and*

$$\left(w^*(R/2), w^*(R)\right) = \begin{cases} \left(\frac{1}{2-p-q}\left[\frac{B}{p-q} - (p+q)\frac{R}{2}\right], w(R/2) + R/2\right) & \text{if} \quad R \leq \frac{2B}{p^2-q^2} \\ \left(0, \frac{B}{p^2-q^2}\right) & \text{otherwise} \end{cases} \tag{12}$$

Proposition 1 establishes two results. First, the total capital of a private equity fund should be invested in different projects. Note that we allow GPs and LPs to write a contract about the exact share allocated to each investment. However, the optimal contract only predicts that GPs should not over-invest in one of the projects. In practice, partnership agreements specify such concentration limits. The second result characterizes the optimal compensation schedule of the GP in a fund where capital is optimally split across projects. In the following, we explain how this split optimally reduces the expected fee of the GP.

We first consider the benchmark where all the capital is invested in the first project, that is $x = 1$. The fund payoff is either 0 or R per unit of investment. The minimum fee $\underline{w}$ the GP must charge in case of success to exert effort can be derived by considering his incentive constraint. When it binds, we have

$$p\underline{w} = q\underline{w} + B \quad \Rightarrow \quad \underline{w} := \frac{B}{p-q} \tag{13}$$

When he receives $\underline{w}$ for a success, the GP weakly prefers to exert effort than to shirk. The expected compensation is then given by $p\underline{w}$.

Consider now the fund with split investment such that $x = 1/2$. Observe that it is possible to compensate the GP for a project independently of the outcome of the other project. The GP would then receive $\frac{1}{2}\underline{w}$ when only one project succeeds and $\underline{w}$ when both projects succeed. His expected payoff would then be

$$W = 2p(1-p)\frac{\underline{w}}{2} + p^2\underline{w} = p\underline{w}$$

With this independent compensation schedule, there is no benefit from diversification. However, Proposition 1 suggests that there exists a more efficient way to incentivize effort. The intuition is that compensating the manager in a state where one project fails is sub-optimal. Under risk-neutrality, it is a well known result that the GP should

only be paid after the outcome that is the most informative about effort exertion. We show in the proof of Proposition 1 that the only relevant deviation is to shirk on both projects. For each relevant unit payoff, we can then define an informativeness ratio

$$\mathcal{I}(y) = 1 - \frac{\mathbf{Pr}[\tilde{y} = y|\text{shirk}]}{\mathbf{Pr}[\tilde{y} = y|\text{effort}]}, \qquad y \in \left\{\frac{R}{2}, R\right\} \tag{14}$$

The higher $\mathcal{I}(y)$, the better signal of effort is payoff y. It is then easier to incentivize the GP by compensating him in this state. Comparing the case of a single success where $y = \frac{R}{2}$ with that of a joint success where $y = R$, we have

$$\mathcal{I}(R) = 1 - \frac{q^2}{p^2} > 1 - \frac{2q(1-q)}{2p(1-p)} = \mathcal{I}(R/2) \tag{15}$$

This means that it is strictly more efficient to compensate the GP after a joint success than after a single success. The GP has more incentives to exert effort when his compensation for one project also depends on the outcome of the other project. The expression for $w^*(R)$ in the second case of Proposition 1 can be simply derived from the incentive constraint of the GP:

$$p^2 w(R) \geq q^2 w(R) + B \tag{16}$$

Finally, observe that when R is small, the GP is also paid after a single success since then $w^*(R/2) > 0$. While it would be optimal to reward the GP only in the joint success state, the monotonicity constraint (7) imposes that

$$w(R) - w(R/2) \leq R/2 \tag{17}$$

so that $w(R)$ cannot exceed $R/2$. When R is small, this constraint would be violated by the unconstrained optimal schedule. The GP then receives in the single success state the lowest compensation compatible with the incentive constraint and the monotonicity constraint.

Proposition 1 allows us to derive the fund size and the GP profit as a function of the deep parameters of the model. Focusing on the case where $R \geq \frac{2B}{p^2-q^2}$, we have

$$W^* = \frac{p^2 B}{p^2 - q^2} \tag{18}$$

$$I^*(\lambda) = \frac{A}{1 - (1-\lambda)p\left[R - \frac{pB}{p^2-q^2}\right]} \tag{19}$$

$$\Pi^*_{GP}(\lambda) = \frac{p^2 B}{p^2 - q^2} I^*_{GP}(\lambda) \tag{20}$$

A first remark is that fund size $I^*(\lambda)$ is increasing with the GP's own contribution to the funds. The ratio between these two variables is sometimes referred to as the equity multiplier. It ensures that GPs have enough skin in the game for LPs to invest in private equity funds. The key insight is that this equity multiplier is decreasing in λ. LPs with lower liquidity risk provide cheaper financing to GPs since they require a relatively low return $\underline{r}(\lambda)$ to commit capital for long-term investments. GPs who face a lower cost of capital can then raise larger funds and increase their profit. Note also that the fund structure increases this equity multiplier since the expected compensation (per unit of fund capital) W^* is lower than with a single project venture. The following Corollary states these findings together with additional comparative statics results.

Corollary 1 (Comparative Statics)
Fund size I^* and GPs' profit Π^*_{GP} are decreasing in LPs' liquidity risk λ, increasing in the projects' payoff R and increasing in the probability of success q when $p - q$ is kept constant.

The effect of an increase in R is intuitive. When projects are more profitable, the payoff to investors net of fees goes up. This attracts more capital and GPs can scale up their funds. The proof shows that the effect of an increase in R on the GPs' profit is even stronger when $R < \frac{2B}{p^2-q^2}$ since an increases in R relaxes the montonicity constraint (17), and allows for a more efficient compensation schedule. When varying the probability of success q under shirking, we fix the difference $p-q$. This allows us to isolate the contribution from a better economic environment from that of having more efficient GPs. When p increases but q does not, an additional effect is that incentives are cheaper to provide. Indeed, the informativeness ratio $\mathcal{I}(R)$ introduced in equation (15) is increasing in the ratio p/q.

Our model can potentially rationalize several empirical relationships between PE fund compensation, fundraising, and returns that have been documented in the previous literature. Fees in PE funds have been shown to vary remarkably little across PE funds and over time, especially when it comes to the carried interest ($w(R)$ in our model), where 94% of the PE funds in Sensoy and Robinson (2013) have a carried interest of exactly 20%. Instead, average funds size increases in periods of high aggregate fundraising (which corresponds to periods when average λ is lower). Funds raised during strong fundraising periods have also been documented to have lower returns (Kaplan and Strömberg 2009), which follows straightforwardly from our model since the return to LPs is determined by their required compensation for liquidity risk λ. Kaplan and Schoar (2005) also find that PE firms raise the size of their funds when previous fund performance has been relatively strong. Our model suggests an a natural explanation for this result, based on the GP's ability to co-invest in the fund. Following

a successful fund, GPs will have earned higher carried interest and will therefore have more wealth A to invest in their next fund. This, in turn, increases the amount of capital $I - A$ they can raise from LPs.[4]

A straightforward extension of our model could also help explain some systematic differences between buyout and VC funds that have been documented in the literature. Our base-line result that GPs respond to a more favorable fundraising environment by increasing fund size rather than fee percentage relies on the underlying investment technology being perfectly scalable, i.e. R and p does does not depend on I. This assumption is more plausible for buyout funds, where a manager who raises a larger fund can simply acquire larger portfolio companies using a similar investment approach. In contrast, a VC manager investing in early-stage start-ups cannot as easily scale up the amount invested in any given company, since start-ups are almost by definition bounded in size. Raising a bigger fund would then force the VC to invest in a larger number of start-ups, but the manager would then also have to hire more investment professionals, limiting the economies of scale. With limited investment scalability, our model would predict that successful GPs should respond by increasing $w(R)$ when they cannot increase I. Consistent with this prediction, Sensoy and Robinson (2013) shows that the variation in carried interest much lower in buyout funds, where only 1% of funds have carried interest above 20%, compared to VC funds with higher carried interest is 10%. Similarly, Metrick and Yasuda (2010) find that buyout managers build on their prior experience by increasing the size of their funds faster than VC managers do, and conclude that the buyout business is more scalable than the VC business. The limited scalability of VC is also supported in Kaplan and Schoar (2005), who find that the sensitivity of fund size to past performance is significantly stronger in buyout compared to VC.

To conclude this section: We have provided model provides a rationale based on moral hazard to explain why GPs bundle several investments into a private equity fund, rather than raising capital on a deal by deal basis. Bundling investments allows LPs to tie the compensation of the GPs to the joint outcome of the projects. This compensation scheme is typical in private equity where GPs' *carry* is calculated based on the total fund performance. The model also shows that the liquidity risk faced by LPs matters. Investors that are less sensitive to liquidity risk require a lower buy and hold return to commit capital for long-term investments. This decreases the cost of capital for GPs who respond by raising larger funds, while keeping fees per unit of capital constant.

[4]Stretching the theory a bit (since we do not have heterogeneous GPs or dynamics in the model), one could imagine that strong performance in the past fund would lead LPs to increase their expectation of R and/or p, which would also lead to an increase in I.

4 Heterogeneous LPs

In our analysis so far, GPs faced a large homogeneous population of LPs with the same probability of a liquidity shock λ. For institutional investors, this probability may capture their investment horizon or the duration of their liabilities. In practice, GPs may thus tap into a pool of investors that differ significantly along these dimensions. To capture this heterogeneity, we allow for two types of LPs, $i \in \{L, H\}$, characterized by the probability λ_i to receive a liquidity shock where

$$\lambda_L < \lambda_H < \hat{\lambda}(p, R) \tag{21}$$

The left inequality ensures that L-LPs are less sensitive to liquidity shocks. The second inequality ensures that GPs would raise capital from H-LPs if they were the only investors in the market, according to Proposition 1. The total resources in the economy M are divided between L-LPs with a share μ_L and H-LPs with a share $\mu_H = 1 - \mu_L$.

The goal of this section is to determine the equilibrium when GPs face heterogeneous LPs. In particular, we will let μ_L vary to capture a shift in the composition of investors from a pool of investors with little exposure to liquidity risk (high μ_L) to a pool of investors with high exposure to liquidity risk (low μ_L). Proposition 1 showed that the fund structure and the compensation schedule are independent of liquidity risk λ. Liquidity risk only affects the equilibrium fund size through the buy and hold return r_{PE} required by investors, as shown by equation (8).

The main difference with an heterogeneous population of LPs is that the value of the equilibrium return r_{PE} is not obvious. Indeed, L-LPs are happy to commit capital as long as $r_{PE} \geq \underline{r}(\lambda_L)$. However, H-LPs who are more exposed to liquidity risk only invest when $r_{PE} \geq \underline{r}(\lambda_H) > \underline{r}(\lambda_L)$. Hence, we expect the equilibrium return r^*_{PE} to depend on the composition of the pool of investors, captured by μ_L. Observe that if H-LPs are ever to invest in private equity, L-LPs should also invest and they would earn a premium $r_{PE} - \underline{r}(\lambda_L)$ over their break-even rate $\underline{r}(\lambda_L)$. Proposition 2 formalizes this intuition and characterizes the equilibrium.

Proposition 2 (Heterogeneous LPs)
Suppose that $\lambda_L < \lambda_H < \hat{\lambda}(p, R)$. There exists two thresholds

$$\underline{\mu}_L = \frac{I^*(\lambda_H) - A}{M}, \qquad \bar{\mu}_L = \frac{I^*(\lambda_L) - A}{M} \tag{22}$$

such that the buy and hold rate of return r^*_{PE} is decreasing in μ_L with

$$r^*_{PE} = \begin{cases} \underline{r}(\lambda_H) & \text{if} \quad \mu_L \leq \underline{\mu}_L \\ \frac{\bar{\mu}_L(A+\mu_L M)}{\mu_L(A+\bar{\mu}_L M)}(1+\underline{r}(\lambda_L)) - 1 & \text{if} \quad \mu_L \in (\underline{\mu}_L, \bar{\mu}_L) \\ \underline{r}(\lambda_L) & \text{if} \quad \mu_L \geq \bar{\mu}_L \end{cases} \tag{23}$$

The fund size is given by $I(r^*_{PE})$ as defined in equation (8) with $W(x) = W^*$. Type L-LPs make a strictly positive profit when $r^*_{PE} > \underline{r}(\lambda_L)$.

Proposition 1 showed that GPs prefer to raise capital from LPs with lower liquidity risk λ_L. When μ_L is high, there is an abundant supply of "liquid" capital. GPs can operate funds at their optimal size $I^*(\lambda_L)$ while offering their beak-even return $r^*_{PE} = \underline{r}(\lambda_L)$ to L-LPs. When μ_L is intermediate, these LPs do not have enough resources to meet the capital demand from LPs. Hence, the market can only clear if the equilibrium rate r^*_{PE} goes up. Observe that H-LPs who now own a larger share of the capital are not willing to invest as long as the rate stays above $\underline{r}(\lambda_H)$. Hence, they cannot compete away the rents of L-LPs who earn a scarcity premium for providing liquid capital. When μ_L goes down further, r^*_{PE} reaches $\underline{r}(\lambda_H)$, H-LPs start supplying capital and the market rate of return stays constant at $r^*_{PE} = \underline{r}(\lambda_H)$. These equilibrium features are illustrated on Figure 2. Note that the profit of GPs is equal to the fund size I^* multiplied by the expected fee W^*.

In our model, the aggregate supply of capital is large but competitive pressure by H-LPs is limited by their liquidity risk. This explains why L-LPs can make a profit in equilibrium. GPs value L-LPs as providers of cheaper capital. When μ_L goes down, this capital becomes scarce and the return to capital provision increases increases for L-LPs. Observe also that this extra return $r^*_{PE} - \underline{r}(\lambda_L)$ increases in the difference $\lambda_H - \lambda_L$, that is how special L-LPs are compared to other investors. Overall, rather than the supply of capital available for private equity, our model stresses the importance of the distribution of this capital among heterogeneous investors.

Our model predicts that investors who are sensitive to liquidity risk do not invest unless μ_L is below a threshold $\underline{\mu}_L$. In other words, GPs do not tap into this capital

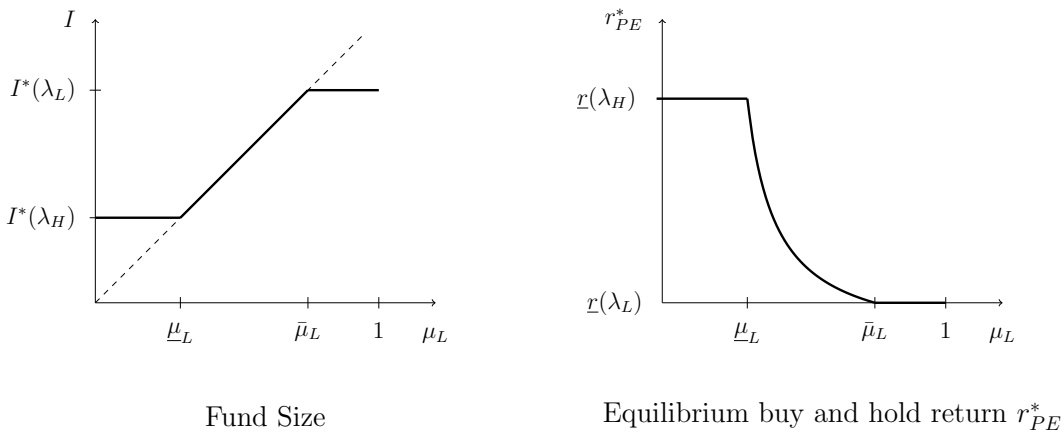

Figure 2: Equilibrium with heterogeneous LPs
μ_L is the share of capital of L-LPs:

pool unless the supply by more patient investors is highly limited. Alternatively, one can fix μ_L to see how other features of the model affect investors' entry into private equity. From Proposition 2, it is clear that both $\underline{\mu}_L$ and $\bar{\mu}_L$ are increasing with q or R. Hence, as profitability increases, pre-existing investors make more profit and new investors with lower tolerance to liquidity risk may enter. Hence a boom in private equity profitability should coincide with a shift in the composition of investors towards LPs more sensitive to liquidity risk.

Finally, we ask whether investors with lower sensitivity to liquidity risk earn higher returns. The monetary return to private equity is equal to $1 + r^*_{PE}$ while the monetary return to cash is 0. Hence, L-LPs earn higher monetary returns when they are the sole investors into private equity, that is when $\mu_L \geq \underline{\mu}_L$. It is perfectly rational for H-LPs to hold only cash because their discount rate for private equity investments is higher than r^*_{PE}. Conditional on investing in private equity however, both types earn the same expected return $1 + r^*_{PE}$. We will show in the next section that with a secondary market, L-LPs do earn a higher monetary return than H-LPs on their private equity investment.

5 Secondary Market

We now introduce a secondary market where LPs can sell their private equity fund claims. LPs who commit capital for long-term investments may value an early exit from

the fund when they face liquidity needs. A secondary market allows these investors to liquidate their participation early by selling their claims to more patient agents. The market for secondary claims opens at date 1 after liquidity shocks are realized. We assume that the claim sold is net of the date 1 capital call.[5] The secondary market is competitive.

In this section, there are again two types $i \in \{L, H\}$ of investors. We also assume that there can be an aggregate liquidity shock $s \in \{B, G\}$ where

$$\pi_G := \mathbf{Pr}[s = G], \qquad \pi_B := \mathbf{Pr}[s = B] = 1 - \pi_G \tag{24}$$

The probability of a liquidity shock λ_{is} for an investor depends both on his type i and on the state of the world s. We assume that

$$\lambda_{iB} \geq \lambda_{iG}, \qquad i \in \{L, H\} \tag{25}$$

so that state B can be interpreted as a bad state since all LPs have a higher probability of a liquidity shock than in state G.[6] Finally, let us define the average probability of a liquidity shock in the population in state $s \in \{B, G\}$ as

$$\bar{\lambda}_s = \mu_L \lambda_{Ls} + (1 - \mu_L)\lambda_{Hs} \tag{26}$$

We first define the interest rate and market prices that allow us to characterize the equilibrium. Like in the previous section, we let r_{PE} denote the net buy and hold return on a private equity claim. The secondary market is characterized by the unit price P_s of a secondary claim in state $s \in \{B, G\}$ at date 1. An upper bound on P_s is the fundamental value of the claim given by $1 + r_{PE}$. To understand this relationship, observe that one dollar invested in a fund yields $1 + r_{PE}$ dollars at date 2. Hence, a date 1, the fundamental price of a unit claim is equal to the final value $1 + r_{PE}$ of this investment. We now characterize the trades in the primary and the secondary market.

Primary Market

With a secondary market at date 1, it is useful to explicit the trade-off faced by

[5] This means that the seller meets the capital call before selling the claim to the buyer. This assumption comes without loss of generality although sellers precisely want to avoid the capital call. If buyers had to finance the capital call themselves, the price of the claim would indeed adjust downward to reflect the associated liability.

[6] Observe that such an aggregate shock would have played no role in the previous sections. When investors are risk-neutral and do not have access to the secondary market, it can easily be shown that the equilibrium of Proposition 1 and 2 only depends on $E_s[\lambda_{is}]$ the expected probability of a liquidity shock across states for each type $i \in \{L, H\}$ of LPs.

investors at date 0. At date 0, each LP of type $i \in \{L.H\}$ keeps c_i in cash and commits f_i to private equity funds. These quantities are related through the budget constraint

$$1 = c_i + f_i \tag{27}$$

To determine his optimal portfolio, an investor compares his expected return on cash denoted r_i^c to his expected return on a private equity investment denoted r_i^f. There are two main novelties with a secondary market. A claim in a private equity fund can be sold at date 1 and cash can be used to purchase those claims. We thus have:

$$1 + r_i^c = \sum_{s=B,G} \pi_s \left(\lambda_{is} + (1 - \lambda_{is}) \frac{1 + r_{PE}}{P_s} \right) \tag{28}$$

$$1 + r_i^f = \sum_{s=B,G} \pi_s \left(\lambda_{is} P_s + (1 - \lambda_{is})(1 + r_{PE}) \right) \tag{29}$$

Let us first explain the expression for the return on private equity in equation (29). Consider a type i investor who commits 1 unit of capital to a private equity fund. At date 1, if the investor is not hit by a liquidity shock (probability $1 - \lambda_{is}$), he obtains the buy and hold payoff $1 + r_{PE}$. Without a secondary market, this would be the only term entering equation (29). However, the investor can now sell his claim at price P_s. It is optimal to do so when hit by a liquidity shock (probability λ_{is}) since the LP does not value payoffs at date 2.[7] Everything else equal, the early exit option through the secondary market increases the return r_i^f from a private equity investment.

Let us now consider a type i investor with one unit of cash in state s, at date 1. When hit by a liquidity shock (probability λ_{is}), the investor consumes the unit of cash. Otherwise, the investor can buy private equity fund claims in the secondary market. The payoff is then $(1 + r_{PE})/P_s$ since one unit of cash buys $1/P_s$ claims and each claim pays off $1 + r_{PE}$. Observe that is always weakly optimal to buy since by definition the claim price P_s can never exceed the fundamental price $1 + r_{PE}$. Hence, as equation (28) shows, cash can have a positive return in the presence of a secondary market. This return is strictly positive if secondary claims sell at a discount in at least one state.

Given the rates of return on cash and private equity funds, r_i^c and r_i^f, the solution to the portfolio allocation problem for a LP of type i is given by

$$(c_i, f_i) = \begin{cases} (0, 1) & \text{if} \quad r_i^f > r_i^c \\ (c, 1 - c),\ c \in [0, 1] & \text{if} \quad r_i^f = r_i^c \\ (1, 0) & \text{if} \quad r_i^c > r_i^c \end{cases} \tag{30}$$

[7] The fact that $\delta = 0$ when a liquidity shock hits is inessential to the result. However, it is important that all types of LPs have the same discount factor conditional on being hit by a liquidity shock

since investors are risk neutral. The supply of capital to private equity is then

$$F = (\mu_L f_L + \mu_H f_H)\, M \tag{31}$$

The demand for capital $I_{sec} - A$ can be derived rewriting equation (8) as

$$I_{sec} - A = \frac{A}{\frac{1+r_{PE}}{pR-W^*} - 1} \tag{32}$$

where the subscript *sec* is used to stress that there is a secondary market for claims.

Secondary Market

The price of secondary claims depends on the supply from LPs who are distressed and on the demand from investors who hold cash and are not hit by a liquidity shock. Given a portfolio allocation $(c_i, f_i)_{i=L,H}$ in the primary market, the supply of claims is given by

$$Q_s = (\mu_L f_L \lambda_{Ls} + \mu_H f_H \lambda_{Hs})M \tag{33}$$

since a type i LP receives a liquidity shock with probability λ_{is} in state s. The demand for claims is given by

$$D_s(P_s) = \begin{cases} M\frac{(1-\lambda_{Ls})\mu_L c_L + (1-\lambda_{Hs})\mu_H c_H}{P_s} & \text{if} \quad P_s < 1 + r_{PE} \\ D \in \left[0, M\frac{(1-\lambda_{Ls})\mu_L c_L + (1-\lambda_{Hs})\mu_H c_H}{P_s}\right] & \text{if} \quad P_s = 1 + r_{PE} \end{cases} \tag{34}$$

The numerator of each fraction is the total resources available to purchase these claims. Only investors who are not hit by a liquidity shock are willing to buy. When the price P_s is strictly below the fundamental value $1 + r_{PE}$, buying claims delivers a strictly positive return. Hence, all investors who have cash supply liquidity. This phenomenon is often referred to as *cash in the market pricing* since claims are not priced according to their fundamental value but rather as a function of the liquidity in the market. Claims then trade at a discount to Net Asset Value (NAV) because there is too little liquidity available to buy secondary claims at par value. Investors must then be compensated with a positive return to provide liquidity. We will say that the secondary market is fully liquid in state s when $P_s = 1 + r_{PE}$ and only partially liquid otherwise. We can now define an equilibrium with a secondary market.

Definition 1 (Equilibrium with secondary market)
An equilibrium is a portfolio $(c_i^*, f_i^*)_{i=L,H}$, a buy and hold return on private equity r_{PE}^* and secondary market prices $(P_s^*)_{s=B,G}$ such that:

1. *LPs portfolio allocation solves equation (30) and GP's demand for capital is given by equation (32).*

2. *Primary and secondary markets clear, that is*

$$I^*_{sec} - A = F^* \tag{35}$$

$$Q^*_s = D^*_s(P_{S^*}), \quad s = B, G \tag{36}$$

where F^*, $I^*_{sec} - A$, Q^*_s and D^*_s are given by equations (31), (32), (33) and (34).

The introduction of a secondary market has two main effects. First, it provides an early exit option for LPs. This mitigates the impact of a liquidity shock and reduces the buy and hold return r_{PE} required by investors. This reduction in the cost of capital allows GPs to raise larger funds. In the limit where fund claims are fully liquid, investors should not require any compensation over cash for investing in private equity. When the secondary market is only partially liquid however, the opportunity cost of cash increases. Hoarding cash allows investors to benefit from potential fire sales in the secondary market. Since this outside option becomes more valuable, investors require a higher return to invest in private equity.

Hence fundraising and returns promised in the primary market depend crucially on the liquidity of the secondary variable. We will show that the outcomes in both markets can be related to a common variable: the availability of liquid capital in the market. To better highlight our results, we will first analyze the economy without aggregate uncertainty and then discuss the effect of an aggregate liquidity shock.

5.1 Idiosyncratic Liquidity Shocks

In this section, investors are only exposed to idiosyncratic liquidity shocks and there is no aggregate uncertainty. We then drop the subscript s corresponding to the state in our notation. To characterize the equilibrium, it will be useful to introduce the variable $I^*(0)$ which is the fund size when LPs do not require any compensation over cash for investing in private equity. We now state our first proposition related to the secondary market.

Proposition 3 (Secondary Market with idiosyncratic liquidity shocks)
Define the liquid capital ratio as

$$LR = \frac{(1-\bar{\lambda})M}{I^*(0) - A} \tag{37}$$

i) If $LR \geq 1$, the buy and hold return on private equity is $r^*_{PE} = 0$, secondary claims trade at par and GPs raise funds of maximal size $I^*(0)$.

ii) If $LR < 1$, the buy and hold return on private equity is

$$1 + r^*_{PE} = \frac{(pR - W^*)I^*}{I^* - A} > 1 \tag{38}$$

where $I^* = A + (1-\bar{\lambda})M$. Secondary claims trade below par at $P^* = 1$

For any type i of LP, the portfolio (c^*_i, f^*_i) is not determined.

The liquidity ratio LR has an intuitive interpretation. The denominator $I^*(0) - A$ can be seen as the maximum borrowing capacity. This is the demand for capital when the required rate of return is $r_{PE} = 0$. The numerator is the mass of investors that are not hit by a liquidity shock at date 1. This variable can be interpreted as a measure of capital that can be committed for the long term. When the ratio is above 1, liquid capital covers the demand from GPs. In this case, liquidity in the secondary market at date 1 is abundant enough for claims to trade at par. Then, investors realize the full value of their investment even when they exit early. Since private equity claims are as liquid as cash, investors do not require any compensation to commit capital and $r^*_{PE} = 0$. From the point of view of GPs, the cost of capital is the same that the cost they would face in an economy where $\lambda = 0$. Observe that the condition $LR \geq 1$ can be rewritten as

$$\mu_L \geq \mu_L^{*,sec} := \frac{I^*(0) - A}{M(\lambda_H - \lambda_L)} - \frac{1-\lambda_H}{\lambda_H - \lambda_L} \tag{39}$$

so that the secondary market is fully liquid if and only if the share of L-LPs in the population of investors is large enough.[8]

Let us suppose now that μ_L decreases below $\mu_L^{*,sec}$ so that $LR < 1$ and assume by contradiction that the secondary market is still fully liquid. From equation (33), the

[8]Observe that the threshold $\mu_L^{*,sec}$ can be either above 1 or below 0. The first case arises when $M - (I^*(0) - A)$ is above but close to 1. The second case arises when $I^*(0) - A \geq (1-\lambda_H)M$.

supply of claims from distressed LPs would be given by

$$S = \bar{\lambda}(I^*(0) - A)$$

Using equation (34), he demand for claims at price $P = 1$ is given by

$$D = (1 - \bar{\lambda})(M - I^*(0))$$

When $LR < 1$, demand is too low given the supply since $D < S$. Hence, secondaries must trade at a discount, that is $P^* < 1 + r^*_{PE}$ for the secondary market to clear. Since liquidity is scarce, liquidity providers must be offered a positive return to make the market. Buyers realize this positive return when buying claims at a discount to NAV.

When secondaries trade at a discount, the buy and hold return r^*_{PE} that GPs offer to LPs in the primary market must be strictly positive. Indeed, cash now delivers a positive return $r^c > 0$ as shown by equation (28) since it can be used to buy claims at a discount in the secondary market. Hence, investors must be promised a strictly positive return r^*_{PE} on their private equity commitment. Since the cost of capital is higher than in a fully liquid market, GPs reduce their fund size. In this case, the lack of liquid capital is the common cause that explains the subdued activity in the primary market and the discount observed in the secondary market.

As we discussed, when $LR \leq 1$, claims trade at a discount to NAV. In our model, this discount is equal to

$$1 - \frac{P^*}{1 + r^*_{PE}} = 1 - \frac{(1 - \bar{\lambda})M}{(pR - W)(1 + (1 - \bar{\lambda})M)} \tag{40}$$

It is straightforward to show that this discount to NAV is increasing with $\bar{\lambda}$ as well as with p, q, and R. The fact that $\bar{\lambda}$ increases the discount to NAV is not surprising since the supply of liquid capital goes down when the average probability of a liquidity shock $\bar{\lambda}$ goes up. The second set of comparative statics shows that the discount to NAV increases when private equity investments are more profitable. When profitability goes up, GPs increase their fund size and LPs commit more capital in equilibrium. Hence, with a fixed supply of liquid capital, the discount to NAV must increase since less cash is chasing more claims in the secondary market. The higher buy and hold return $1 + r^*_{PE}$ earned in normal times compensates for the discount LPs have to concede in the secondary market in bad times.

Proposition 3 also shows that the equilibrium does not fully pin down the portfolio of LPs. In particular, a lower commitment to private equity by L-LPs can be compensated by a higher commitment from H-LPs. Such a substitution would increase the supply of claims sold by distressed investors in the secondary market. However,

the demand from L-LPs simultaneously increases since they have more resources for secondary trades after decreasing their commitments in the primary market. At the equilibrium price $P^* = 1$, these two effects exactly compensate. Hence, H-LPs may invest in the primary market for all values of μ_L. This result is in stark contrast with our findings in Proposition 2 where these investors did not commit capital unless $\mu_L \leq \underline{\mu}_L$. Interestingly, higher participation by H-LPs is made possible by L-LPs who are the main providers of liquidity in the secondary market.

The analysis with a secondary market delivers another new result. Remember that in the absence of this market, all private equity investors earn the same monetary return on their private equity investment which is simply equal to the buy and hold return r^*_{PE}. This result does not hold anymore because investors actively manage their private equity portfolio in the secondary market. The return for patient L-LPs is higher than that for H-LPs when $LR < 1$ since patient LPs tend to sell less often their PE claims in the secondary market at a discount. Hence, the realized monetary return on a private equity portfolio is strictly higher for L-LPs when secondary claims trade at a discount.[9] This result speaks to the debate about the persistence of high returns for some categories of LPs. Our model suggests that some LPs earn higher returns because they are less exposed to liquidity risk. Hence our explanation does not require LPs to possess skills in cherry-picking the GPs.

Finally, we analyze the welfare effects of a secondary market for different categories of investors. Except for the most impatient H- LPs, the welfare impact is ambiguous. The Proposition below presents the most salient results.

[9]It is important to note that our consideration is about monetary returns as opposed to returns measured in utils which are naturally lower for H-LPs, also in the absence of a secondary market.

Proposition 4 (Welfare effects)
The introduction of a secondary market has the following effects:

1. *The welfare of H-LPs weakly increases and strictly increases when $\mu_L < \mu_L^{*,sec}$.*

2. *There exists $\hat{\mu}_L > \underline{\mu}_L$ such that L-LPs' welfare decreases when $\mu_L < \hat{\mu}_L$ if*

$$pR - W^* \geq \frac{\lambda_L(1-\lambda_H)}{\lambda_H(1-\lambda_L)} \tag{41}$$

3. *The welfare of GPs strictly increases if*

$$pR - W^* \geq \frac{\lambda_H - \lambda_L}{1-\lambda_L} \tag{42}$$

When condition (42) does not hold, GP's welfare can go down if $p \geq 3q$ for intermediate values of μ_L in an interval that contains $\mu_L^{*,sec}$.

The first result that H-LPs benefit from the introduction of a secondary market is not surprising. In the absence of such market, Proposition 2 showed that these impatient investors make zero profit in expectation. With a secondary market, they can make a positive expected profit whenever $\mu_L < \mu_L^{*,sec}$, that is when secondaries trade at a discount. In this case, even cash delivers a strictly positive net return because it can be used to purchase secondary claims at a discount.

To welfare effect for L-LPs is ambiguous. Observe first that if λ_L is sufficiently close to λ_H, then condition (41) cannot hold, given that $pR - W^* < 1$ by Assumption (4). This is intuitive since both types of investors are then very similar. The L-LPs should reap the same benefits than H-LPs from the introduction of a secondary market. The most interesting case arises when condition (41) holds, that is when λ_L is sufficiently smaller than λ_H. Then, L-LPs' welfare necessarily goes down for low values of μ_L. As we showed in Proposition 2, without a secondary market, L-LPs pocket a premium for providing liquid capital when μ_L is low. With a secondary market however, the competition from H-LPs is more intense because PE claims become partially liquid. Higher competition from H-LPs reduces the profit of L-LPs who lose their role as special providers of liquid capital. Interestingly, L-LPs ultimately destroy their own rents because they provide the secondary market liquidity that allows H-LPs to compete in the primary market.

It is possible to show that when M is sufficiently large, L-LPs always lose from

the introduction of a secondary market. Suppose that $\mu_L^{*,sec} > 0$, which arises if $(1-\lambda_H)M > I^*(0) - A$. Then, according to Proposition 3, the secondary market is fully liquid for any value of μ_L and no investor makes a profit. Without a secondary market, L-LPs instead made a positive profit when $\mu_L < \bar{\mu}_L$. It can easily be shown that $\bar{\mu}_L$ can be strictly positive when $\mu_L^{*,sec} > 0$. Collectively, H-LPs provide a quantity $\mu_H M(1-\lambda_H)$ of liquid capital which may be large when M is large although the probability of a liquidity shock λ_H is very high.

Finally, it may seem that GPs should benefit from a secondary market. This is the case when the secondary market is fully liquid since the cost of capital is then $r_{PE}^* = 0$. When the secondary market is only partially liquid however, the welfare effect is ambiguous. To see, this let us write the market return r_{PE} using the fact that LPs of type i are indifferent between cash and private equity in equilibrium. Using equations (28) and (29), we have

$$r_{PE} = \frac{r_i^c - \lambda_i(P-1)}{1-\lambda_i} \tag{43}$$

Without a secondary market, $P = 0$, while $P = 1$ with a secondary market. Hence, keeping the return on hoarding cash constant, equation shows that a secondary market reduces the return required by any investor for private equity. However, with a secondary market, r_i^c is not zero when secondaries trade at a discount. Keeping P constant, the higher opportunity cost of cash implies that LPs require a higher return r_{PE} so that the cost of capital goes up. The first effect generally dominates. However, when condition (42) does not hold and $p > 3q$, the second effect may dominate. Interestingly, this last condition means that GPs are very efficient at improving project returns. In this case, GPs would prefer to shut down the secondary market.

Finally, we observe that a secondary market can have a negative impact on GPs and L-LPs only with heterogeneous investors. Indeed, when $\lambda_L = \lambda_H$, condition (41) for a negative welfare result for L-LPs cannot hold and condition (42) for a positive result for GPs always holds.

5.2 Aggregate Liquidity Shock

We now consider the case where an aggregate liquidity shock hits the economy. In our previous analysis, all investors faced idiosyncratic liquidity shocks. In practice, these shocks tend to be correlated and correspond to market downturns when many asset classes become illiquid. For simplicity, we consider the case when the secondary market is illiquid in the B state only. This is an endogenous outcome when λ_{LG} and λ_{HG} are not too large and we assume for simplicity that

$$\lambda_{LG} = \lambda_{HG} = 0 \tag{44}$$

We assume again that L-LPs are the most patient investors, which implies the following ranking $\lambda_{LB} < \lambda_{HB}$.[10]

Most of the qualitative results of Proposition 3 and Proposition 4 with idiosyncratic shocks also apply in this environment. We thus stress the new results that only arise in the presence of an aggregate liquidity shock.

Proposition 5 (Aggregate Liquidity Shock)
Let $\lambda_{LG} = \lambda_{HG} = 0$. Secondaries trade at par in state G. They trade at a discount in state B if

$$LR_B := \frac{I^*(0) - 1}{(1 - \bar{\lambda}_B)M} \leq 1 \tag{45}$$

Patient investors' portfolio weight on private equity is strictly lower, that is $f_L^* < f_H^*$.

Conditional on a state, condition (45) for secondaries to trade at a discount is similar to that obtained in Proposition 3. However, in the B state, the liquid capital ratio tends to be lower since more LPs are hit by the shock. By construction, secondaries trade at par in state G. However, it is possible to show that the discount to NAV is always higher in the B state when condition (25) holds, that is if state B is worse for all investors. As our previous analysis showed, the opportunity cost of cash goes up when secondaries trade at a discount. In this case, investors expect to earn a premium for providing liquidity in the bad state. This increases the cost of capital for GPs and funds must be operated below their maximum capaciy $I^*(0)$.

Unlike in Proposition 3, we can show that patient investors tend to invest relatively less in private equity than impatient investors when there is an aggregate liquidity shock. This might be surprising because these investors are less exposed to fire sale risks in the secondary market. However, the relative return on cash is greater for them. To see this, it is useful to derive the extra return an investor of type i earns from holding cash rather than investing in private equity. Using equations (28) and (29), we have

$$r_i^c - r_i^f = -\pi_G r_{PE} + \pi_B(1 - P_B^*)\left[\lambda_{iB} + (1 - \lambda_{iB})\frac{1 + r_{PE}^*}{P_B^*}\right] \tag{46}$$

It can be shown that in equilibrium $P_B^* < 1$. The first term of (46) shows that in the good state, cash delivers a lower return. Since the market if fully liquid, all investors indeed earn the buy and hold extra return r_{PE} on a private equity commitment while

[10] When relaxing Assumption (44), it can be shown that the investors who earn a higher profit are those for which λ_B is small. The fact that $E_s[\lambda_s]$ is small is not a sufficient condition because investors benefit primarily when they are not too exposed to the bad liquidity state. In other words, there is no benefit from providing liquidity in an already liquid market.

cash has zero net return. In the bad state however, cash has two advantages. When the investor is hit by a liquidity shock, the return on cash of 0 exceeds the negative return on private equity since $P_B^* < 1$. When the investor is not by this shock, he would rather have cash to buy secondary claims at a discount. Since the expression in equation (46) is decreasing in λ_{iB}, L-LPs invest more in cash and less in the primary private equity market than H-LPs. To sum up, L-LPs over invest in the asset that give them a competitive edge. When the secondary market is fully liquid in state G, this competitive edge only materializes in state B where cash dominates.[11]

6 Conclusion

This paper provides a model of delegated fund investment where investors are subject to liquidity risk. We derive the optimal partnership whereby GPs manage LPs' capital in a fund for a series of long term investments. Fund managers receive compensation based on the overall performance of the fund, in line with the carry earned by GPs. Our model stresses the importance of liquidity risk faced by LPs. Investors exposed to liquidity shocks require a premium over liquid assets to commit capital to private equity funds. This implies that GPs prefer to raise capital from LPs with low exposure to these shocks and that these LPs can earn a scarcity premium on their capital.

Our model delivers original predictions about the effects of a secondary market for LPs' claims in private equity funds. Secondary market provides liquidity to distressed investors but few investors are able to provide this liquidity in bad times. Our model thus explains trades at a discount below Net Asset Value based on aggregate market illiquidity. Investors who are heavily exposed to liquidity shocks benefit from this early exit option provided by the secondary market. As a result, they increase their capital commitment in the primary market, thereby competing away the rents of investors with a low exposure to liquidity shocks. When secondaries trade at a discount, investors have incentives to decrease their private equity commitment to keep dry powder for cheap secondary deals. We show that this effect can be strong enough to reduce GPs' profit after the introduction of the secondary market.

[11] This effect would not arise in the environment with an idiosyncratic shock because then cash and private equity commitments are identical assets. This can be seen from equation (46) by setting $\pi_G = 0$ and $P_B^* = 1$.

Appendix

A Proofs

A.1 Proof of Proposition 1

We first introduce additional notation for this proof. Let us denote by $w_F^{SS}(x)$ the GP's compensation at date 2 when both projects succeed. We let $w_F^{FS}(x)$ (resp. $w_F^{SF}(x)$) denote the GP's compensation when the first (resp. the second) project fails but the other project succeeds. The GP's expected compensation is thus given by

$$W(x) = p^2 w^{SS}(x) + p(1-p)\Big[w^{FS}(x) + w^{SF}(x)\Big] \tag{47}$$

Our objective is first to derive the compensation schedule that minimizes the expected compensation $W(x)$ for a given value of x. Then, we want to show that the minimum over values of x is reached for $x = \frac{1}{2}$. By symmetry, it is enough to consider only the case $x \geq \frac{1}{2}$. In this case, the monotonicity constraint (7) implies the following inequalities

$$0 \leq w^{SS}(x) - w^{SF}(x) \leq R(1-x) \tag{48}$$

$$0 \leq w^{SF}(x) - w^{FS}(x) \leq R(2x-1) \tag{49}$$

$$0 \leq w^{FS}(x) \leq R(1-x) \tag{50}$$

The proof is in several steps. We first show that GPs do not receive a positive compensation when one project fails unless the right inequality of the monotonicity constraint (48) binds. We then show that the expected compensation of the GP is minimized for $x = \frac{1}{2}$ considering the different cases for R highlighted in Proposition 1.

A.1.1 Optimality of joint compensation

We will show that unless the right inequality of constraint (48) binds, we can set $w_F^{FS}(x) = w^{SF}(x) = 0$ without loss of generality. We first need to derive the incentive constraints to exert effort on each project. At date 1, a GP who worked at date 0 exerts effort if

$$\begin{aligned} & p^2 w^{SS}(x) + p(1-p)\big[w^{SF}(x) + w^{FS}(x)\big] \\ & \geq pqw^{SS} + p(1-q)w^{SF}(x) + (1-p)qw^{FS}(x) + B(1-x) \end{aligned} \tag{51}$$

which we can express as $F_1 \geq B(1-x)$ where F_1 is implicitly defined by (51). It will also be useful to write the off-equilibrium payoff at date 1 of a GP who shirked at date 0. This payoff is given by

$$\tilde{V}_1(w(x), x) = \max\Big\{pqw^{SS}(x) + q(1-p)w^{SF}(x) + (1-q)pw^{FS}(x), \\ q^2 w^{SS} + q(1-q)\big[w^{SF}(x) + w^{FS}(x)\big] + B(1-x)\Big\} \tag{52}$$

since the GP might either exert effort on date 1 project or shirk again. Finally, the incentive constraint at date 0 writes

$$p^2 w^{SS}(x) + p(1-p)\big[w^{SF}(x) + w^{FS}\big] \geq \tilde{V}_1(w(x), x) + Bx \tag{53}$$

which we rewrite as $F_0 \geq Bx$ where F_0 is implicitly defined by (53).

Suppose then that the right inequality of (48) does not bind and $w^{SF}(x) > 0$. We can then decrease $w^{SF}(x)$ without affecting any constraint while keeping the GP's expected fee $W(x)$ constant. Suppose first that $w^{FS}(x) = 0$ and consider the following change in the compensation schedule

$$\Delta w^{SF}(x) = -p\epsilon, \qquad \Delta w^{SS}(x) = (1-p)\epsilon$$

Since $w^{FS}(x) = 0$ and $w^{SF}(x) > 0$ by assumption, the monotonicity constraint (49) is still satisfied for $\epsilon > 0$ small enough. Let us then show that (51) and (53) still hold. For this, it is enough to verify that the changes ΔF_0 and ΔF_1 are positive. We have

$$\begin{aligned} \Delta F_1 &= p(p-q)(\Delta w^{SS}(x) - \Delta w^{SF}(x)) = p(p-q)\epsilon > 0 \\ \Delta F_0 &\geq (p-q)(p\Delta w^{SS}(x) - (1-p)\Delta w^{SF}(x)) \\ &= (p-q)(p(1-p)\epsilon - p(1-p)\epsilon) = 0 \end{aligned}$$

Finally, we need to check that the GP expected payoff is the same. We have

$$\Delta W = p^2 \Delta w^{SS}(x) + p(1-p)\Delta w^{SF}(x) = p^2(1-p)\epsilon - p^2(1-p)\epsilon = 0$$

which proves the claim when $w^{SF}(x) = 0$.

To finish the proof, we need to consider the case when $w^{FS}(x) > 0$. Similar steps show that the following change in the compensation schedule

$$\Delta w^{SF}(x) = \Delta w^{FS}(x) = -p\epsilon, \qquad \Delta w^{SS}(x) = 2(1-p)\epsilon$$

for $\epsilon > 0$ small enough. also leaves the GP's expected payoff unchanged while still satisfying the incentive constraints (51) and (53). We will now show that the expected compensation is minimized for $x = \frac{1}{2}$.

A.1.2 Optmality of split investment

We now prove that it is always (weakly) optimal to split investment equally across projects. To prove this result, we distinguish two cases.

Case 1) $R \geq \frac{2B}{p^2-q^2}$.

We guess and verify that the right inequality of monotonicity constraint (48) does not bind. Step 1 shows that we can set $w_F^{FS}(x) = w^{SF}(x) = 0$ without loss of generality. The incentive constraint at date 1, equation (51) becomes

$$w^{SS} \geq \frac{B(1-x)}{p(p-q)} \tag{54}$$

The incentive constraint at date 0, equation (53), becomes

$$w^{SS} \geq \begin{cases} \frac{B}{p^2-q^2} & \text{if } x \in \left[\frac{1}{2}, \frac{p}{p+q}\right] \\ \frac{Bx}{p(p-q)} & \text{if } x > \frac{p}{p+q} \end{cases} \tag{55}$$

Given that $x \geq \frac{1}{2}$, it is clear that the incentive constraint at date 0, equation (55) is binding rather than inequality (54). This proves the claim in the main text that the relevant deviation for a GP is to shirk twice rather than only once. It follows that the expected compensation is minimized at $x = \frac{1}{2}$. In fact, one can find that any value $x \in [\frac{q}{p+q}, \frac{p}{p+q}]$ minimizes the expected compensation which is equal to

$$w^{SS,*} = \frac{B}{p^2-q^2}$$

Our initial claim that the monotonicity constraint does not bind is verified since $\frac{R}{2} \leq w^{SS,*}$. This corresponds to the second case in Proposition 1.

Case 2) $R < \frac{2B}{p^2-q^2}$

Observe that in what follows, it must be that the right inequality of monotonicity constraint (48) binds that is

$$w^{SS}(x) = w^{SF}(x) + R(1-x) \tag{56}$$

Using equation (56), we can rewrite the incentive constraint at date 1, equation (51) as

$$(p-q)\left\{pR(1-x) + (1-p)w^{FS}(x)\right\} \geq B(1-x) \tag{57}$$

To prove the results in this case, we use a series of claims.

Claim 1. *$\tilde{V}_1(w(x), x)$ is equal to its second argument*

Proof: Suppose by contradiction that it is not the case. Then, we have

$$(p-q)\Big[qR(1-x) + (1-q)w^{FS}(x)\Big] > B(1-x) \tag{58}$$

Observe first that (58) implies that the incentive constraint at date 1, equation (57) holds. Given the upper bound on R, equation (58) may hold only if $w^{FS}(x) > 0$. But then, $w^{FS}(x)$ should be decreased until either (58) is violated or the right hand side of monotonicity constraint (49), that is

$$w^{SF}(x) - w^{FS}(x) \leq R(2x-1) \tag{59}$$

holds as an equality. Indeed, when (58) holds, decreasing $w^{FS}(x)$ relaxes the incentive constraint at date 0, equation (53).

To prove that $\tilde{V}_1(w(x), x)$ is equal to its second argument, we are left to show that when (59) holds as an equality, then (58) is violated. Suppose it is not the case. We can rewrite the incentive constraint at date 0, equation (53), as

$$(p-q)\Big[pR(1-x) + w^{SF}(x) - pw^{FS}(x)\Big] \geq Bx$$

Using equation (59), it becomes

$$w^{FS}(x) \geq \frac{1}{1-p}\left[\frac{Bx}{p-q} - R(2x-1) - pR(1-x)\right] \tag{60}$$

Inequality (60) is implied by (58) but since (60) is the relevant constraint, it means that $w^{FS}(x)$ should be decreased until (58) is violated, which proves our claim. Using the result that $\tilde{V}_1(w(x), x)$ is equal to its second argument, the incentive constraint at date 0 can be rewritten

$$(p-q)\Big[(p+q)R(1-x) + w^{SF}(x) + (1-p-q)w^{FS}(x)\Big] \geq B \tag{61}$$

Claim 2. *The split $x = 1/2$ is optimal.*

Proof: We can now show our result that the minimum expected compensation is attained in $x = \frac{1}{2}$. The relevant incentive constraints are equation (61) at date 0 and (57) at date 1 as well as the monotonicity constraints (49) and (50). Observe first that either (57) binds or (59) binds or $w^{FS}(x) = 0$. Indeed, otherwise, the following change in the compensation schedule.

$$\Delta w_F^{SF}(x) = (1-p-q)\epsilon, \qquad \Delta w_{LT}^{FS}(x) = -\epsilon$$

leaves (61) unchanged and increases the expected compensation by

$$\Delta W = p(1-p-q)\epsilon - p(1-p)\epsilon = -pq\epsilon < 0$$

Case 2)-i) Date 1 incentive constraint binds

Suppose first that (57) binds. Since the expected compensation W is an increasing function of $w^{SF}(x)$, the incentive constraint at date 0, equation (61) should bind, that is

$$w^{SF}(x) = \frac{B}{p-q} - (p+q)R(1-x) - (1-p-q)w^{FS}(x)$$

or $w^{SF}(x) = w^{FS}(x)$. If $w^{SF}(x) = w^{FS}(x)$, the incentive constraint at date 0 becomes

$$w^{SF}(x) \geq \frac{1}{2-p-q}\left[\frac{B}{p-q} - (p+q)R(1-x)\right]$$

where the right hand side is increasing in x. Hence, in order to minimize W, it is optimal to choose $x = 1/2$. If the first statement is true, then constraint (59) imposes that

$$w^{FS}(x) \geq \frac{1}{2-p-q}\left[\frac{B}{p-q} - (p+q)R(1-x) - R(2x-1)\right] \tag{62}$$

where the right hand side is decreasing in x.Since to minimize W, $w^{SF}(x)$ should be as small as possible, equation (63) and thus (59) should hold as an equality. The analysis of the case where (59) holds as an equality shows again that it is not possible to improve over the split $x = 1/2$.

Case 2)-ii) $w^{FS}(x) = 0$

Suppose now that $w^{FS}(x) = 0$ but that (57) is slack. Then $w^{SF}(x)$ is pinned down using (61) as an equality, that is

$$w^{SF}(x) = \frac{B}{p-q} - (p+q)R(1-x)$$

and must satisfy equation (49), that is $w^{SF}(x) \leq R(2x-1)$ given that $w^{FS}(x) = 0$. These equations are compatible only if $x \geq \underline{x} > \frac{1}{2}$ where

$$\frac{B}{p-q} - (p+q)R(1-\underline{x}) = R(2\underline{x}-1)$$

In the range $[\underline{x}, 1]$, it is optimal to set $x = \underline{x}$ since the expected compensation

$$W = p^2R(1-x) + pw^{SF}(x) = \frac{pB}{p-q} - pqR(1-x)$$

is increasing in x. The expected compensation at $x = \underline{x}$ is given by

$$W = \frac{pB}{p-q} - \frac{qp}{2-p-q}\left[R - \frac{B}{p-q}\right]$$

This is the same as the expected compensation in Lemma **??** and thus does not improve on the split with $x = 1/2$

Case 2)-iii) Monotonicity constraint (59) *binds*

Suppose finally that $w^{FS}(x) \geq 0$ and that the incentive constraint at date 1, equation (57), is slack but that (59) holds as an equality. Then, the incentive constraint at date 0, equation (61) can be rewritten as

$$w^{FS}(x) \geq \frac{1}{2-p-q}\left[\frac{B}{p-q} - (p+q)R(1-x) - R(2x-1)\right] \tag{63}$$

The condition that $w^{FS}(x) \geq 0$ is equivalent to $x \leq \underline{x}$. Using that the incentive constraint (63) should bind, the expected compensation is equal to

$$W(x) = p^2R(1-x) + p(2x-1)R + p(2-p)w^{FS}(x)$$

Using equation (63), it is easy to see that W is constant over $[1/2, \underline{x}]$. This shows that the expected compensation is minimized for any value $x \in [1/2, \underline{x}]$ and in particular for $x = \frac{1}{2}$ when, the compensation schedule is given by

$$w^{SF} = w^{FS} = \frac{1}{2-p-q}\left[\frac{B}{p-q} - (p+q)\frac{R}{2}\right], \qquad w^{SS} = \frac{R}{2} + w^{FS}$$

This finishes the analysis of all possibles cases and the proof of Proposition 1.

A.2 Proof of Corollary 1

– $\Pi^*_{GP}(\lambda)$ *and* $I^*(\lambda)$ *are decreasing in* λ

Observe first that $I^*(\lambda)$ is a decreasing function of λ since λ only enters negatively at the denominator. The result about $\Pi^*_{GP}(\lambda)$ immediately follows since Π_{GP} is increasing in I and λ only affects Π_{GP} through I.

– $\Pi_{GP}(\lambda)^*$ *and* $I^*(\lambda)$ *are increasing in* R

The result for $I^*(\lambda)$ in the case $R \geq \frac{2B}{p^2-q^2}$ is immediate since then R only appears negatively at the denominator of $I^*(\lambda)$ in expression (19). The result for Π^*_{GP} again

follows form the observation that R only affects Π^*_{GP} through $I^*(\lambda)$. When $R < \frac{2B}{p^2-q^2}$, let us rewrite the fund size as

$$I^*(\lambda) = \frac{1}{1-(1-\lambda)\left[pR - W^*\right]} = \frac{1}{1-(1-\lambda)\frac{p(2-p)}{2-p-q}\left[R - \frac{B}{p-q}\right]} \tag{64}$$

where we used the compensation schedule in Proposition 1 to derive W^*. Observe that $I^*(\lambda)$ is also increasing in R in this case.The derivative is in fact higher since and increase in R relaxes the monotonicity constraint. Finally, we can write the expected profit of the GP as

$$\Pi^*_{GP}(\lambda) = \frac{W^*}{1-(1-\lambda)\left[pR - W^*\right]}$$

Observe that *i)* $\Pi^*_{GP}(\lambda)$ is increasing in R and decreasing in $W^*(R)$ and *ii)* $W^*(R)$ is decreasing in R. This proves that the total derivative of $\Pi^*_{GP}(\lambda)$ with respect to R is strictly positive.

– $\Pi^*_{GP}(\lambda)$ *and* $I^*(\lambda)$ *are increasing in* q *when* $p = q + \alpha$ *with* α *constant*

Let us focus first on the case when $R \geq \frac{2B}{p^2-q^2}$. Then we can write the fund size as

$$I^*(\lambda) = \frac{1}{1-(1-\lambda)(q+\alpha)\left[R - \frac{B(q+\alpha)}{\alpha(2q+\alpha)}\right]}$$

Since

$$\frac{\partial(q+\alpha)}{\partial q} > 0 \qquad \text{and} \qquad \frac{\partial\left(\frac{q+\alpha}{2q+\alpha}\right)}{\partial q} < 0$$

the fund size $I^*(\lambda)$ is increasing in q. Using expression (64) and substituting p by $q+\alpha$, we obtain a similar result when $R < \frac{2B}{p^2-q^2}$.

By a slight abuse of notation, let us now write the GPs' profit as a function of q, we have

$$\Pi^*_{GP}(q) = \frac{W(q)}{1-(1-\lambda)\left[(q+\alpha)R - W(q)\right]}$$

so that Π^*_{GP} is increasing in q if and only if

$$W'(q) + (1-\lambda)R(W(q) - (q+\alpha)W'(q)) \geq 0$$

Let us consider first the case where $R \geq \frac{2B}{p^2-q^2}$. Since then $W'(q) > 0$, it is enough to show that $W(q) - (q+\alpha)W'(q) \geq 0$. This is true since

$$\begin{aligned} W(q) - (q+\alpha)W'(q) &= \frac{(q+\alpha)^2}{\alpha(2q+\alpha)}B - (q+\alpha)\frac{2q}{\alpha(2q+\alpha)^2}B \\ &= \frac{(q+\alpha)^2}{\alpha(2q+\alpha)^2}(2q+\alpha-2q) \geq 0 \end{aligned}$$

A.3 Proof of Proposition 2

The demand for capital from GPs can be derived using equation (8) and replacing $W(x)$ by its optimal value W^*. We obtain

$$D(r_{PE}) = I - A = \frac{A}{\frac{1+r_{PE}}{pR-W^*} - 1}$$

The supply of capital from investors denoted S is given by the following step functions

$$S(r_{PE}) = \begin{cases} 0 & \text{if} \quad r_{PE} < \underline{r}(\lambda_L) \\ S \in [0, \mu_L M] & \text{if} \quad r_{PE} = \underline{r}(\lambda_L) \\ \mu_L M & \text{if} \quad r_{PE} \in (\underline{r}(\lambda_L), \underline{r}(\lambda_H)) \\ S \in [\mu_L M, M] & \text{if} \quad r_{PE} = \underline{r}(\lambda_H) \\ M & \text{if} \quad r_{PE} > \underline{r}(\lambda_H) \end{cases}$$

Observe that the first and fifth cases for $S(r_{PE})$ cannot be part of an equilibrium. Indeed, at $r_{PE} < \underline{r}(\lambda_L)$, the demand from GPs is strictly positive. The case $r_{PE} > \underline{r}(\lambda_H)$ is ruled out because the demand from GPs is capped at $D(0) = I^*(0) - A$ which we assume to be strictly lower than M.

We are not left to examine the remaining three cases. The equilibrium is $r^*_{PE} = \underline{r}(\lambda_L)$ if the demand at this rate belongs to $[0, \mu_L M]$. This condition is equivalent to

$$I^*(\lambda_L) - A \leq \mu_L M \quad \Leftrightarrow \quad \mu_L \geq \bar{\mu}_L = \frac{I^*(\lambda_L) - A}{M}$$

The equilibrium is $r^*_{PE} = \underline{r}(\lambda_H)$ if the demand at this rate belongs to $[\mu_L M, M]$. This condition is equivalent to

$$I^*(\lambda_H) - A \geq \mu_L M \quad \Leftrightarrow \quad \mu_L \leq \underline{\mu}_L = \frac{I^*(\lambda_H) - A}{M}$$

Finally, in the last case, the last case $r^*_{PE}(\underline{r}(\lambda_L), \underline{r}(\lambda_H))$ obtains when the solution to

$$D(r_{PE}) = \mu_L M$$

lies in $(\underline{r}(\lambda_L), \underline{r}(\lambda_H))$. This equation above is equivalent to

$$\begin{aligned} 1 + r_{PE} = (pR - W^*)\frac{A + \mu_L M}{A} &= \frac{I^*(\lambda_L) - A}{I^*(\lambda_L)} \frac{A + \mu_L M}{A}(1 + \underline{r}(\lambda_L)) \\ &= \frac{\bar{\mu}_L(A + \mu_L M)}{\mu_L(A + \bar{\mu}_L M)}(1 + \underline{r}(\lambda_L)) \end{aligned}$$

where the last line follows from the definition of $\bar{\mu}_L$. It is straightforward to see that the condition $r^*_{PE}(\underline{r}(\lambda_L), \underline{r}(\lambda_H))$ when r^*_{PE} is determined by the equation above can be shown to be equivalent to $\mu_L \in (\underline{\mu}_L, \bar{\mu}_L)$. This concludes the proof.

A.4 Proof of Proposition 3

The first step if to show that the price of secondary claims is $P = 1$. The result follows from a simple argument that neither cash, nor private equity should be dominated as an investment. For any type i LP, the payoff tree for each asset can be written as follows:

Figure 3: Payoff for one unit invested in the asset

For any type i of LP, the expected return on PE is thus

$$1 + r_i^f = P(1 + r_i^c)$$

Unless $P = 1$, one asset is dominated which leads to a contradiction. For instance if $P > 1$, all investors would only invest in private equity. But then, there is no demand for claims in the secondary market so that the market clearing price would be 0, a contradiction. This first result also shows that any LP is indifferent between cash and a private equity commitment at date 1. This proves our final result that the portfolio allocation of LPs is not determined as long as as the primary market clears.

Observe that market clearing in the primary market does not impose any constraint. Indeed, since LPs are indifferent between cash and private equity commitments, the supply of capital is given by $F \in [0, M]$. By definition, the demand for capital from GPs is bounded above by $I^*(0) - A < M$. Hence the equilibrium is pinned down by market clearing in the secondary market.

In the secondary market, the supply of claims is given by

$$Q = (\mu_L f_L \lambda_L + \mu_H f_H \lambda_H)\, M$$

Using equation (34), the demand for claims at price $P = 1$ is given by

$$D \in [0, M(1 - \lambda_L)\mu_L(1 - f_L) + M(1 - \lambda_H)\mu_H(1 - f_H)]$$

Two cases are then possible. First, secondaries trade at par if $D > Q$, which is equivalent to

$$\begin{aligned}(\mu_L f_L + \mu_H f_H)\, M &\leq (1-\lambda_L)\mu_L M + (1-\lambda_H)\mu_L M \\ I^*_{sec} - A = F &\leq (1-\bar{\lambda})M\end{aligned}$$

where the second line follows from market clearing in the primary market. But if secondaries trade at par, we have $1 + r_{PE} = P = 1$ so that $I^*_{sec} = I^*(0)$. Hence, the condition above is equivalent to $LR \geq 1$. This proves the first case of Proposition 3.

We are left to determine r^*_{PE} and thus I^*_{sec} when secondaries trade at a discount, that is when $LR < 1$. In this case, at the market price of $P' = 1$, supply equal demand. Building on our previous derivations, the condition $Q = S$ is equivalent to

$$I^*_{sec} - A = (1-\bar{\lambda})M \tag{65}$$

We can then determine the equilibrium value of r^*_{PE} by using the GPs' demand for capital, equation (32). Simple algebra establishes that r^*_{PE} is indeed given by equation (38). This concludes the proof.

A.5 Proof of Proposition 4

We denote Π^{sec}_i and Π_i the utility of an LP of type i with and without a secondary market respectively. Similarly, we denote Π^{sec}_{GP} and Π_{GP} the corresponding quantities for GPs. Finally, to avoid confusion we let $r^{*,sec}_{PE}$ and r^*_{PE} be the equilibrium buy and hold return respectively with and without a secondary market.

We have

$$\Pi_H = \max\{1, (1-\lambda_H)(1 + r^*_{PE})\} = 1$$

where the second equality follows from the result in Proposition 3 that $r^*_{PE} \leq \underline{r}(\lambda_H)$. Hence, it immediately follows that H-LPs cannot lose from the introduction of a secondary market. They strictly benefit whenever the return on cash is strictly positive, that is $r^c_H > 0$. We showed in Proposition 3 that this arises when $\mu_L \leq \mu^{*,sec}_L$. This proves our result for H-LPs.

To prove the claims for L-LPs, by continuity, we only need to show that under condition (41), we have that $\Pi^{sec}_L < \Pi_L$ for $\mu_L = 0$. Proposition 3 shows that

$$\Pi_{L|\mu_L=0} = \frac{1-\lambda_L}{1-\lambda_H} > 1$$

We only need to consider the case $\mu^{*,sec}_L > 0$ since otherwise $Pi^{sec}_L = 0$ for all values of μ_L, according to Proposition 3. Then, when $\mu_L = 0$, we have that

$$\Pi^{sec}_{L|\mu_L=0} = \lambda_L + (1-\lambda_L)(pR - W^*)\frac{A + (1-\lambda_H)M}{(1-\lambda_H)M}$$

We now show that $\Pi^{sec}_{L|\mu_L=0} \leq \Pi_{L|\mu_L=0}$ is implied by condition (41). We have

$$\begin{aligned}
& 0 \leq \Pi_{L|\mu_L=0} - \Pi^{sec}_{L|\mu_L=0} \\
\Leftrightarrow \quad & 0 \leq 1 + \frac{\lambda_H - \lambda_L}{1-\lambda_H} - 1 - \frac{1-\lambda_L}{1-\lambda_H}\left((pR - W^*)\left[1 - \lambda_H + \frac{A}{M}\right] - (1-\lambda_H)\right) \\
\Leftarrow \quad & 0 \leq \lambda_H - \lambda_L - \lambda_H(1-\lambda_L)\,(1 - pR + W^*) \\
\Leftrightarrow \quad & 0 \leq \lambda_H(1-\lambda_L)(pR - W^*) - \lambda_L(1-\lambda_H)
\end{aligned}$$

where the third line follows from assumption (5) and the last line is equivalent to condition (41).

Finally, we prove the last claim for GPs. Observe that the GPs' profit is equal to the expected fee W^* multiplied by the equilibrium size of the fund. Hence, it is enough to compare I^* and $I^{*,sec}$. Fund size is given in each case by

$$I^* = \begin{cases} I^*(\lambda_H) & \text{if} \quad \mu_L \leq \underline{\mu}_L \\ \mu_L M & \text{if} \quad \mu_L \in (\underline{\mu}_L, \bar{\mu}_L) \\ I^*(\lambda_L) & \text{if} \quad \mu_L \geq \bar{\mu}_L \end{cases}, \qquad I^{*,sec} = \begin{cases} A + (1-\bar{\lambda})M & \text{if} \quad \mu_L \leq \mu_L^{*,sec} \\ I^*(0) & \text{if} \quad \mu_L > \mu_L^{*,sec} \end{cases}$$

Hence, the necessary and sufficient conditions for $I^*_{sec} > I^*$ to hold for all μ_L are

$$I^{*,sec}_{|\mu_L=0} > I^*_{|\mu_L=0}, \qquad I^{*,sec}_{|\mu_L=\bar{\mu}_L} > I^*_{|\mu_L=\bar{\mu}_L}$$

We prove that the first condition always holds. We have

$$I^{*,sec}_{|\mu_L=0} = \min\{I^*(0), A + (1-\lambda_H)M\}, \qquad I^* = I^*(\lambda_H)$$

Hence, the desired inequality holds since

$$\begin{aligned}
(1-\lambda_H)M - (I^*(\lambda_H) - A) &= (1-\lambda_H)\left(M - \frac{(pR - W^*)}{1-(pR-W^*)}\right) \\
&= (1-\lambda_H)\,(M - I^*(0) + A)
\end{aligned}$$

and the last expression is positive under assumption (5). This proves the first result.

For the second condition, it is enough to show that $(1-\bar{\lambda}) > \bar{\mu}_L$ when condition

(42) holds. Condition (42) implies that

$$\begin{aligned}
& 0 \leq 1 - \frac{\lambda_L}{1-\lambda_H+\lambda_L} - 1 + \frac{\lambda_L}{1-(1-\lambda_L)(pR-W^*)} \\
\Leftrightarrow \quad & 0 \leq \frac{1-\lambda_H}{1-\lambda_H+\lambda_L} - (1-\lambda_L)\frac{1-(pR-W^*)}{1-(1-\lambda_L)(pR-W^*)} \\
\Leftrightarrow \quad & 0 \leq \frac{1-\lambda_H}{1-\lambda_H+\lambda_L} - \frac{I^*(\lambda_L)-A}{I^*(0)-A} \\
\Rightarrow \quad & 0 \leq \frac{1-\lambda_H}{1-\lambda_H+\lambda_L} - \bar{\mu}_L
\end{aligned}$$

where the last inequality obtains under assumption (5) and is equivalent to the desired condition (42).

To complete the proof, we must show that the fund size can decrease. Given our previous analysis, it is enough to show that condition (42) does not always hold. A necessary condition for (42) to be violated is that $(pR - W^*) \leq \lambda_H$. Since we assumed that $\lambda_H \leq \hat{\lambda}(p, R)$, we need to show that

$$pR(pR - W^*) \leq pR - 1 \tag{66}$$

may hold under our maintained assumptions. Observe that assumptions (3) and (4) together imply that $B \in [\frac{p^2-q^2}{p^2}(pR-1), (p-q)R]$. This interval is non-empty if

$$(p+q)(pR-1) \leq p^2R \quad \Leftrightarrow \quad R \leq R_{max} := \frac{p+q}{pq}$$

Since condition (66) is easier to satisfy for higher values of B, we can replace B by its highest possible value $(p-q)R$ to obtain a necessary condition. This yields

$$\begin{aligned}
p^2R^2\left(1 - \frac{p}{p+q}\right) &\leq pR - 1 \\
qp^2R^2 - (p+q)pR + (p+q) &\leq 0
\end{aligned}$$

This inequality may hold if the discriminant is positive that is if

$$\Delta := p^2(p+q)(p-3q)$$

is positive, that is if $p \geq 3q$. Then the lowest solution to the corresponding second order equation is given by

$$R_1 = \frac{p(p+q) - \sqrt{\Delta}}{2qp^2} < \frac{p+q}{pq} = R_{max}$$

This shows, as required, that when $p \geq 3q$ and condition (42) does not hold, there is a range of parameter values where GPs' welfare can go down.

A.6 Proof of Proposition 5

To prove the first part, we build on the results of Proposition 3. We need to show that in the bad state the secondary market can clear at price $P = 1$ when $I^{*,sec} = I^*(0)$. This condition holds if demand exceeds supply at this price, that is if

$$\begin{aligned}(\mu_L \lambda_{BL} f_L + \mu_H \lambda_{BH} f_H) M &\leq (1 - \lambda_{BL}) \mu_L (1 - f_L) M + (1 - \lambda_{BH}) \mu_H (1 - f_H) M \\ (\mu_L f_L + \mu_H f_H) M &\leq (1 - \lambda_{BL}) \mu_L M + (1 - \lambda_{BH}) \mu_L M \\ I^*(0) - A &\leq (1 - \bar{\lambda}_B) M\end{aligned}$$

To prove the second part, let us consider equation (46) that gives the extra return of cash over private equity commitment for each type i of LP. Observe that it must be that $P_B^* < 1$. Otherwise, cash is strictly dominated by private equity commitments for all investors. This cannot arise in equilibrium since then the supply of capital M exceeds the maximum demand $I^*(0) - A$ for capital from GPs. But then if $P_B^* < 1$, we have

$$r_L^c - r_L^f > r_H^c - r_H^f$$

Observe that the second term cannot be strictly positive since otherwise no investor would commit capital to private equity funds. hence, two cases are possible. If $r_H^c - r_H^f = 0$, L-LPs strictly prefer cash and $f_L^* = 0$. H-LPs are indifferent between cash and and private equity but market clearing requires that $f_H^* > 0$. If $r_H^c - r_H^f < 0$, then we have that $f_H^* = 1$, since H-LPs strictly prefer private equity. But then, market clearing imposes that $f_L^* < 1$. This proves the second result and concludes the proof.

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