Dynamic Coordination with Flexible Security Design

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Abstract

Entrepreneurs obtain funding liquidity by issuing securities backed by the current period dividend and resale price of a long-lived collateral asset. They are privately informed about the collateral quality. Higher (lower) resale price lowers (increases) adverse selection and makes the asset a good (lousy) collateral. Conversely, a good (lousy) collateral has high (low) resale price. When only equity can be issued, this dynamic feedback between the asset price and collateral quality can lead to self-fulfilling prices and multiple equilibria: When asset price is high (low), equity is liquid (illiquid) and real output is high (low). Optimal flexible security design, which involves short-term liquid collateralized debts, eliminates multiple equilibria fragility and improves welfare through inter-temporal coordination. When security design is rigid, multiple equilibria reemerge. Comparative statics generate rich dynamic properties of haircuts and interest rates.

Keyword: Liquidity; Dynamic Price Feedback; Security Design; Multiple Equilibria; Self-fulfilling Prices; Financial Fragility; Haircut; Dynamic Feedback Runs; Repo; Portfolio Repo; Asset-Backed Security; Collateral; Limited Commitment; Adverse Selection.

JEL classification: G10, G01

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1 Introduction

In this paper, we identify a new source of financial fragility in studying the classical financing problem of entrepreneurs who have access to productive opportunities but lack funding to implement them. In our setting, entrepreneurs face two commonly observed frictions in obtaining the funding liquidity. The first is that entrepreneurs cannot pledge the output from their projects to obtain funding. To overcome this non-pledgbility constraint, they borrow against or sell securities backed by a long-lived collateral asset. In addition to this friction, there is also short-lived asymmetric information about the quality of the collateral asset in term of its dividend payment at the current period and potential borrowers are better informed about its quality than potential lenders. The key observation is that the resale price of the long-lived asset can ameliorate the resulting adverse selection problem. More importantly, the dependency of the level of adverse selection on the asset price generates a dynamic feedback between asset price and liquidity. Higher (lower) resale price lowers (increases) adverse selection and makes the asset a good (lousy) collateral. Conversely, a good (lousy) collateral has high (low) resale price. When the set of available financial instruments is restricted, this dynamic feedback leads to fragility in liquidity provision and asset price volatility. This fragility comes from a new source: self-fulfilling asset prices.

We characterize optimal security design in this setup and one of the implementations of optimal security design involves short-term collateralized debt. Haircuts on these short-term collateralized debts reflect heavily the severity of adverse selection but their interests rates are less so. We find that the optimal flexible security design eliminates the fragility and improves liquidity and welfare. This is achieved through inter-temporal coordination that operates through the dynamic feedback effect. Suppose that in this period collateralized debt is illiquid because the resale price is low. The low resale price reflects the fact that the collateralized debt is illiquid next period, which in turn reflects the fact that the collateralized debt is illiquid in the period after that, and so forth. Given that the security design is flexible and is adjusted in every period, current borrowers increase the haircut and decrease the face value of the debt contract in order to lower the adverse adverse selection and make sure that the debt becomes liquid, and they believe that all future borrowers will do so too. The funding raised from selling liquid debt allows entrepreneurs to engage in productive activities which in turns makes the collateral asset more valuable and hence increases its price in every period. This increase in asset price triggers the dynamic price feedback: it lowers adverse selection which allows a decrease in the haircut of the liquid debt and an increase in the amount of funding raised. This process continues until the haircut of the debt cannot be decreased any further since the resulting debt might contain too much adverse selection for it to remain liquid. Since adverse selection is short-lived and recurring, flexible security design is essential. When security design is not updated frequently enough, fragility reemerges as dynamic coordination fails leading to 'slow-forming' dynamic feedback runs that result in evaporation of liquidity and collapse in asset price. The current understanding of the crisis in the shadow banking system of overnight repurchase agreements, asset-backed securities, broker-dealers and investments during the 2008-09 Great Recession is that it is a classic bank run a la Diamond and Dybvig (1983). However, this popular explanation ignores the fact that most of the securitized products and the short-term funding instruments of these shadow banks are backed by the resale prices of the assets on their balance sheet (in addition to dividend/interest payments). Our model shows that this funding model operating on the dynamic price feedback effect potentially increases the amount of funding liquidity and real output in the economy. The culprit of the fragility we have observed is the failure of inter-temporal coordination due to the rigidity of securitization contracts. Flexible securitization in fact eliminates fragility.

The two key frictions in the funding market faced by the potential borrowers in the our model are commonly observed. First, limited pledgeability may result from non-contractibility of the cash flows and lack of commitment of the borrowers to divert the cash flows for private consumption. The ability to pledge collateral assets helps borrowers obtain necessary liquidity. Second, the quality of collateral assets is often subject to adverse selection which limits the effectiveness of these assets in raising liquidity. One potential source of adverse selection is the incentive of borrowers to tamper with the collateral quality. Historically, borrowers have debased collateral assets, eg., by reducing the metallic content of coins below their face value, or using counterfeits to obtain more funding. In recent times, collateral quality has been subject to questioning because of the possibility that borrowers might pledge it multiple times. In addition, since borrowers hold collateral assets on their balance sheets, they may have an informational advantage about the quality of these assets.

In our setup, borrowers' desire for liquidity stems from the fact that they have access to a productive technology and need funding to take advantage of it. Broadly speaking, the model captures any situation where it is more efficient to allocate funding to borrowers who might have better investment opportunities (eg., entrepreneurs or financial institutions), or who have immediate consumption needs (eg., liquidity constrained consumer or firms). Due to aforementioned frictions borrowers are unable to obtain this liquidity directly. Instead, they sell securities backed by the cashflow from a long-lived collateral asset to raise liquidity. We consider a dynamic setting where the quality of the collateral asset (captured by the distribution of its dividend payoff) varies period by period. Collateral is of either high or low quality, where the dividend distribution of the high quality collateral first-order stochastically dominates that of the low quality.[1] The borrowers are privately informed about the current period quality at the beginning of each period.[2]

We then explore the implications of optimal security design for liquidity provision in this dynamic environment with adverse selection. We begin with a benchmark case where borrowers are restricted to selling asset-backed *equity* for liquidity.Given this limitation, the economy might exhibit fragility in terms of possible multiple (dynamic stationary) equilibria for the same underlying technology and beliefs. The logic behind the multiple equilibria in the benchmark case is based on a dynamic feedback between the resale price and the level of adverse selection. The key to understanding the feedback is that higher resale price for the collateral asset allows borrowers to exchange the asset-backed equity claims for more immediate funding. As a result when resale price is high (low) enough both high and low (only low) quality assets are used to back equity claims resulting in lower (higher) adverse selection. Conversely, when adverse selection is low (high), lenders are willing to exchange more (less) funding for equity claims backed by the asset which leads to higher (lower) resale price for the asset.

This dynamic price feedback leads to three possible equilibrium regions in this economy. There is a 'separating' region where productivity is low and adverse selection is severe. In this region, high-quality borrowers choose to retain their asset-backed equity claims. Since only low quality borrowers are selling equity claims and engaging in production, the equity price today is indeed low, the economic output is limited, and the asset resale price in turn is depressed. There is also a 'pooling' region where productivity is high and adverse selection is mild. In this region, both types borrow against their equity claims to employ the productive technology, the equity price is high, the output is large and the asset price is, in turn, booming. For the intermediate values of productivity and adverse selection, there are multiple equilibria where both separating and pooling equilibria coexist.

Next we turn to optimal security design. It is well understood in the literature that in a static economy optimal security design improves liquidity. In a dynamic economy, we demonstrate that optimal security design also eliminates the multiple equilibria fragility. To our knowledge this new role for security design has not been discovered till now. To state this result more explicitly, let us first describe our notion of liquid vs. illiquid security. We call a security liquid if both borrower types sell it. A liquid security commands a higher price so more funding can be raised by borrowers to scale up production. We call a security illiquid if only the low type sells it. An illiquid security has a lower price so less funding can be

[1]To focus on the role of the asset as collateral, we assume that the asset is not an input in the production process and the dividend process is exogenous.

[2]That is, there is adverse selection about the quality of the collateral asset between borrowers and lenders at the beginning of each period before any borrowing and production takes place.

raised by low type borrowers to scale up production. Our main result on optimal security design shows that there is a unique stationary dynamic security design equilibrium where the optimal design involves a *short-term liquid collateralized* debt tranche, and the residual *illiquid* equity tranche.[3]

In the optimal security design, the issuer chooses the face value of the debt as large as possible in order to raise the maximum amount of funding liquidity. As the face value increases, the debt tranche incorporates more of the high dividend states. If the face value is too high, the high quality borrowers, who know these states are likely, might prefer to retain the debt tranche rather than pooling with the low quality borrowers to get a discounted price for these states. Hence the security design pushes the face value of the debt up to the point where the high quality borrowers are indifferent between selling at discount to engage in more productive technology versus retaining the asset-backed debt tranche.[4] A key point is that the tranche always incorporates the resale price of the collateral. As the collateral price increases, selling the debt tranche becomes more attractive to the high quality type, allowing the security designer to increase the face value of the collateralized debt.

The dynamic security design equilibrium Pareto dominates the separating equilibria in the equity-only benchmark case and selects the pooling equilibrium in the multiple equilibria range. To see why, suppose a debt tranche backed only by the future resale price is introduced. Since this debt is free of adverse selection problem, both borrower types will issue it to take advantage of the productive technology. The asset price rises due its collateral role in securing immediate funding. The higher asset price will allow the borrowers to increase the face value of debt further by incorporating some of the high dividend states. The face value will increase until the high quality borrowers become indifferent between selling the liquid debt tranche versus retaining it. In the separating equilibrium region of the equity-only benchmark case, this process leads to a liquid debt tranche that is traded by both types and improves the welfare of the borrowers. In the multiple equilibria region it selects the pooling equilibrium – that is, issuers sell the entire equity-like "pass-through" debt. In this unique security design equilibrium, both liquidity and production output are higher than the eliminated separating equilibrium.

We show this uniqueness result hinges on the assumption that borrowers have the flexibility to adjust the security design at the beginning of each period. In practice, security contract terms may not be updated frequently because of administrative costs or simply inattention. When contract terms are rigid in the sense that the face value of the contract does not get updated at the beginning of every

[3]In the model, dividend is independently distributed over time, so that the adverse selection problem only exists at the beginning of each period.

[4]Selling the debt tranche generates value through the technology multiplier. However, selling it is less attractive for the high type as he must pool with the low type and accept a lower price.

period, a run equilibrium through the dynamic price feedback might emerge and the liquidity of the security market may deteriorate.[5] Essentially during such a run, the asset value and the asset price drop, increasing the severity of adverse selection about the quality of the collateral. This makes the previously liquid debt tranche illiquid, which in turn justifies the drop in the asset price. Had the design been flexible, borrowers would redesign the security in this event by lowering the debt threshold to make sure that the debt tranche liquid. This will push the asset price up, triggering the dynamic price feedback and leading to a full recovery of prices and the debt threshold. However, when the design is rigid, the drop in asset price can be self-fulfilling.

The dynamic feedback runs unfold slowly and are marked by two stages. The initial stage is when the liquid security design equilibrium switches to the run equilibrium where the pessimistic belief of an upcoming sunspot is triggered. The second stage is when a sunspot actually hits the economy. When the economy enters a run equilibrium, haircut of the debt tranche immediately increases, because investors anticipate that it may be illiquid when a sunspot hits the economy. At the same time, the asset price and the volume of the debt tranche are also lowered. When the sunspot hits, they decrease further, while haircut may increase further. The drop in funding liquidity is severe when the sunspot hits, because the debt tranche backed by high quality collateral stops circulating entirely. Our model of runs also maps out the recovery process of asset-backed funding liquidity. When the contract terms are updated, the update restores investors' sentiment about the liquidity of this market, the price and the volume recover partially, to the levels right after equilibrium switch. The fluctuation driven by sunspots may take place repeatedly, until the sunspot equilibrium cannot be sustained as an equilibrium. It is possible that the asset-backed liquidity evaporates completely (eg., in the asset-backed commercial paper market) if adverse selection becomes so severe that only low quality borrowers are in the market.

Besides optimal security design, our model also has implications for asset prices. Asset price in our model is more than the sum of the discounted future dividends because the collateral asset commands a liquidity price premium. Borrowers pledge the asset's dividend flows and resale price to overcome pledgeability constraints and raise funding for production. The liquidity premium reflects a technology multiplier since the funds are more valuable when the technology is more productive. This connection between productivity and liquidity premium might seem *counter-intuitive* because the long-lived asset is

[5]Runs in our setup are dynamic feedback runs and hence distinct from bank runs as in Diamond and Dybvig (1983), or the type of repo runs caused by systemic asset firesales as in Martin, Skeie, and Von Thadden (2012) or due to repo market microstructure features, liquidity need of the lenders as well as the capital position of the borrowers as in Martin, Skeie, and Von Thadden (2014), or the collateral crisis due to the endogenous information production studied in Gorton and Ordonez (2014).

not a direct input in the production technology per se and serves only as a collateral to obtain funding liquidity. This theoretical finding about the technology multiplier might speak to the meteoric rise of asset price during the productivity boom we observed during the mid 2000s and in general pro-cyclical patterns of asset price premium.

Next we turn to the specific applications of this framework of optimal security design with the dynamic price feedback. In particular, we focus on developing detailed implications for one implementation of the optimal security in our model, the short-term repo contract. The optimal security in our model is liquid, short-term, and collateralized debt, which resembles the commonly observed short-term repo contract. Under the repo interpretation, the model generates unique testable predictions on the contract terms such as interest rates and haircut as well as the collateral asset price. For example, our model predicts that the magnitude of repo haircut has two components: the productivity of the borrower's technology, and the value of the equity tranche relative to the value of the collateral. The first component arises because borrowers, who price the collateral asset, value the liquidity service the asset provides, while lenders, who price the loan, does not value the liquidity service. It reflects heterogeneous valuation over the collateral assets among agents. This component relates to the difference in opinion literature on leverage starting with Geanakoplos and Zame (2002), Geanakoplos (2003), Fostel and Geanakoplos (2012), and Simsek (2013). The second component arises because of information friction (and/or adverse selection). This component has been emphasized by Dang, Gorton, and Holmström (2011) and Gorton and Ordonez (2014). Interestingly, the repo rate in our model is free of the adverse selection risk since repo debt is liquid and both high and low quality borrowers participate in this market. Nevertheless, repo debt is risky and repo rate is determined by the default risk of the repo contacts (which is related to the face value of the repo contract) and the demand for funding liquidity (which is related to the productivity). Our model also generates predictions on commonly used portfolio repos, which are repo contracts backed by a portfolio of collateral assets. It predicts that when the fraction of safe asset in the collateral pool increases, repo contract terms improve since the level of adverse selection is lowered.

Since the framework of security design with dynamic price feedback effects is quite general, we conclude by discussing some of the potential applications. We demonstrate that the model can be adapted to the setting of security lending, where the gain from trades comes from different valuations in securities. We also discuss the potential new insights for the pecking order theory of financing with the dynamic price feedback. In this case, firms pledge the resale price in addition to the dividend when issuing securities and the resale price is endogenous in the types of securities being offered. We also discuss the application of this framework for understanding the collateral multiplier role of monetary policies.

The structure of the paper is as follows. In section 2, we review related literature. In section 3, we lay out the basic setup. In section 4, we describe the security design problem. In section 5, we study the baseline case where security design is restricted to equity. In sections 6, we solve for the optimal security design and study its equilibrium properties including uniqueness and runs. In section 7, we explore one of the implementations of the optimal security, short-term repo contracts, and associated economic implications in details. Section 8 concludes and discusses potential applications.

2 Related Literature

The seminal work of Akerlof (1970) started the literature on lemons market to study the impact of adverse selection on trade volume and efficiency. There is a long lineage of security design literature including Leland and Pyle (1977); Myers and Majluf (1984a); DeMarzo and Duffie (1995); and DeMarzo and Duffie (1999) that examine informed sellers' incentive to issue optimal security to signal asset quality. For example, in DeMarzo and Duffie (1999), a closely related paper, retaining equity is a signal for quality. By comparison, ours is a competitive screen model and securities are designed to screen issuer types. Therefore implications are different. In ours, both borrower types issue debt while only the poor quality type issues equity. Moreover, extending the static setup to a dynamic environment allows us to discover that security design helps to mitigate adverse selection problem not only by increasing the amount of liquidity but also by eliminating fragility.

Our result that both borrower types issue debt and debt is liquid is reminiscent of the finding in Gorton and Pennacchi (1990) where they find that low-information-intensity (debt-like) securities protect sellers from the risk of selling only high-quality assets when trading with an informed buyer. Boot and Thakor (1993) also find that the optimal security design is implementable by a liquidity debt contract and an equity contract, and others. However, the motivation is to stimulate information production using information sensitive securities. This literature has now progressed to incorporate endogenous asymmetric information in optimal security design problem such as Yang (Forthcoming); Dang, Gorton, and Holmström (2013); and Farhi and Tirole (2015). That information friction affects moneyness of an asset has also been studied by Lester, Postlewaite, and Wright (2012) and Li, Rocheteau, and Weill (2012).

There has also emerged a literature on heterogeneous information and security design such as Ellis, Piccione, and Zhang (2017). Under diverse beliefs, however, there is no fragility under dynamic environment. There will be speculative premium under diverse beliefs but it is difficult to investigate financial fragility unless exogenous changes in beliefs are introduced. With adverse selection as in our model, the changes in market liquidity or "beliefs" can be endogenous.

By studying optimal collateral-backed security design and funding liquidity, our paper is also related to a long line of collateral literature in money and macroeconomics starting with the seminar work of Kiyotaki and Moore (1997) and recent studies on the prevalence of the use of repo contracts in funding financial institutions such as Geanakoplos and Zame (2002), Geanakoplos (2003), Fostel and Geanakoplos (2012), Simsek (2013), and Gottardi, Maurin, and Monnet (2017). Increasingly attempts are made to incorporate financial frictions in the macroeconomic models or studying macroeconomic implication of financial friction such as collaterals to understand the boom and bust cycles. The recent papers include but are not limited to Gorton and Ordonez (2014); Kuong (2017); Parlatore (Forthcoming); and Miao and Wang (2018). Kurlat (2013) and Bigio (2015) study financial frictions that arise endogenously from adverse selection in a dynamic production economy.

Our paper is also closely to Plantin (2009), Chiu and Koeppl (2016), Donaldson and Piacentino (2017); and Asriyan, Fuchs, and Green (2017), where multiple equilibria is dynamic in nature. Although Asriyan, Fuchs, and Green (2017) focuses on sentiment-driven multiple equilibria and differs from ours in setup and implication, the insight that asset price and liquid is closely linked is very close to ours. Our insight on the implication of security design on the dynamic fragility is important contribution to this literature.

3 The Model Setup

In this economy, there are two types of agents. All agents have access to a "basic technology" to produce a consumption good. This basic technology produces one unit of the consumption good using one unit of labor. A broad interpretation of the basic technology is that it captures the "outside option" of the agents and the associated benefit is the opportunity cost of undertaking other technologies or investments. In addition, one type has access to a "special technology" to produce an intermediate good. This technology is constant-returns-to-scale and allows the agent to produce one unit of the intermediate good from one unit of labor. However, the intermediate good does not provide direct utility. The other type possesses a "productive technology" that produces a consumption good using the intermediate good through a constant returns-to-scale technology. This technology is highly productive because an input of one unit of intermediate good generates $z > 1$ units of consumption good, and we term it the z-technology. We call the agent who has the ability to produce the intermediate good the type I agent and the agent who possesses the z-technology the type O agent.[6]

The assumption on the technologies in the economy is made to capture the gain from trade of intermediate goods between the two types of agents. Intermediate goods can be interpreted as any inputs to the z-technology such as capital, equipments, or intermediate products. The O type agents would like to borrow as much intermediate goods as possible from the I type agents to engage in the productive z-technology. In the theory, type O agents emerge naturally as borrowers and type I agents as lenders because type O agents need funding from type I agents to take advantage of the highly productive z-technology. In this way the model parsimoniously captures any situation where it is more efficient to allocate funding to borrowers who might have better investment opportunities (eg., entrepreneurs or financial institutions), or who have immediate consumption needs (eg., liquidity constrained consumer or firms).

Timing. The economy is set in discrete time and lasts forever. Each period has three dates. At date 1, the intermediate good is produced by agent I. At date 2, consumption good is produced via the z-technology using the intermediate good and/or the basic technology using labor. At date 3, consumption takes place. Any leftover intermediate or consumption good perishes at the end of the period.[7]

Utilities and discounting. An agent's utility in period t is given by $U_t(x, l) = x - l$ where x is the amount of consumption good consumed and l is the amount of labor supplied by the agent. There is no discounting between sub-periods. Agents discount periods at a rate β, with $0 < \beta < 1$.

Productive Asset and Asymmetric Information. There is an asset in the economy, which pays s units of dividend in terms of consumption good at date 3. The total supply of the asset is A. With probability λ, the dividend of the asset follows distribution $F_L \in \Delta[s_L, s_H]$, with $0 \leq s_L < s_H$. With probability $1 - \lambda$, it follows distribution $F_H \in \Delta[s_L, s_H]$. We assume that F_H first order stochastically dominates F_L. The quality, denoted by $Q \in \{H, L\}$, represented by λ ($Q = L$ with probability λ), is i.i.d. over time. More generally, asset quality could be persistent over time. We will consider that case in later sections.

We assume that agent O's production of the consumption good is not pledgeable. Without this friction, in any period, agent O would borrow unlimited amount of intermediate goods from the I agents at date 1, produce unlimited amount of consumption goods at date 2, and pay back the I agents at

[6]Here, O stands for owner since, as we show later, agent O will own a productive asset and I stands for investor since agent I will invest in asset based securities.

[7]The framework of dynamic analysis is borrowed from Lagos and Wright (2005).

date 3.[8] Given the nonpledgeability assumption, agent O cannot promise to pay back at date 3, and hence cannot borrow from the I agents at date 1. The asset provides a way for the O agent to partially overcome this friction by providing liquidity since it can be used as collateral to back up agent O 's promise to pay back. If agent O owns the asset, she can borrow intermediate goods from the I agents at date 1 using both the dividend and the resale value of the asset at date 3 as collateral. If agent O does not fulfill her promise, I agents can seize the collateral asset.

However, the use of this collateral asset for liquidity service is limited by an additional friction in our economy, which is asymmetric information. We assume that the quality of the collateral asset is privately observed by agent O at the beginning of the period (i.e, at date 1 of each period). This introduces an adverse selection problem which plays a key role in our analysis. The assumption that agent O is better informed of the collateral asset's quality can be motivated or micro-founded in several ways. As demonstrated later, agent O would purchase all collateral assets in equilibrium because her need of liquidity to kick start the z-technology. Consequently, she has a stronger incentive to acquire information on the collateral asset. Empirically, one can also motivate the superior information advantage of the asset owner by the fact that the asset quality can be easily tempered with by the owners. Historically, borrowers have debased collateral assets, eg., by reducing the metallic content of coins below their face value, or using counterfeits to obtain more funding. In recent times, collateral quality has been subject to questioning because of the possibility that borrowers might pledge it multiple times. In addition, since borrowers hold collateral assets are on their balance sheets, they may have an informational advantage about the quality of these assets.

In this asymmetric information environment, the asset provides only limited amount of liquidity since the amount that agent O can borrow is bounded by the expected dividend and the resale value of the asset. As we will show in our baseline case, when agent O has superior information about the asset quality, resulting adverse selection tightens the liquidity constraint. In this case, agents' expectations about the asset price can make the adverse selection problem more or less severe, leading to multiple equilibria.

In this environment with adverse selection, agent O can improve liquidity available at the beginning of the each period by optimally designing securities which are used to exchange for the intermediate goods at date 1 and deliver consumption good payments at the end of each period. A security, hence is a state-contingent promise at date 1 of consumption good payment at date 3. Denote the payoff from security j at state s to be $y^j(s)$. Because agent O cannot commit to pay, the security must be

[8] Recall that the returns to scale of the z-technology are $z > 1$.

backed by the dividend and the ex-dividend price of the asset, denoted by ϕ_t. The set of all feasible asset backed securities at time t for a given price ϕ_t is $\mathcal{I}_t(\phi_t) \subseteq \{y : y(s) \leq s + \phi_t, \forall s \in [s_L, s_H]\}$. The set $\mathcal{I}_t(\phi_t)$ captures any potential exogenous restrictions on the set of feasible securities. One possible set, $\mathcal{I}_t(\phi_t) = \{y : y(s) = s + \phi_t, \forall s \in [s_L, s_H]\}$, consists of only a single "pass-through" security which promises the dividend and resale value of the collateral asset. A second possibility, $\mathcal{I}_t(\phi_t) = \{y : y(s) \text{ increasing in } s, y(s) \leq s + \phi_t, \forall s \in [s_L, s_H]\}$, is the set of all monotone securities backed by the collateral asset. The monotonicity restriction is motivated by realism since the payoff from any loan collateralized directly by the asset or any other collateralized loan is increasing in s.

A security design is a finite selection of securities that are backed by the asset.

Definition 1. Given the asset price ϕ_t, a *security design* consists of a finite set of securities $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$.

When $j \in \mathcal{J}_t(\phi_t)$, we say that security j is available.

Trading environment. There are two types of markets in this economy. After state is realized at the end of each period t (ie., at date 3), a *centralized* market for the collateral asset opens for trading. The asset price, denoted by ϕ_t, is determined in this centralized market.

In addition, at the beginning of each period t (ie., at date 1), there are *decentralized* markets for intermediate goods. We may interpret this market as an over-the-counter funding market. Specifically, for each available security, there is a decentralized sub-market where agent O meets at least two randomly chosen I agents to trade asset-based securities in exchange for intermediate goods. We assume that agent Is simultaneously make price offers per unit of the security. Agent O then observes the price offers and decides the quantity of the security to allocate to each agent I. Since each unit of the security must be backed by one unit of the asset, the total quantity of the security sold by agent O must be less than or equal to the amount of asset owned by agent O in that period. If agent O decides to sell a positive amount of the security, she allocates the amount of the security that she would like to sell to the agent I who offers the higher price. If several agent Is are tied for the highest offer, agent O equally splits the amount that she would like to sell between them.

The following figure summarizes the time and events in this setup.

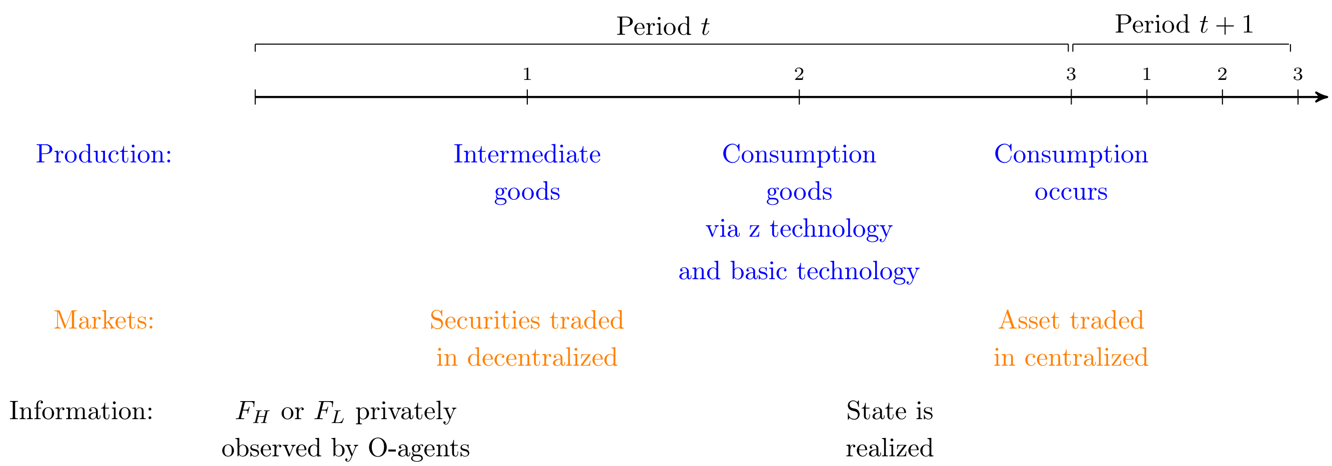

Figure 1: Timeline

4 Security Design Problem

4.1 Defining the security design problem

A few notations are in order before the definition. We denote the value function of agent O at date 1 of period t by $V_{o,t}(a, \mu_{o,t})$ and her value function at date 3 of period t by $W^s_{o,t}(c, a)$ at state s, where c is the amount of consumption goods, a is the amount of the asset that she brings into period t, and $\mu_{o,t}$ indicates the set of available securities. Similarly, we denote the value function of agent I in submarket for asset j by $V^j_{I,t}(a)$ and his value function in the last sub-period of period t to be $W^s_{I,t}(c, a)$. The security design problem is defined based on the characterization of these value functions.

Agent O's value function at date 3 of period t is solved by

$$W^s_{o,t}(c, a) = \max_{x,l,\tilde{a}\geq 0} x - l + \beta V_{o,t+1}(\tilde{a}), \quad (1)$$

$$s.t. x + \phi_t \tilde{a} = c + (s + \phi_t)\, a + l.$$

Likewise, Agent I's value function at date 3 of period t is solved by

$$W^s_{I,t}(c, a) = \max_{x,l,\tilde{a}\geq 0} x - l + \beta V_{I,t+1}(\tilde{a}), \quad (2)$$

$$s.t. x + \phi_t \tilde{a} = c + (s + \phi_t) a + l.$$

From (1) and (2), it is easy to see that $W^s_{o,t}$ and $W^s_{i,t}$ are linear in c and a since

$$W^s_{o,t}(c,a) = c + (s + \phi_t)\, a + W^s_{o,t}(0,0), \tag{3}$$

$$W^s_{I,t}(c,a) = c + (s + \phi_t)\, a + W^s_{i,t}(0,0). \tag{4}$$

Next, we characterize $V^j_{I,t}(a)$. We denote by $y^j_t(s)$ the payoff of asset j in state s and assume that that the high type values the security weakly more than the low type, i.e., $E_L y^j_t(s) \leq E_H y^j_t(s)$.[9] We denote by R^j_t the ratio of the expected value of the security under the low versus the high distribution, i.e., $R^j_t \equiv E_L y^j_t / E_H y^j_t$ As this ratio increases, the expected values of the asset under the low versus the high distribution become closer, and hence the adverse selection problem becomes less severe.

Recall that agent Is simultaneously make price offers per unit of the security, agent O observes the price offers and decides how much of the security to allocate to each agent I.[10] Hence, in principle, the value function depends on the offer that agent I makes in sub-market j.

Consider an arbitrary agent I who participates in submarket j and bids for security y^j_t at per-unit price q^j_t. If this is the highest bid, he receives $a^Q_t(q^j_t) \in [0,a]$ units of security j and pays $q^j_t a^Q_t(q^j_t)$ units of intermediate goods in return. Agent I's expected payoff from this bid is given by:

$$\begin{aligned} V^j_{I,t}(a) &= \int \lambda \left[-q^j_t a^L_t(q^j_t) + W^s_{I,t}(a^L_t(q^j_t) y^j_t(s), a)\right] dF_L(s) \\ &\quad + \int (1-\lambda) \left[-q^j_t a^H_t(q^j_t) + W^s_{I,t}(a^H_t(q^j_t) y^j_t(s), a)\right] dF_H(s) \\ &= \int \lambda \left[-q^j_t a^L_t(q^j_t) + a^L_t(q^j_t) y^j_t(s)\right] dF_L(s) + \int (1-\lambda) \left[-q^j_t a^H_t(q^j_t) + a^H_t(q^j_t) y^j_t(s)\right] dF_H(s) \\ &\quad + \int W^s_{i,t}(0,a)\, d\left[\lambda F_L(s) + (1-\lambda) F_H(s)\right] \end{aligned} \tag{5}$$

where the second equality is obtained by substituting (4) using the fact that $W^s_{i,t}$ are linear in c and a.[11] If q^j_t is not the highest bid, this agent's expected payoff is given only by the third term of the second equality since he is not allocated any of the security in sub-market j.

The winning bid q^j_t must satisfy two conditions. First, due to Bertrand competition I agents make zero surplus in expectation. This means that q^j_t must equal the expected value of a unit of the security given the expectation of I agents about the quantities that will be sold by the two types. Second, these expectations must be incentive compatible in the sense that if I agents anticipate that a given type of the O agent will sell a positive amount of the security at per-unit price q^j_t, that type must find it profitable

[9] This assumption is automatically satisfied for monotone securities.

[10] In this formulation agent O has all the bargaining power, but this is not crucial for any of our results.

[11] If q^j_t ties with $k-1$ other highest bids, his expected payoff is as above except that $a^Q_t(q^j_t)$ terms are now divided by k.

to sell the security given the price. The next proposition shows that among the prices that satisfy the zero surplus and incentive compatibility conditions, Bertrand competition selects the highest one.

Proposition 1. *If $R_t^j > \zeta \equiv 1-(z-1)/\lambda z$, in submarket j the price of the security is $q_t^j = \lambda E_L y_t^j + (1-\lambda) E_H y_t^j$ and $a_t^L(q_t^j) = a_t^H(q_t^j) = a$. If $R_t^j < \zeta$ then the price of the security is $q_t^j = E_L y_t^j$ and $a_t^L(q_t^j) = a$ and $a_t^H(q_t^j) = 0$.*[12]

Proof. Let $\overline{q}^j = \lambda E_L y_t^j + (1-\lambda) E_H y_t^j$. Note that $z\overline{q}^j - E_H y_t^j \gtreqless 0$ iff $R_t^j \gtreqless \zeta$.

Consider the case $R_t^j > \zeta$. Suppose that the equilibrium price q_t^j is strictly less than $\overline{q}$. In this case an I agent can deviate and bid $\overline{q} - \epsilon$ where $\epsilon > 0$. For ϵ small enough, $z(\overline{q} - \epsilon) - E_H y_t^j > 0$. Hence at this price both types sell a units of the security and the deviation generates strictly positive surplus. This means that the equilibrium price must be at least $\overline{q}$. At price $\overline{q}$ or above both types will sell a units of the security, hence the only price that is consistent with zero profit condition is $q_t^j = \overline{q}$.

Now consider the case $R_t^j < \zeta$. In this case high type will sell the security only if q_t^j is sufficiently larger than $\overline{q}$. However, at prices above $\overline{q}$, I agents make negative profit. Hence equilibrium price must be below $\overline{q}$. If q_t^j is below $\left(E_L y_t^j\right)/z$ then neither type sells the security. In this case, one of the I agents can deviate and bid $E_L y_t^j - \epsilon$ where $\epsilon > 0$. For ϵ small enough, $z\left(E_L y_t^j - \epsilon\right) - E_L y_t^j > 0$ so the low type sells the security and the deviating agent makes strictly positive surplus. If q_t^j is at least $\left(E_L y_t^j\right)/z$ but less than $E_L y_t^j$ then the low type sells the security to the I agents who bid that price. In this case, one of the I agents who bids $E_L y_t^j$ or less can deviate and bid slightly above q_t^j. This agent then buys the security alone and increases her surplus. At prices greater than equal to $E_L y_t^j$ (and below $\overline{q}$), the low type alone sells a units of the security. Hence the only price that is consistent with zero profit condition is $q_t^j = E_L y_t^j$. □

This proposition shows that when R_t^j is above the threshold ζ, the adverse selection problem is not too severe and both types sell a units of the security. In this case the security price is the pooling price $q_t^j = \lambda E_L y_t^j + (1-\lambda) E_H y_t^j$. When R_t^j is below the threshold, the adverse selection problem is severe and only the low type sells a units of the security. In this case the security price is the separating price $q_t^j = E_L y_t^j$.

[12] When $R_t^j = \zeta$ there are multiple equilibria. In particular both pooling and separating (and even semi-separating) equilibria are possible. To simplify exposition in this knife edge case we will select the pooling equilibrium. To see why there are multiple equilibria, suppose I agents bid $E_L y_t^j$, the low type sells a units and I agents make zero profit. Since $R_t^j = \zeta$, to attract the high type, an I agent must deviate to bidding at least $\lambda E_L y_t^j + (1-\lambda) E_H y_t^j$. But this deviation is not profitable since by deviating an I agent can not make positive surplus. Hence both $E_L y_t^j$ and $\lambda E_L y_t^j + (1-\lambda) E_H y_t^j$ can be sustained as equilibrium bids.

Now we are ready to state the optimal security design problem. Agent O chooses security design $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$ to maximize

$$\begin{aligned} V_{o,t}(a) &= \lambda \int W_{o,t}^{s}\left(\sum_{j\in\mathcal{J}_t(\phi_t)} a_t^L(j)\left[zq_t^j - y_t^j(s)\right], a\right) dF_L(s) \\ &+ (1-\lambda)\int W_{o,t}^{s}\left(\sum_{j\in\mathcal{J}_t(\phi_t)} a_t^H(j)\left[zq_t^j - y_t^j(s)\right], a\right) dF_H(s) \\ &= \lambda \int \left\{\sum_{j\in\mathcal{J}_t(\phi_t)} a_t^L(j)\left[zq_t^j - y_t^j(s)\right] + a\left(s+\phi_t\right)\right\} dF_L(s) \\ &+ (1-\lambda)\int \left\{\sum_{j\in\mathcal{J}_t(\phi_t)} a_t^H(j)\left[zq_t^j - y_t^j(s)\right] + a\left(s+\phi_t\right)\right\} dF_H(s) + W_{o,t}^{s}(0,0), \end{aligned} \tag{6}$$

subject to

$$\sum_{j\in\mathcal{J}_t(\phi_t)} y_t^j(s) \leq s+\phi_t, \forall s, \tag{7}$$

$$q_t^j = \begin{cases} \lambda E_L y_t^j + (1-\lambda)E_H y_t^j, & \text{if } R_t^j \geq \zeta, \\ E_L y_t^j, & \text{if } R_t^j < \zeta, \end{cases} \tag{8}$$

$$a_t^L(j) = a \text{ and } a_t^H(j) = \begin{cases} a, & \text{if } R_t^j \geq \zeta, \\ 0, & \text{if } R_t^j < \zeta. \end{cases} \tag{9}$$

The security design is done ex-ante, before Agent O learns the asset quality. At the security design stage, Agent O simply decides which sub-markets are open for trading but she cannot commit to trading in a given sub-market. The first constraint ensures that the security design is feasible in the sense that Agent O should be able to fulfill her promises in every sub-market and in all states. The second and third constraints say that prices and quantities in the decentralized security markets must be the equilibrium outcomes characterized in Proposition 1.

We now state the equilibrium definition for the dynamic security design problem.

Definition 2. A dynamic equilibrium consists of asset prices ϕ_t, security design $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$ and security prices q_t^j for each $j \in \mathcal{J}_t(\phi_t)$ such that (i) $\mathcal{J}_t(\phi_t)$ solves the security design problem (6), (ii) security price q_t^j satisfies equation (8) and (iii) ϕ_{t-1} solves the Euler equation given by:

$$\phi_{t-1} = \beta\left[z\left(\sum_{j\in P_t} q_t^j + \lambda \sum_{j\in\mathcal{J}_t(\phi_t)\backslash P_t} q_t^j\right) + (1-\lambda)\sum_{j\in\mathcal{J}_t(\phi_t)\backslash P_t} E_H y_t^j\right], \tag{10}$$

where $j \in P_t \subseteq \mathcal{J}_t(\phi_t)$ iff $R_t^j \geq \zeta$.

5 The Baseline: Fragility of the Dynamic Lemons Market

In this section, we consider the benchmark case where Agent O secures liquidity only by selling Agent I the collateral asset at the beginning of each period in exchange for intermediate goods as inputs for the z-technology. We demonstrate that this economy is fragile and exhibits dynamic multiplicity in prices. That is, we show that for a given price path there might be multiple equilibria in the decentralized markets.

For this benchmark case we use the notion of equilibrium in Definition 2 except that we take the collateral asset as the only available security. That is, we set $\mathcal{I}_t(\phi_t) = \{y : y(s) = s + \phi_t, \forall s \in [s_L, s_H]\}$. The optimization problem in (6) becomes trivial since there is only a single feasible security which is the asset itself, but in equilibrium (8) and (10) must still be satisfied. The payoff of the collateral asset in state s is $s + \phi_t$. Hence, by (8) the price of the collateral asset in the decentralized market is given by $q_t^P = \phi_t + \lambda E_L s + (1-\lambda) E_H s$ if $(E_L s + \phi_t)/(E_H s + \phi_t) \geq \zeta$ and $q_t^S = \phi_t + E_L s$ otherwise. Using (10) we obtain the price of the collateral asset in the centralized market as

$$\phi_t = \begin{cases} \beta z q_{t+1}^P, & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} \geq \zeta, \\ \beta \left[z \lambda q_{t+1}^S + (1-\lambda)(\phi_{t+1} + E_H s) \right], & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} < \zeta. \end{cases} \tag{11}$$

We focus on the stationary equilibrium from this point on and drop the time index. Plugging q^P and q^S into (11) we observe that a pooling equilibrium, in which both types of agent O sell the asset in the decentralized market for the intermediate goods, exists if and only if

$$\frac{E_L s + \phi^P}{E_H s + \phi^P} \geq \zeta, \tag{12}$$

where asset price in the pooling equilibrium is given by,

$$\begin{aligned} \phi^P &= \beta z (\phi^P + \lambda E_L s + (1-\lambda) E_H s), \\ \phi^P &= \frac{\beta z \left(\lambda E_L s + (1-\lambda) E_H s \right)}{1 - \beta z}. \end{aligned} \tag{13}$$

Similarly, a separating equilibrium in which only the low type of agent O sells the asset in the decentralized market for the intermediate goods, exists if and only if

$$\frac{E_L s + \phi^S}{E_H s + \phi^S} < \zeta, \tag{14}$$

where asset price in the separating equilibrium is given by,

$$\begin{aligned} \phi^S &= \beta \left[\lambda z \left(\phi^S + E_L s \right) + (1-\lambda) \left(\phi^S + E_H s \right) \right], \\ \phi^S &= \frac{\beta \left[\lambda z E_L s + (1-\lambda) E_H s \right]}{1 - \beta(\lambda z + 1 - \lambda)}. \end{aligned} \tag{15}$$

Note that the pooling price is always higher than the separating price:

$$\phi^P = \frac{\beta z \left[\lambda E_L s + (1-\lambda) E_H s\right]}{1-\beta z} > \phi^S = \frac{\beta \left[\lambda z E_L s + (1-\lambda) E_H s\right]}{1-\beta(\lambda z + 1 - \lambda)}.$$

Furthermore, the discounted value of future dividends is $\beta \left(\lambda E_L s + (1-\lambda) E_H s\right) / \left(1-\beta\right)$. It is easy to see that, since $z > 1$, the price for the asset is strictly higher than the discounted value of future dividends in both scenarios. The difference is justified by the collateral service provided by the asset. The pooling price is higher because the collateral service is more valuable in the pooling equilibrium of the decentralized market as both types use the collateral to purchase the intermediate goods. Moreover, when z is higher, there is more demand for collateral which justifies a higher asset price.

Since the pooling price is higher than the separating price, for the same underlying parameters, there may be multiple price equilibria. That is, the separating price ϕ^S may be consistent with a separating equilibrium and the pooling price ϕ^P may be consistent with a pooling equilibrium in the decentralized market.

Corollary 1. *The condition for price multiplicity when agent O uses the productive assets as collateral for intermediate goods is*

$$1 - \frac{z-1}{z} \frac{1}{\lambda\left(1-\beta\right)} < \frac{E_L s}{E_H s} < 1 - \frac{z-1}{z} \frac{1}{\lambda\left(1-\beta+\beta\left(1-\lambda\right)\left(z-1\right)\right)}. \tag{16}$$

Proof. By Proposition 1 the condition for price multiple equilibria is,

$$\frac{E_L s + \phi^S}{E_H s + \phi^S} < \zeta \leq \frac{E_L s + \phi^P}{E_H s + \phi^P}.$$

Plugging for ϕ^S and ϕ^P we obtain the condition for multiplicity in 16. Hence for intermediate values of $E_L s / E_H s$ both price equilibria exist. □

The existence of multiple price equilibria is due to a dynamic price feedback effect. If agents anticipate the asset to be traded in a pooling equilibrium in the decentralized market, the asset price is high. In turn, when the price is high, the high type O agent is willing to pool. Conversely, if agents anticipate the asset to be traded in a separating equilibrium in the decentralized market, the asset price is low. In turn, when the price is low, the high type O agent keeps the asset. The beliefs are self-fulfilling. Next we show security design helps to eliminate this type of fragility in the economy.

6 Dynamic Security Design

In this section we first restriction attention to monotone securities and solve the security security design problem given in ((6)). We show that agent O can use security design to overcome the fragility of the price equilibrium that arises when agents can only trade the underlying asset. At the end of this section, we show the uniqueness of equilibrium does not depend on the restriction of issuing monotone securities. It also obtains when borrowers issue Arrow securities against the dividend payment and the resale value of the asset.

6.1 Solving for Optimal Security Design

As a preliminary step, we first show that optimal security design involves at most two securities. One security is always liquid and the other one is illiquid.

Lemma 1. *If two securities y^j and y^k are both liquid (illiquid) then $y^j + y^k$ is also liquid (illiquid). Moreover, if a security design involves y^j and y^k, replacing the two securities by their combination $y^j + y^k$ is also a feasible security design and provides the same payoff to agent O. Hence, w.l.o.g. we can restrict attention to security design that involves at most two securities, a liquid and an illiquid one.*

According to 1, we focus on security design with at most one liquid and one illiquid tranches. Note that when a liquid and an illiquid security are combined the resulting security might be liquid (illiquid) and strictly improve (lower) agent O's payoff. In fact, whenever there are two tranches and the liquidity constraint is not binding for the liquid tranche, that is if $E_L y^j > \zeta E_H y^j$, it is always possible to combine part of the illiquid tranche with the liquid one and strictly improve agent O's payoff. We state this observation in the next lemma.

Lemma 2. *If a security design is optimal and involves a liquid and an illiquid tranche, the liquid tranche must satisfy the liquidity constraint with equality.*

Proof. From Lemma 1 we can restrict attention to security design with two tranches. Suppose a given security design involves two tranches where y^j is liquid which satisfies liquidity constraint strictly and y^k is illiquid. Now take $y^j + \epsilon y^k$ and $(1 - \epsilon)\, y^k$. This is a feasible design. If ϵ is small enough $y^j + \epsilon y^k$ is still liquid, and the new design improves agent O's payoff. □

Additionally, the feasibility constraint must always hold with equality. To see this suppose there are two securities y_t^j and y_t^k where the former is liquid and the latter is illiquid and some of the dividend is not incorporated into either security, that is, $y_t^j(s) + y_t^k(s) < s + \phi$ for a positive measure of states. If the unused portion of the dividends is incorporated into the illiquid tranche then there are two possibilities. If the illiquid tranche becomes liquid, agent O's payoff increases since both types can now use the new security to borrow. If the illiquid tranche remains illiquid, it still increases agent O's payoff since the low type can borrow more and the high type is still not trading the illiquid portion. Given Lemma 1 and the fact that the feasibility constraint must be binding, the designer's problem can be simplified into choosing a liquid tranche $y(s)$ and an illiquid tranche $s + \phi - y(s)$. Computing the prices of the two tranches from (8) and plugging into (6) the optimal security design simplifies to:

$$\max_{y(s)} (z-1)\left[\lambda(E_L s + \phi) + (1-\lambda)E_H y(s)\right] \tag{17}$$

$$s.t. s + \phi - y(s) \geq 0, \forall s,$$

$$E_L y(s) - \zeta E_H y(s) \geq 0,$$

$$y(s) \text{ is weakly increasing on } [s_L, s_H]. \tag{18}$$

The first constraint above is the feasibility constraint and requires $y(s)$ to be backed by the underlying asset in every state. The second is the pooling constraint and guarantees that the high O type agent sells a units of security y.

Clearly the liquid tranche in an optimal security design must satisfy $y(s) \geq \phi$ for all $s \in [s_L, s_H]$. Following Ellis, Piccione, and Zhang (2017), we write the monotone security $y(s)$ as:

$$y(s) = \phi + s_L + \int_{s_L}^{s} x(j)dj,$$

where $x(j) \geq 0$ for all $j \in [s_L, s_H]$.[13] Let $\widetilde{F}_Q(s) = 1 - F_Q(s)$ for $Q \in \{L, H\}$ and $s \in [s_L, s_H]$. Then,

$$E_Q y(s) = \phi + s_L + \int_{s_L}^{s_H} \widetilde{F}_Q(j)x(j)dj.$$

Hence, the optimal security design problem (17) is equivalent to the following:

$$\arg\max_x \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds, \tag{19}$$

$$s.t. \int_{s_L}^{s} x(j)dj \leq s - s_L, \forall s \in [s_L, s_H], \tag{20}$$

$$\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi \geq 0, \tag{21}$$

$$x(s) \geq 0, \forall s \in [s_L, s_H] \tag{22}$$

[13] In our analysis we restrict attention to securities that can be written as the sum of an absolutely continuous increasing function and countably many jump points.

In the above problem, (20) corresponds to the feasibility constraint, (21) corresponds to the pooling constraint and (22) guarantees that the security is monotone.

The next proposition shows that, as long as $f_L(s)/f_H(s)$ is decreasing, the optimal liquid tranche is a debt contract with face value $D = \phi + \delta$.

Proposition 2. *Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. The optimal security is a unique standard debt contract y_D such that*

$$y_D(s) = \phi + \min(s, \delta),$$

for some $\delta \in (s_L, s_H]$.

To prove this proposition we use the Lagrangian for the optimization problem (19) and proceed in three steps. First, we show that when the dividend is above a cutoff $x(s) = 0$ or equivalently y must be flat. Second, we show that feasibility constraint must be binding at s whenever $x(s) > 0$. In other words, $y(s) = \phi + s$ whenever the liquid security is increasing, thus it promises the resale price and all of the dividend in such states. Finally, we show that there is a unique cutoff below which $x(s) > 0$ and above which $x(s) = 0$. The proof also shows that the optimal security cannot have jump points. Together, these steps imply that the optimal liquid security must be a debt contract.

6.2 Characterizing the Optimal Liquid Security

Suppose the designer anticipates that the liquid security will be liquid as long as $E_L y / E_H y \geq \zeta$. In this case, the prices of equity and liquid debt are

$$q_E = \int_{\delta}^{s_H} \widetilde{F}_L(s)ds,$$
$$q_D = \phi + s_L + \lambda \int_{s_L}^{\delta} \widetilde{F}_L(s)ds + (1-\lambda)\int_{s_L}^{\delta} \widetilde{F}_H(s)ds$$

Using Proposition 2 and plugging for ζ, the optimization problem in (19) and the associated constraints (20)-(22) can be simplified as

$$\max_{\delta \in [s_L, s_H]} \int_{s_L}^{\delta} \widetilde{F}_H(s)ds \tag{23}$$

subject to

$$(z-1)\left[\phi + s_L + \lambda \int_{s_L}^{\delta} \widetilde{F}_L(s)ds + (1-\lambda)\int_{s_L}^{\delta} \widetilde{F}_H(s)ds\right] \geq \lambda \int_{s_L}^{\delta} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds, \tag{24}$$

where the constraint is the condition for the both O types to pool and issue the liquid debt.

To complete the characterization of optimal liquid security we solve for the equilibrium given in Definition 2. Note that the equilibrium boils down to solving (23) to find the optimal debt threshold level $\delta \in (s_L, s_H]$ given the asset price ϕ and determining the asset price ϕ in the centralized market through the corresponding Euler equation:

$$\phi = \beta \left\{ z\left(q_D + \lambda q_E\right) + (1-\lambda) \int_{\delta}^{s_H} \widetilde{F}_H(s) ds \right\}. \tag{25}$$

Proposition 3. *Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. If $E_L s / E_H s < 1 - (z-1)/(z\lambda(1-\beta))$, there is a unique equilibrium where the debt threshold $\delta \in (s_L, s_H)$ and the asset price ϕ are solutions to the following two equations:*

$$\phi = \frac{z}{z-1} \lambda \int_{s_L}^{\delta} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s) \right] ds - \int_{s_L}^{\delta} \widetilde{F}_H(s) ds - s_L \tag{26}$$

$$\phi = \frac{\beta}{1-\beta z} \left\{ z\left[\lambda E_L s + (1-\lambda) E_H s\right] - (1-\lambda)(z-1) \int_{\delta}^{s_H} \widetilde{F}_H(s) ds \right\} \tag{27}$$

Otherwise, there is a unique equilibrium where $\delta = s_H$ and $\phi = \frac{\beta}{1-\beta z} z\left[\lambda E_L s + (1-\lambda) E_H s\right]$. Moreover, in the former case the equilibrium of the security design problem strictly Pareto dominates all equilibria of the case where only the asset can be used as collateral. In the latter case, the equilibrium of the security design problem strictly Pareto dominates the separating equilibrium of the case where only the asset can be used as collateral and replicates the pooling equilibrium.

The formal proof of the proposition is in the Appendix. We provide an intuitive discussion of this result and the economics mechanism behind it in the next subsection. The following corollary follows immediately from Proposition 3.

Corollary 2. *Under the welfare improving security design equilibrium, there is non-trivial tranching when $E_L s / E_H s < 1 - (1-\zeta)/(1-\beta)$.*

Note that this condition is the same condition for the left boundary of multiple equilibria region in (16) indicating that security design improves the liquidity of the unique separating regime when only equity is allowed to be traded.

6.3 Discussion of the Unique Equilibrium under Optimal Security Design

In this section we compare the results from Section 5 and the optimal security design problem of Section 6.2 and discuss the underlying economic mechanism. There are two important differences with the dynamic lemons market when the optimal security design is introduced: 1) there is non-trivial welfare

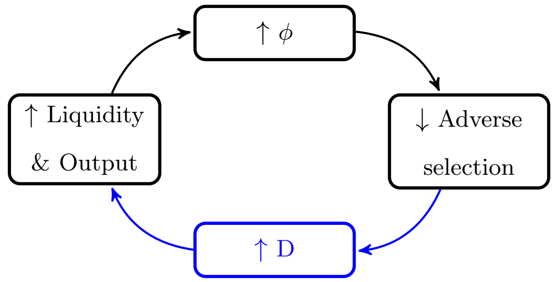

Figure 2: Asset Price ϕ and Liquid Debt Face Value $\phi + D$

improving tranching in the separating equilibrium region; and 2) the pooling equilibrium is selected as the unique equilibrium in the multiple equilibria region.

Figure 2 illustrates the feedback loop between the asset price and the face value of debt that leads to these differences. The face value of the liquid debt always incorporate the resale price, ϕ, in addition to the debt threshold backed by dividend, δ, that is, $D = \phi + \delta$. As the face value increases, more of the dividend states are pledged as collateral, more funds are raised for the productive sector and the real output goes up. This in turn leads to an increase in the collateral asset price ϕ. A higher asset price is incorporated into the face value of debt alleviating the adverse selection problem (ie., the adverse selection ratio for the debt, R, is now lower) and allowing even more dividend to be pledged as collateral.

To understand this mechanism, we revisit the equilibrium construction in the optimal security design equilibrium. Suppose that the security designer sells a liquid debt tranche with a face value $D = \phi + \delta$ and an illiquid equity tranche. Note that the liquid debt tranche incorporates the resale price of the asset in the face value since both types of debt issuers' promise to return ϕ to the creditors is credible. Recall that the asset price ϕ in the centralized market, after substituting for the prices of the debt and equity tranches q_D and q_E, is given by the Euler equation:

$$\phi = \frac{\beta}{1 - \beta z} \left\{ z \left[\lambda E_L s + (1 - \lambda) E_H s \right] - (1 - \lambda)(z - 1) \int_{\delta}^{s_H} \widetilde{F}_H(s) ds \right\}, \tag{28}$$

where it is immediate that ϕ is increasing in δ.

For any δ let $\phi(\delta)$ be the asset price in the centralized market satisfying (28). Let $\underline{\phi} = \phi(s_L)$ and $\phi^P = \phi(s_H)$. Recall from Section 5 that ϕ^S is the asset price when only the low type sells the asset and high type retains both the resale price and the dividend. In contrast, the asset price calculation in (28) assumes that both types of borrowers sell (liquid) debt claims backed by the future resale price at the minimal as collateral. As a result, $\underline{\phi} > \phi^S$. On the other hand, $\phi(s_H)$ is the same as the pooling price ϕ^P. To see this, note that ϕ^P is calculated assuming that both types use the resale price and the entire dividend of the asset as collateral which is equivalent to setting the face value of the liquid debt contract to $\phi^P + s_H$. The solid line in Figure 3 depicts the function $\phi(\delta)$.[14]

Next consider the designer's choice of debt threshold, δ, as a function of the asset price ϕ. Optimal security design chooses δ as large as possible making sure that the debt tranche is liquid. As δ increases, the debt tranche incorporates more of the high dividend states. If δ is too high, the high type, who knows that those states are likely, might prefer to retain the debt tranche rather than pooling with the low type. Hence, the security design can push up δ to the point where the high type is indifferent between selling or retaining the debt. As the asset price increases, selling the debt tranche becomes more attractive to the high type, allowing the security designer to increase δ. Recall $\delta(\phi)+\phi$ is the optimal face value of debt given the asset price ϕ.[15] The dash dotted line in Figure 3 depicts the function $\delta(\phi)$ for the case $E_L s/E_H s < 1 - (1-\zeta)/(1-\beta)$.[16] The figure illustrates that no matter how low the asset price is, as long as tranching is feasible, optimal security design involves a debt tranche that incorporates some dividend. That is, $\delta\left(\underline{\phi}\right) > s_L$. This is a robust feature of security design that holds regardless of underlying parameters. Also note that in the region depicted, adverse selection is severe, and even when the asset price is as high as possible, high type prefers to retain the equity tranche. That is, $\delta\left(\phi^P\right) < s_H$.

Using these two curves, $\phi(\delta)$ and $\delta(\phi)$, we can find the equilibrium values (δ^*, ϕ^*). The equilibrium is where the two curves intersect, ie., when $\phi^* = \phi(\delta^*)$ and $\delta^* = \delta(\phi^*)$. As Figure 3 shows, when $E_L s/E_H s < 1 - (1-\zeta)/(1-\beta)$, the equilibrium debt threshold $\delta^* \in (s_L, s_H)$. This explains the first difference in the results of the two sections.

Perhaps more interesting is the case when $E_L s/E_H s > 1 - (1-\zeta)/(1-\beta)$ given in Figure 4 where the second difference arises. In this case, adverse selection is less severe and $\delta(\phi)$ function is shifted to the right as the same asset price can sustain a higher face value of the liquid debt. When the asset price is above a threshold denoted by $\widehat{\phi}$, optimal security design incorporates all dividend states s_H to the face value of debt, which is captured by the vertical part of $\delta(\phi)$ function. The two curves intersect only at the upper right corner, $\left(s_H, \overline{\phi}\right)$. As a result, there is a unique equilibrium for the security design problem and it involves setting the debt threshold $\delta^* = s_H$.

The scenario depicted in Figure 4 may seem surprising since, as we illustrated in Section 5, without

[14] Note that ϕ is strictly increasing for $D \in [s_L, s_H)$, $\partial\phi/\partial D$ is decreasing and is zero at $D = s_H$.

[15] $D(\phi)$ is constructed as the unique solution to the following equation for a given ϕ

$$\mathcal{T}_\phi(D) = (z-1)\left[\phi + s_L + \int_{s_L}^{D} \widetilde{F}_H(s)ds\right] - z\lambda \int_{s_L}^{D} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds = 0$$

whenever there is a solution in $[s_L, s_H]$. If there is no solution, ie, if $\mathcal{T}(s_H) > 0$, then $D(\phi) = s_H$.

[16] Recall that this is the left boundary of multiple equilibria region in 16. In this region adverse selection leads to a unique separating equilibrium without security design.

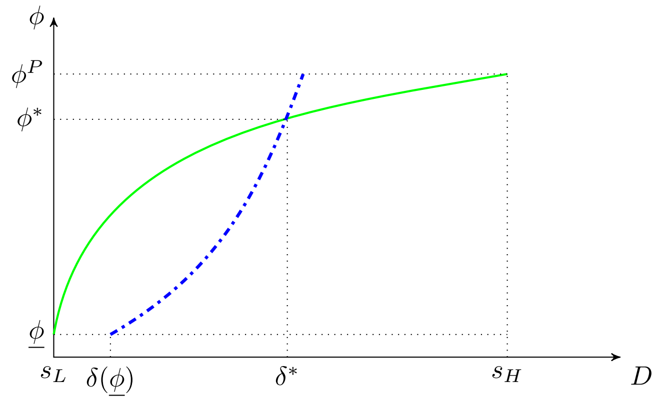

Figure 3: $\phi(\delta)$ and $\delta(\phi)$ when $E_L s/E_H s < 1 - (z-1)/(s\lambda(1-\beta))$.

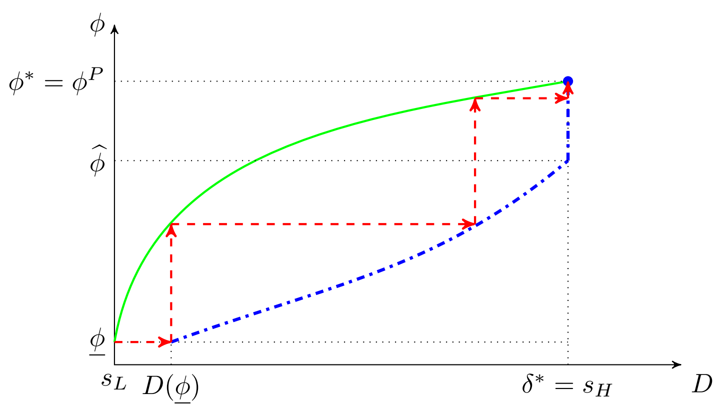

Figure 4: $\phi(\delta)$ and $\delta(\phi)$ when $E_L s/E_H s > 1 - (z-1)/(s\lambda(1-\beta))$.

the possibility of security design there are multiple equilibria in part of this region. With security design, we obtain a unique equilibrium in which Agent O sells the entire "pass-through" debt tranche in a pooling equilibrium. To understand this, note that without security design the high type decides among only two options: whether to use the resale price and dividend of the asset as collateral versus to retain both parts. The outcome depends on the asset price. In the good equilibrium $\phi = \phi^P$ and the high type sells the asset. In the bad equilibrium, $\phi = \phi^S$ and the high type retains the asset. The bad equilibrium cannot survive with security design because even if the asset price were ϕ^S, the optimal security design would be able to improve this separating equilibrium by creating a liquid debt tranche with the face value ϕ^S, which in turns would increase the asset price above ϕ^S. Both graphs in Figures 3 and 4 in fact show that the equilibrium asset price in the optimal security design equilibrium is not less than $\underline{\phi} = \phi(s_L) > \phi^S$ (since the face value of the liquid debt is never below $\underline{\phi} + s_L$). Given the increase in the asset price to $\underline{\phi}$ from ϕ^S, the high type's participation constraint is relaxed, this leads to the optimal security design to incorporate more of the dividend into the debt tranche (that is $\delta > s_L$). A higher δ will increase the asset price ϕ and so on, triggering the dynamic price feedback. This unravelling process is illustrated in Figure 4 with the dashed arrows. As the figure shows when the asset price is $\underline{\phi}$, face value of the debt rises to $\underline{\phi} + \delta\left(\underline{\phi}\right)$. When the face value of the debt increases to $\underline{\phi} + \delta\left(\underline{\phi}\right)$, the asset price further increases, and so on. The process ends when price rises to ϕ^P.

The uniqueness of equilibrium does not depend on the restriction to issuing monotone securities. The following proposition shows that if borrowers can issue Arrow securities against the dividend payment and back liquid securities by pledging the resale value of the asset, there always exists a unique equilibrium.

Proposition 4. *Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. The optimal security design under Arrow securities has two tranches, the liquid tranche $y_{1t}(s)$ and illiquid tranche $y_{2t}(s)$.*

$$y_1(s) = \phi + s_L + (s - s_L)\mathbb{I}(s \leq \delta),$$
$$y_2(s) = (s - s_L)\mathbb{I}(s > \delta).$$

If $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$, there is a unique equilibrium for the optimal design, where $\delta \in (s_L, s_H)$ and the asset price ϕ are solutions to the following two equations:

$$\phi = \frac{\int_{s_L}^{\delta} sdF_H(s) - z\int_{s_L}^{\delta} sdF_\lambda(s)}{z-1} \tag{29}$$

$$\phi = \frac{\beta}{1-\beta z}\left[z\int sdF_\lambda(s) - (z-1)(1-\lambda)\int_{\delta}^{s_H} sdF_H(s)\right] \tag{30}$$

where $F_\lambda(s) = \lambda F_L(s) + (1-\lambda)F_H(s)$. Otherwise, there is a unique equilibrium where $\delta = s_H$ and $\phi = \frac{\beta}{1-\beta z}z\left[\lambda E_L s + (1-\lambda)E_H s\right]$.

As in Proposition 3, in the former case, the equilibrium of the security design problem strictly Pareto dominates all equilibria of the case where only the asset can be used as collateral. In the latter case, the equilibrium of the security design problem strictly Pareto dominates the separating equilibrium of the case where only the asset can be used as collateral and replicates the pooling equilibrium.

6.4 Contract Rigidity and Sunspot Runs

Our main model shows that optimal security design eliminates fragility and improves welfare. In this section, we show that this strong result relies on the assumption that borrowers have the flexibility to design securities and change the terms of the contracts at the beginning of each period. In practice, contract terms may not be updated daily because of associated administrative costs or simply inattention. Next, we show that the induced rigidity may be a crucial source of fragility.

To capture rigidity, in any period, we allow the economy to be in one of two regimes, 0 or 1. The transition between these two regimes depends on whether security design is flexible or rigid as well as the arrival of a sunspot which indicates negative sentiment. Specifically, if in the beginning of period t contract design is rigid and a sunspot occurs then the economy moves from regime 0 to regime 1. This event happens with probability $\chi \geq 0$. Once the economy enters regime 1, it returns to regime 0 only if the design is updated. That is, if the economy is in regime 1 at time $t-1$, then it returns to regime 0 if the design becomes flexible in the beginning of period t. This event happens with probability $1-\gamma$ where $\gamma > \chi$. Note that our main model is a special case where $\gamma = 0$.

In this more general model, as will become clear later, the value of the asset depends on the regime $i \in \{0,1\}$ which we denote by v_i. The asset price, which is determined in the centralized market at the end of each period, takes into account the possible values of the asset in the following period. Hence, it is distinct from the asset value and depends on the regime. We denote the asset price in regime i by ϕ_i. Similarly, security prices depend on the asset price and hence are also regime dependent. We denote the price of security j in regime i by q_i^j. In addition, even though the security design remains the same in both regimes, because the asset price changes, the adverse selection ratios of securities will also change. We denote the adverse selection ratio of security j in regime i by R_i^j. The formal definition of a sunspot equilibrium which modifies Definition 2 to allow for these regime dependencies is given next.

Definition 3. Let $i \in \{0,1\}$ denote the regime. A dynamic stationary equilibrium with rigidity consists of asset values and prices v_i and ϕ_i, security design (with monotone securities) $\mathcal{J}(\phi_0) \subseteq \mathcal{I}(\phi_0)$ and security prices q_i^j for each $j \in \mathcal{J}(\phi_0)$ such that

(i) asset prices in states 0 and 1 are given by $\phi_0 = (1-\chi)v_0 + \chi v_1$ and $\phi_1 = \gamma v_1 + (1-\gamma)v_0$,

(ii) $\mathcal{J}(\phi_0)$ solves the security design problem (6),

(iii) security price q_i^j satisfies equation (8) for $i \in \{0,1\}$,and

(iv) asset value v_i solves the Euler equations given by:

$$v_i = \beta \left[z \left(\sum_{j \in P_i} q_i^j + \lambda \sum_{j \in \mathcal{J}(\phi_0) \setminus P_i} q_i^j \right) + (1-\lambda) \sum_{j \in \mathcal{J}(\phi_0) \setminus P_i} E_H y^j \right], \tag{31}$$

where $j \in P_i \subseteq \mathcal{J}(\phi_0)$ iff $R_i^j \geq \zeta$.

We call such an equilibrium a run equilibrium if $v_0 > v_1$.

Note that agent O designs the securities under the asset price ϕ_0. This is because security design is only possible if the economy is in regime 0 in the beginning of a given period.[17] Under the assumption that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s, following steps similar to the one in the main model, the optimal security is still a standard debt contract. To distinguish the debt cutoff of the contract under rigidity from the one in the main model we denote it by δ_0.

The equilibrium characterized in Proposition 3 remains an equilibrium even with rigidity. To see why, suppose the asset value is the same under the two regimes, ie, $v_0 = v_1$, then the asset prices do not depend on the regime, ie, $\phi_0 = \phi_1$. Consequently, $R_0^j = R_1^j$ and the set of liquid securities are the same in the two states, that is, $P_0 = P_1$. Specifically, the debt tranche remains liquid under both regimes which in turn justifies the fact that the asset value does not depend on the regime.

Due to the dynamic price feedback, under rigidity a run equilibrium is also possible. In this equilibrium, the asset value and the asset price drop whenever the economy enters regime 1. As a result, the adverse selection ratio R_i^j of the debt tranche decreases, triggering a run where the debt tranche that was previously liquid becomes illiquid, which in turn justifies the drop in the asset value and the asset price.[18] Had the design been flexible, agent O would redesign the security in this event by lowering the debt threshold to make sure that the debt tranche remains liquid. This would push the asset price up, and as we discussed before, this process would automatically lead to a full recovery of prices and the debt threshold. However, when the design is rigid, the drop in asset price can be self-fulfilling.

The following proposition characterizes the run equilibrium in which the debt tranche becomes illiquid when economy is in state 1.

[17] By definition the asset price in the centralized market is ϕ_0 at the end of a period in which the economy is in regime 0.

[18] During the run the price of the debt tranche drops to $q_{1D} = \phi_1 + s_L + \int_{s_L}^{\delta_1} \tilde{F}_L(s)ds$.

Proposition 5. *There exists a cutoff* $\Gamma(\chi, \gamma)$ *which is increasing in* γ *and* χ *with*

$$\Gamma(\chi, \gamma) > 1 - \frac{z-1}{z\lambda(1-\beta)}$$

such that whenever

$$\frac{E_L s}{E_H s} < \Gamma(\chi, \gamma), \tag{32}$$

(i) There exists a run equilibrium, i.e., $v_0 > v_1$ *and* $\phi_0 > \phi_1$. *(ii) Both values* v_0 *and* v_1*, and hence prices* ϕ_0 *and* ϕ_1 *are lower than the price* ϕ *in the equilibrium without a run. (iii) The debt threshold* δ_0 *in the run equilibrium is strictly lower than* δ *without a run. Consequently, welfare in the run equilibrium is Pareto dominated by the equilibrium without a run.*

Recall that in our optimal security design model a passthrough security arises as part of the unique equilibrium when $\frac{E_L s}{E_H s}$ exceeds $1 - \frac{z-1}{z\lambda(1-\beta)}$ which is also the cutoff for the existence of a pooling equilibrium in the benchmark case when only (the equity claim of) the collateral can be pledged to obtain interim liquidity. The above proposition implies that even in this region, with rigidity it can be optimal to create a non-trivial debt tranche.

7 Implementation of the Optimal Security Design as a Short-Term Repo Contract

In this section, we describe how the optimal security can be implemented as a repo contract in our model, and discuss its economic implications.

7.1 Terms of Repo

The optimal security design in our model can be implemented by a one-period repo contract, which is liquid, and an equity-like contract, which is illiquid. In this implementation, the repo borrower pledges the resale price of the collateral as well as any interim cash flow generated by the collateral (eg. dividend, or accrued interest payment) to obtain repo debt. As in the standard repo contracts, the repo lender returns the collateral at the end of each period if the borrower pays back the repo debt with interest, but keeps the collateral if the borrower fails to do so. The borrower can also choose to issue equity by pledging the residual cashflow from the collateral (ie., the remaining cash flow – if there is any – once repo debt obligations including principal and interest are paid off).

The terms of the contract are endogenous. Therefore, our theory offers a perspective on how adverse selection affects of the terms of short-term repo contracts backed by long-term assets. The face value of the repo contract is

$$D = \phi + \delta.$$

The expected value of the repo contract for the lender is

$$q_D = \phi + s_L + \int_{s_L}^{\delta} \left[\lambda \tilde{F}_L(s) + (1-\lambda)\tilde{F}_H(s)\right] ds.$$

The value of collateral underlying the repo contract at the beginning of a period to the productive borrowers is

$$\phi/\beta = z\phi + z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{\delta}^{s_H} \tilde{F}_H(s)ds$$

The last term reflects the loss of value from the illiquid equity tranche.

We are now ready to state the terms of the repo contract, including repo rate, R, and haircut, h. The definition of repo rate is straightforward:

$$R = \frac{\text{face value}}{\text{expected loan value}} - 1 = \frac{D - q_D}{q_D}. \tag{33}$$

When the expected quality of the debt contract is low relative to the face value, the repo rate is high. The asset quality might have two opposing effects on repo rate. When asset quality worsens (improves), expected loan value is lower (higher), leading to a high (low) repo rate. At the same time, the face value of the debt might be adjusted down (up), implying a lower (higher) likelihood of default and resulting in a lower (higher) repo rate.

The definition of repo haircut in our model is:

$$\begin{aligned} h &= 1 - \frac{\text{expected loan value}}{\text{collateral value}} = \frac{(z-1)q_D + \lambda z q_E + (1-\lambda)\int_{\delta}^{s_H} \tilde{F}_H(s)ds}{\phi/\beta} \\ &\approx \underbrace{(z-1)}_{\text{productivity}} + \underbrace{\frac{\int_{\delta}^{s_H}\left[\lambda \tilde{F}_L(s) + (1-\lambda)\tilde{F}_H(s)\right] ds}{\phi/\beta}}_{\text{equity/collateral}}, \text{ if } z \text{ is close to 1.} \end{aligned} \tag{34}$$

It is immediate that repo haircut has two components: the productivity of the borrower's technology, and the value of the equity tranche relative to the value of the collateral. The first component arises because borrowers, who price the collateral asset, value the liquidity service the asset provides, while lenders, who price the loan, does not value the liquidity service. The term $z - 1$ is the net value of the liquidity service. It reflects heterogeneous valuation over the collateral assets between lenders and borrowers in our model. This component is similar to the difference-of-opinion explanation of haircut in Geanakoplos (2003) and Fostel and Geanakoplos (2012). The second component arises because of information friction. It reflects how adverse selection affects the repo contract. This component has been emphasized by Dang, Gorton, and Holmström (2011) and Gorton and Ordonez (2014).

7.2 Properties of Repo Contracts

In this section, we look at properties of the repo contracts with three examples.

Example 1: two-point distribution. Suppose that the high (low) quality asset pays one unit of dividend with probability π_H (π_L) and pays zero otherwise. Assume $0 < \pi_L < \pi_H < 1$. In this example, the debt contract becomes very simple. No matter what the realization of the dividend, it pays the resale price ϕ. In addition, it pays δ units of the dividend if the dividend is one.

In this case, we obtain closed form solutions for the face value of the debt, $\delta + \phi$, and the asset price, ϕ, as follows:

$$\delta = \frac{1}{\frac{z}{z-1}\lambda\left(\pi_H - \pi_L\right) - \pi_H}\phi,$$
$$\phi = \frac{\beta\left[z\lambda\pi_L + (1-\lambda)\pi_H\right]}{1 - \beta z - \beta\frac{(1-\lambda)(z-1)\pi_H}{\frac{z}{z-1}\lambda(\pi_H-\pi_L)-\pi_H}}.$$

We can use these expressions to show that both the resale price and the debt cutoff δ are decreasing in the probability that the asset is low quality, i.e. $\frac{d\delta}{d\lambda} < 0$ and $\frac{d\phi}{d\lambda} < 0$. Moreover, the effect of productivity z can also be easily assessed. Because $\frac{\partial\delta}{\partial z} > 0$, more debt is created in good times. The sensitivity of asset price to productivity is amplified by the endogenous security design.

The terms of the repo contract can also be expressed in closed form and allow us to examine the determinants of repo rates and haircuts in this particular example. The repo rate is expressed as

$$R = \left[\frac{1-\pi_H}{\lambda(\pi_H - \pi_L)} + 1\right](z-1)\,. \tag{35}$$

It is immediate that repo rate is increasing in the productivity of technology z which measures the demand for liquidity from the productive borrowers. It is also clear that in this particular example, repo rate is increasing in λ. That is, a worsening (improving) asset quality leads to a lower (higher) repo rate, indicating that the face value of the repo debt drops (increases) significantly to eliminate (incorporate) risky states. To give another perspective on how repo rate is related to the riskiness of the cashflow and information frictions, we rewrite the above expression using the incentive constraint of the high quality seller of the repo contract, which is

$$zq_D = \pi_H \delta + \phi,$$

and obtain

$$R = \frac{\phi + \delta}{q_D} - 1 = \underbrace{\frac{\phi + \delta}{\phi + \pi_H \delta}}_{\text{Cashflow Riskiness}} \underbrace{z}_{\text{Productivity}} - 1. \tag{36}$$

Taking repo debt face value $\phi + \delta$ as given, (36) implies that the interest rate depends on the riskiness of the high quality assets directly. The degree of information friction plays an indirect role through debt face value. In fact, if the high quality asset pays dividend for sure, (35) implies that the repo rate R is $z - 1$. In this extreme case the repo rate is insensitive to changes in asset quality and driven purely by the productivity from the productive borrowers, which measures their liquidity demand. This example illustrates that in our model, repo rates capture more the demand for liquidity and cashflow riskiness of the repo contract and less so asset quality. This is due to the nature of security design: both high and low quality borrowers participate in the repo market and repo debts are free from adverse selection.

The repo haircut in this example can be expressed as

$$h = 1 - \beta + \frac{\beta}{1 - \frac{\lambda(\pi_H - \pi_L)}{(z-1)[(1-\lambda)\pi_H + \lambda \pi_L]}}. \tag{37}$$

Suppose z and β are close to 1, from (37) we then have

$$h \simeq \underbrace{(z-1)}_{\text{Productivity}} \left[1 - \underbrace{\frac{\pi_H}{\lambda(\pi_H - \pi_L)}}_{\text{Information Friction}} \right] + 1 - \beta.$$

It demonstrates again the two components in repo haircut highlighted in equation 34: one is related to the liquidity services of the collateral due to the technology productivity z and the other is related to the ratio of equity tranche over the collateral asset which is pinned down by the information friction $\lambda(\pi_H - \pi_L)$. In this particular example, $\frac{\partial h}{\partial \lambda} = \frac{\pi_H}{\lambda^2(\pi_H - \pi_L)}(z-1) > 0$. That is, as the asset quality deteriorates, haircut monotonically increases. Furthermore, haircut is also increasing in the quality difference between high and low type $\pi_H - \pi_L$, a measure of severity of adverse selection. This example shows again that the haircut of a repo contract is a robust indicator of information frictions over the asset quality, reflecting the magnitude of adverse selection, while the interest rate reflects the cashflow riskiness of the repo contract.

Example 2: Markov process for asset quality and project productivity In this example, we introduce Markov processes for asset quality and project productivity. Assume that the aggregate state x follows a Markov process, and parameters such as asset quality λ and productivity z are functions of the state: where $x \sim G(\cdot)$,

$$x_{t+1} \begin{cases} = x_t, & \text{with probability } \rho, \\ \sim G(x), & \text{with probability } 1-\rho. \end{cases}$$

We characterize the stationary Markov equilibrium. Given the state x, denote the end-of-period value of asset price if tomorrow's state is x to be ϕ_x and the face value of the liquid debt contract $E_x\phi + \delta_x$, where $E_x\phi \equiv \rho\phi_x + (1-\rho)E\phi$, $E\phi \equiv \int \phi_x dG(s)$. From optimal securitization decisions, summarized by equation (A.1),

$$\begin{aligned} E_x\phi &= \rho\phi_x + (1-\rho)E\phi \\ &= \frac{z_x}{z_x-1}\lambda_x \int_{s_L}^{\delta_x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{\delta_x} \widetilde{F}_H(s) ds - s_L, \end{aligned}$$

we have

$$\phi_x = E\phi + \frac{1}{\rho}\left\{\frac{z_x}{z_x-1}\lambda_x \int_{s_L}^{\delta_\lambda} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{\delta_x} \widetilde{F}_H(s) ds - s_L - E\phi\right\}. \tag{38}$$

From the Euler equation at the end of the period, after the quality of the asset in the period is revealed,

$$\phi_x = \frac{\beta}{1-\beta\rho z_x}\left\{z_x\left[\lambda_x E_L s + (1-\lambda_x)E_H s + (1-\rho)E\phi\right] - (1-\lambda)(z_x-1)\int_{\delta_x}^{s_H} \widetilde{F}_H(s) ds\right\} \tag{39}$$

(38) and (39) solve jointly (δ_x, ϕ_x) for all states.

Suppose $z_x = z$ and $\lambda_x = x \in [0,1]$. Then, with the process, the quality distribution is persistent over time but may change with probability $1-\rho$. When the quality distribution changes, the distribution parameter λ will be drawn from distribution G. We focus on the stationary Markov equilibrium. Security design and asset price depends on the quality distribution λ. Figure 5 illustrates how the collateral value, face value, interest rate and haircut of the repo contract respond to shocks to quality distribution.

The upper-left side subfigure show the liquid premium in the asset price, defined as difference between the actual asset price, ϕ_x, and the value of the asset without providing liquidity services, denoted φ_x, as a percentage of φ_x.

$$\varphi_x = \frac{\beta\left[(1-\rho)E\varphi + \lambda_x E_L s + (1-\lambda_x)E_H s\right]}{1-\beta\rho}, \quad E\varphi = \int \varphi_x dG(x).$$

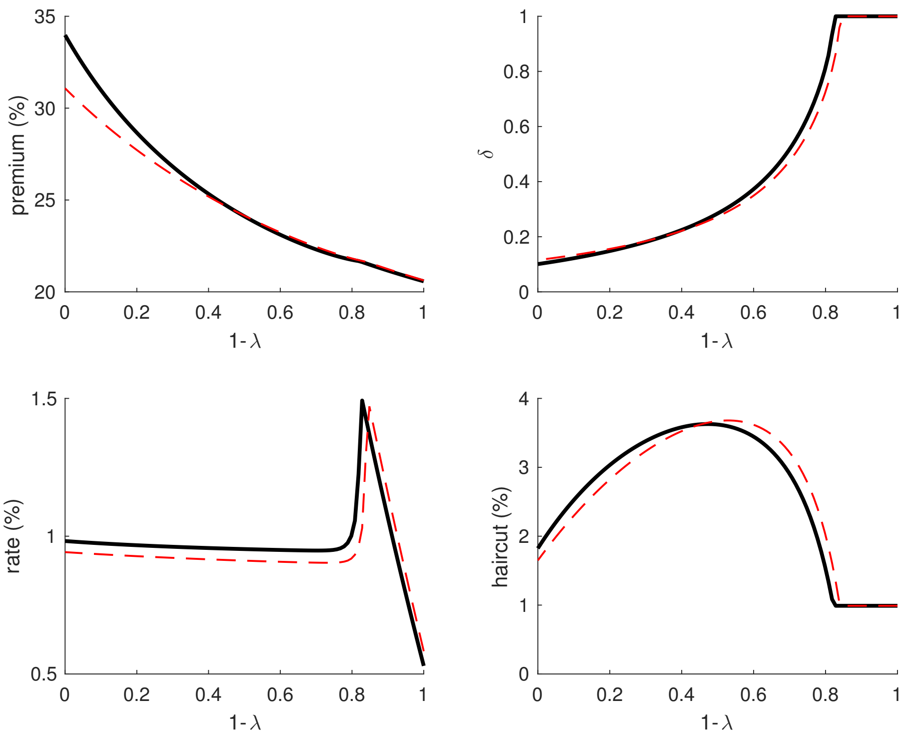

Figure 5: Asset quality, asset price and terms of the repo contract. The parameters for the numerical examples are as follows: high quality asset dividend follows a Beta distribution with $(a, b) = (10, 1)$ and low quality asset dividend follows a Beta distribution with $(a, b) = (0.1, 1)$, $\lambda \sim U[0, 1]$, $\beta = .95$, $z = 1.01$. The solid lines are drawn with $\rho = .95$ and the dashed lines with $\rho = .90$.

It reflects the liquidity value of the asset. When λ increases, both δ and φ decrease so that the value of liquidity services provided by a collateral decreases. But because both high and low quality assets provide some liquidity service, the liquidity value of collateral decreases more slowly than φ_x. This is why the liquidity premium decreases in $1-\lambda$. This shows the liquidity gain from security design is higher for assets of lower average quality.

Both the haircut (bottom right subfigure) and the repo rate (bottom left subfigure) change non-monotonically in λ. For haircut, this is because the value of the equity tranche is non-monotonic. When the asset quality is on average good (high $1-\lambda$), the information friction is small enough that there is no need to tranche the cash flow. In this case, no illiquid equity tranche is created. Then, according to (34), the haircut then only reflects heterogeneous valuation over liquidity between borrowers and lenders. When the asset quality is on average bad, the repo tranche is also very likely to default and the value of the equity tranche is small in that case. When the asset quality is in the intermediate range, the adverse selection is severe and hence the ratio of equity tranche to the asset is high, resulting in large haircuts.

The interest rate on the repo contract, when there is a non-trivial equity tranche, is for the most part decreasing in asset quality, reflecting the declining default probability and loss from default. The uptick in the repo rate reflects the opposing effect of changing asset quality mentioned previously when discussing equation (33): the face value of the debt might increase faster and incorporate more risky dividend states relative to the expected value of repo debt as asset quality improves.

We observe that in this example when the illiquidity induced by adverse selection is strong (high λ), haircut is very sensitive to changes in λ while the repo rate barely responds. This is qualitatively consistent with empirical observations during the repo runs where there were rare changes in repo rates but haircut skyrocketed.

The red dash lines correspond to lower persistence of the Markov process. When the quality distribution is less persistent, the collateral value is less responsive to the current productivity. When the high quality state is less persistent, adverse selection becomes more severe in that state, face value and repo rate decrease and haircut increases in that state.

Alternatively, suppose $\lambda_x = \lambda$ and $z_x = (1-x)z_L + xz_H$. Figure 6 illustrates that when firms are productive, the asset price is high, more repo contracts are issued, repo rate increases and repo haircut decreases.

Notice that a percentage point increase in productivity leads the collateral asset to increase in value by more than 10 percentage points. This amplification of productivity shocks reflects the dynamic feedback effect between the future collateral value and liquidity of the current market. Future collateral

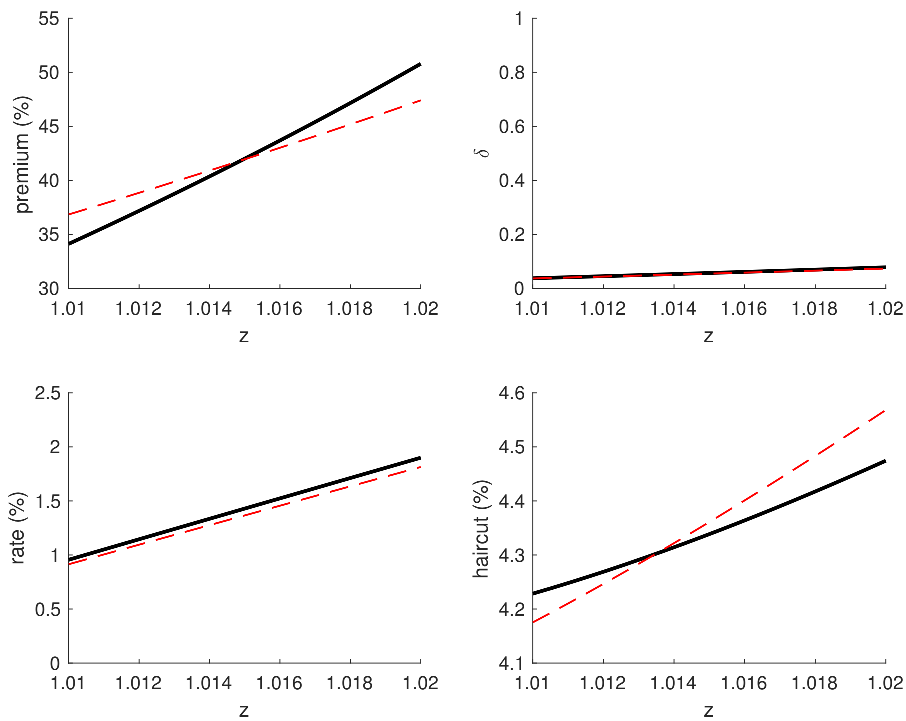

Figure 6: Productivity, asset price and terms of the repo contract. The parameters for the numerical examples are as follows: high quality asset dividend follows a Beta distribution with $(a, b) = (10, 1)$ and low quality asset dividend follows a Beta distribution with $(a, b) = (0.1, 1)$, $\lambda = 0.99$, $\beta = .95$, $z \sim U[1.01, 1.02]$. The solid lines are drawn with $\rho = .95$ and the dashed lines with $\rho = .90$.

value increases in future productivity. This reduces adverse selection in the current market, further increasing the collateral value. As productivity increases, the face value of the repo contract backed by a collateral increases, its interest rate increases and its haircut decreases. As before, the red dash lines correspond to lower persistence of the Markov process.

Example 3: Portfolio repo to improve asset quality In this example, we illustrate the observation that pooling safe assets with the collateral asset exposed to information frictions can improve the liquidity of the repo market. To derive analytical results, we use the two-point distribution in Example 1. Denote the fraction of safe assets in the asset pool to be ω. To keep the example tractable, we assume that the asset pool pays 0 or 1 with probability. This simplifying assumption is not essential. It also arises naturally if we think of the asset pool as being held by a special purpose vehicle (SPV) that issues debt contract with face value 1 against the pool. Then, the owners of the debt issued by the SPV face a sequential service constraint when it defaults.

Given ω and the quality of the collateral Q, the probability that the pool pays 1 is $\omega + (1-\omega)\pi_Q$. When there is a non-trivial debt tranche in the optimal design, we can show that both asset price and the debt threshold of the asset pool are increasing in ω.

$$\delta = \frac{1}{(1-\omega)\left[\frac{z}{z-1}\lambda\left(\pi_H - \pi_L\right) - \pi_H\right] - \omega}\phi,$$

$$\phi = \frac{\beta\left[\omega\left(z\lambda + (1-\lambda)\right) + (1-\omega)\left(z\lambda\pi_L + (1-\lambda)\pi_H\right)\right]}{1 - \beta z - \beta\frac{(1-\lambda)(z-1)[\omega + (1-\omega)\pi_H]}{(1-\omega)\left[\frac{z}{z-1}\lambda(\pi_H - \pi_L) - \pi_H\right] - \omega}}.$$

This implies that the portfolio repo improves the liquidity of the collateral asset.

The liquidity improvement also shows up in the term of the repo contract. Let the interest and haircut of the pool be R^ω and $h^\omega$and those of the standalone collateral asset be R^0 and h^0.

$$R^\omega = R^0 = \left[\frac{1-\pi_H}{\lambda(\pi_H - \pi_L)} + 1\right](z-1) \tag{40}$$

It is immediate that the interest rate does not respond to ω which is what we have learned in Example 1: the interest rate does not respond to information frictions.

The haircut is

$$h^\omega = 1 - \beta + \frac{\beta}{1 - \frac{\lambda(1-\omega)(\pi_H - \pi_L)}{(z-1)[\omega + (1-\omega)((1-\lambda)\pi_H + \lambda\pi_L)]}}.$$

Suppose z and β are close to 1, we show that

$$h^\omega \simeq 1 - \beta + (z-1)\left[1 - \frac{\frac{\omega}{1-\omega} + \pi_H}{\lambda\left(\pi_H - \pi_L\right)}\right] = h^0 - \frac{\omega(z-1)}{(1-\omega)\lambda(\pi_H - \pi_L)}. \tag{41}$$

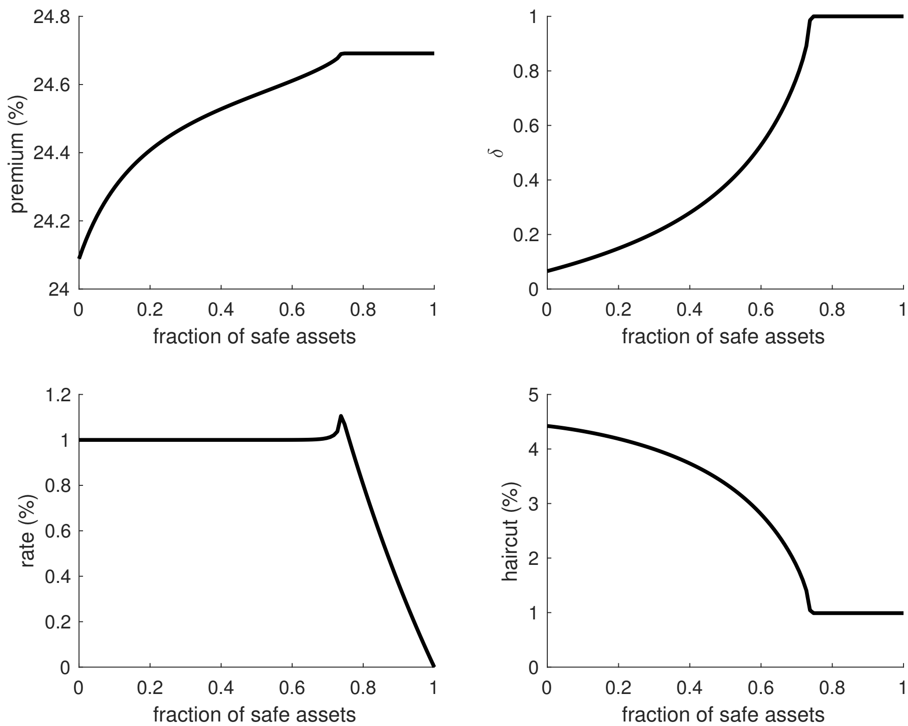

Figure 7: Fraction of safe assets, asset price and terms of the portfolio repo contract. The parameters for the numerical examples are as follows: high quality asset dividend follows a Beta distribution with $(a, b) = (10, 1)$ and low quality asset dividend follows a Beta distribution with $(a, b) = (0.1, 1)$, $\lambda = 0.9$, $\beta = .95$, $z = 1.01$.

The haircut decreases in ω implying that pooling the collateral asset with safe assets reduces information friction.

The qualitative results in the analytical example holds more generally. With the parametrization in Example 2, Figure 7 illustrates comparative statics of the equilibrium as the fraction of the safe assets – each of which pays 1 unit of dividend deterministically each period – increases.

When haircut is positive, or $\delta < 1$, the owner of the collateral sells the repo contract rather than the whole asset. In this case, increasing the fraction of safe assets decreases haircut, increases repo volume and the asset price. In this example, the repo rate is not sensitive to the portfolio composition. This is because the probability that the risky collateral is of high quality is low. The risk premium in the repo rate does not change very much. When the fraction of safe assets is very high, the liquidity of the risky collateral is completely restored, the whole asset is traded without delay.

This result is consistent with empirical findings. In particular, Julliard et al. (2018) find that repo contracts backed by a portfolio including AAA rated assets receive (statistically significant) 0.9% to 1.15% lower haircut compared with repo contracts without any AAA rated assets, controlling for counterparty and collateral characteristics.

7.3 Repo Runs

In section 6.4 we discussed how contract rigidity may be a crucial source of fragility. Take the overnight repo market as an example. Our results imply that when borrowers are able to update the terms of overnight repo contracts each day, the market is robust to run. In practice, the haircut of a repo contract is determined by the value-at-risk assessment of the collateral assets. This assessment is not performed continuously for the bank's risk management team to revise the haircut frequently which introduces some amount of rigidity in the updating of the contracts. Hence, proposition 5 implies that repo market might be susceptible to runs.

Recall from 6.4 that with rigidity there are two possible regimes 0 and 1. Under the repo interpretation, the terms of the repo contract are rigid in the sense that the book value of the repo contract, D, does not depend on the regime. But, because in a run the asset price decreases to a lower level, ϕ_1, the effective debt threshold increases from δ_0 to $\delta_1 \equiv \min(s_H, \delta_0 + \phi_0 - \phi_1)$.[19] When investors receive a sunspot during an episode of rigidity, the repo contract becomes illiquid. Only owners with low quality assets trade the securities. In this scenario, denote the effective interest rate and haircut to be R_1^S and h_1^S respectively and the price of the debt tranche in regime i by q_{iD}.

$$R_1 = \frac{D - q_{1D}}{q_{1D}}, \tag{42}$$

$$h_1 = 1 - \frac{q_{1D}}{\phi_1/\beta}, \tag{43}$$

where

$$q_{1D} = \phi_1 + s_L + \int_{s_L}^{\delta_1} \tilde{F}_L(s)ds.$$

In the appendix, we show $q_{1D} < q_{0D}$. Then, repo rate increases when the sunspot arrives, $R_1 > R_0$. The response of haircut is indeterminate because both $\phi_1 < \phi_0$ and $q_{1D} < q_{0D}$.

The dynamics in a typical repo run is illustrated in Figure 8. In the figure before the moment marked by the equilibrium switch the economy is a "good" equilibrium in which even when the security is rigid agents expect that the repo tranche will be liquid and the asset price will be high. After the switch a repo

[19] By part (i) of proposition 5, $\phi_0 > \phi_1$ so, $\delta_1 > \delta_0$.

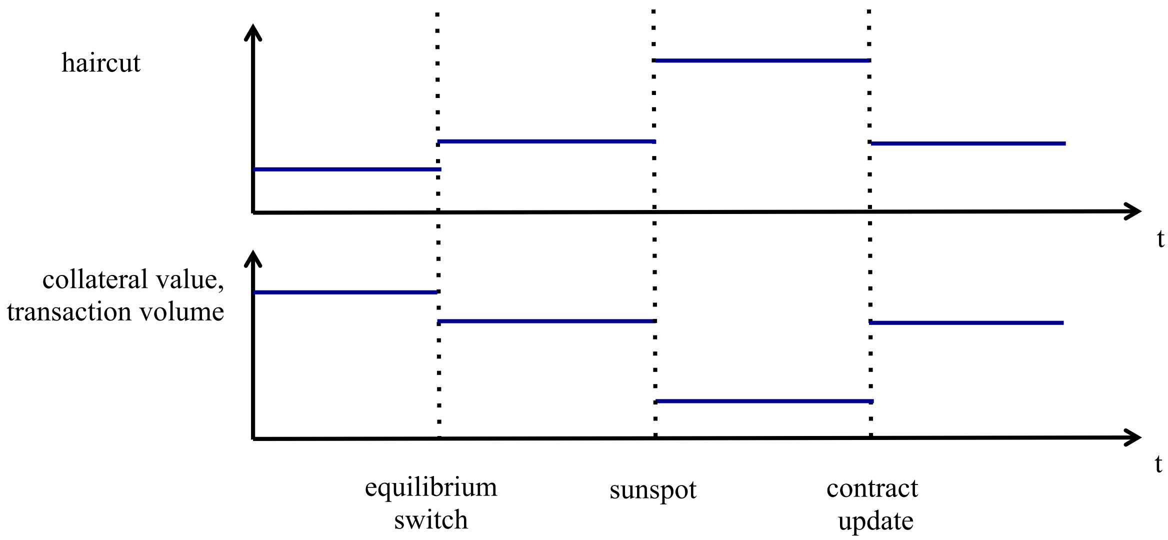

Figure 8: Dynamics of Security run.

run typically takes two stages. First, the equilibrium switches to the sunspot equilibrium described in Proposition 5. Once the economy enters a sunspot equilibrium, haircut of the repo contract immediately increases, because investors anticipate that the repo contract will be illiquid when a sunspot hits the economy. At the same time, the asset price and the repo volume decrease. When the sunspot actually hits the economy, asset price and the repo volume decrease further. The repo rate increases further while the repo haircut may also increase. This occurs despite that the face value of repo debt remains unchanged due the contract rigidity. The drop in repo volume and the asset price is higher when the sunspot hits, because the repo backed by high quality collateral stops circulating entirely. When the contract terms are updated, the update restores investors' sentiment about the liquidity of the repo market, the price and the volume recover partially, to the levels right after equilibrium switch. The fluctuation driven by sunspots may take place repeatedly as long as the economy remains in the sunspot equilibrium.

Notice that the equilibrium switch can be triggered either by a switch of self-fulfilling beliefs from the equilibrium without repo run to an equilibrium with repo run, or by a small shift in the fundamental. Suppose the fundamental of the economy, represented by asset quality λ or productivity z, is initially such that condition (32) does not hold. As the fundamental deteriorates and condition (32) holds, even if the change in fundamentals is very small, a sunspot equilibrium might emerge, leading to a discontinuous drop in market liquidity and the asset price.

8 Conclusion and Discussions

Our paper studies optimal security design in a dynamic lemons setup. We show that the implementation of optimal security design involves short-term liquid collateralized debt. Because optimal security design helps coordinate investors' inter-temporal decisions, the dynamic lemons market under optimal security design is robust to multiple-equilibrium fragility induced by inter-temporal mis-coordination. We show a dynamic run might occur when contract terms update infrequently. We also explore economic implications of an implementation of optimal security: short term repos, and derives dynamic equilibrium properties of repo rates, haircuts and volume, and also aggregate funding liquidity over the productivity and the asset quality cycles.

We conclude by discussing a few potential applications of this framework of dynamic price feedback with security design to highlight its generality. One immediate application could be on security lending. In the setup of the baseline model, there is a gain to trade since the cash borrowers can use cash to generate more cashflow (ie., at a multiplier z) than the cash lenders. We can modify this setup and assume also that borrowers value collaterals at a discount, u, relative to the cash lenders. Now there is another gain from trade: the cash borrowers have incentive to lend out the asset at a lower price due to their low private valuation. In this case, the haircut on the collateral asset might turn negative. The fluctuation of multiplier z and value discount u might explain time-varying haircut for some firms.[20] Another potential application is on the dynamic pecking order of financing. The classical pecking order theory of Myers and Majluf (1984b) is in a static environment where security issuers are more informed about the future dividend states and can only pledge the dividend when issuing securities. By allowing issuers to pledge also future resale price of the security as in our dynamic price feedback framework, the adverse selection environment might change which could potentially reverse the static pecking order of financing. In fact, equity might emerge as the most liquid and desired form of financing when the borrowing firm has a highly productive project and suffers relatively less adverse selection regarding its interim dividend cashflows.[21]Finally, the insight that creating a liquid trance from collateral assets exposed to adverse selection will trigger the dynamic feedback of higher asset prices and hence a larger liquid tranche could offer new perspectives of how to implement effective monetary policies. For example, quantitative easing,

[20]During the financial crisis of 2018-2019, AIG changed from charging a haircut to paying a haircut Peirve (2017). According to our theory, it might be due to its time varying need for external cash relative to securities.

[21]Fulghieri, Garćia, and Hackbarth (2016) have also extended the static pecking order model to incorporate dynamic considerations by giving the issuing firm access to a growth option for dynamic considerations and have also generalized the distribution assumption of dividend cashflows. Our dynamic price feedback framework will add an additional dimension of endogenous security price to this line of literature.

instead of viewed as a plain liquidity injection, could act as a catalyst in this dynamic price feedback and create more pledgeable liquid tranches from illiquid collateral assets, ie., a liquid collateral multiplier, when the economy suffers severe adverse selection. We leave these applications for future works.

References

Akerlof, G. A. (1970). "The Market for" Lemons": Quality Uncertainty and the Market Mechanism". *The Quarterly Journal of Economics* 84.3, pp. 488–500.

Asriyan, V., W. Fuchs, and B. Green (2017). "Liquidity Sentiments". *Working paper.*

Bigio, S. (2015). "Endogenous liquidity and the business cycle". *American Economic Review* 105.6, pp. 1883–1927.

Boot, A. W. A. and A. V. Thakor (1993). "Security Design". *The Journal of Finance* 48.4, pp. 1349–1378.

Chiu, J. and T. V. Koeppl (2016). "Trading dynamics with adverse selection and search: Market freeze, intervention and recovery". *The Review of Economic Studies* 83.3, pp. 969–1000.

Dang, T. V., G. Gorton, and B. Holmström (2011). "Haircuts and Repo Chains". *Manuscript. Columbia University.*

— (2013). "The information sensitivity of a security". *Working paper.*

DeMarzo, P. M. and D. Duffie (1995). "Corporate incentives for hedging and hedge accounting". *The review of financial studies* 8.3, pp. 743–771.

— (1999). "A liquidity-based model of security design". *Econometrica* 67.1, pp. 65–99.

Diamond, D. W. and P. H. Dybvig (1983). "Bank Runs, Deposit Insurance, and Liquidity". *Journal of Political Economy* 91.3, pp. 401–419. DOI: 10.1086/261155. eprint: https://doi.org/10.1086/261155. URL: https://doi.org/10.1086/261155.

Donaldson, J. R. and G. Piacentino (2017). "Money runs". *Working papper, Washington University in St. Louis.*

Ellis, A., M. Piccione, and S. Zhang (2017). "Equilibrium Securitization with Diverse Beliefs". *Working paper.*

Farhi, E. and J. Tirole (2015). "Liquid bundles". *Journal of Economic Theory* 158.PB, pp. 634–655.

Fostel, A. and J. Geanakoplos (2012). "Tranching, CDS, and Asset Prices: How Financial Innovation Can Cause Bubbles and Crashes". *American Economic Journal: Macroeconomics* 4.1, pp. 190–225. DOI: 10.1257/mac.4.1.190.

Fulghieri, P., D. Garćia, and D. Hackbarth (2016). *Asymmetric information and the pecking (dis)order*. Tech. rep. UNC.

Geanakoplos, J. (2003). "Liquidity, default, and crashes endogenous contracts in general". In: *Advances in economics and econometrics: theory and applications: eighth World Congress*. Vol. 170.

Geanakoplos, J. and W. Zame (2002). *Collateral and the enforcement of intertemporal contracts*. Tech. rep. Yale University working paper.

Gorton, G. and G. Ordonez (2014). "Collateral crises". *American Economic Review* 104.2, pp. 343–78.

Gorton, G. and G. Pennacchi (1990). "Financial intermediaries and liquidity creation". *The Journal of Finance* 45.1, pp. 49–71.

Gottardi, P., V. Maurin, and C. Monnet (2017). "A theory of repurchase agreements, collateral re-use, and repo intermediation".

Julliard, C. et al. (2018). *UK Repo Haircut*. Tech. rep. London School of Economics.

Kiyotaki, N. and J. Moore (1997). "Credit cycles". *Journal of Political Economy* 105.2, pp. 211–248.

Kuong, J. C.-F. (2017). "Self-fulfilling fire sales: Fragility of collateralised short-term debt markets".

Kurlat, P. (2013). "Lemons markets and the transmission of aggregate shocks". *American Economic Review* 103.4, pp. 1463–89.

Lagos, R. and R. Wright (2005). "A unified framework for monetary theory and policy analysis". *Journal of political Economy* 113.3, pp. 463–484.

Leland, H. E. and D. H. Pyle (1977). "Informational asymmetries, financial structure, and financial intermediation". *The journal of Finance* 32.2, pp. 371–387.

Lester, B., A. Postlewaite, and R. Wright (2012). "Information, liquidity, asset prices, and monetary policy". *The Review of Economic Studies* 79.3, pp. 1209–1238.

Li, Y., G. Rocheteau, and P.-O. Weill (2012). "Liquidity and the threat of fraudulent assets". *Journal of Political Economy* 120.5, pp. 815–846.

Martin, A., D. Skeie, and E.-L. Von Thadden (2012). "The Fragility of Short-Term Funding Markets". *Journal of Economic Theory* 149. DOI: 10.2139/ssrn.2170882.

— (2014). "Repo runs". *Review of Financial Studies* 27.4, pp. 957–989.

Miao, J. and P. Wang (2018). "Asset bubbles and credit constraints". *American Economic Review* 108.9, pp. 2590–2628.

Myers, S. C. and N. S. Majluf (1984a). "Corporate financing and investment decisions when firms have information that investors do not have". *Journal of Financial Economics* 13.2, pp. 187–221.

Myers, S. C. and N. S. Majluf (1984b). "Corporate Financing and Investment Decisions when Firms have Information that Investors Do Not Have". *Journal of Financial Economics* 13.2, pp. 187–221.

Parlatore, C. (Forthcoming). "Collateralizing liquidity". *Journal of Financial Economics.*

Peirve, H. (2017). *Securities lending and the Untold Story in the Collapse of AIG.* Tech. rep. George Mason University, Mercatus Center Working Paper.

Plantin, G. (2009). "Learning by holding and liquidity". *Review of Economic Studies* 76.1, pp. 395–412.

Simsek, A. (2013). "Belief disagreements and collateral constraints". *Econometrica* 81.1, pp. 1–53.

Yang, M. (Forthcoming). "Optimality of debt under flexible information acquisition". *Review of Economic Studies.*

A Appendix

A.1 Proof of Lemma 1

Proof. If two securities, y^j and y^k, are both liquid, $E_L y^j \geq \zeta E_H y^j$ and $E_L y^k \geq \zeta E_H y^k$. Then combining the two security retains liquidity. Similarly, combining two illiquid securities results in an illiquid security. To see the second statement, first note that replacing the two securities with their combination is clearly feasible. In addition, when y^j, y^k and $y^j + y^k$ all trade in a pooling (separating) equilibrium, q^{jk}, the price of $y^j + y^k$, is the sum of q^j and q^k, the prices of y^a and y^b. Now consider the liquid case. Ignoring the irrelevant terms, Agent O's payoff when the two securities are separate is:

$$\lambda \int \left\{ a\left[zq^j - y^j(s)\right] + a\left[zq^k - y^k(s)\right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a\left[zq^j - y^j(s)\right] + a\left[zq^k - y^k(s)\right] \right\} dF_H(s)$$

and when they are combined is:

$$\lambda \int \left\{ a\left[zq^{jk} - \left(y^j(s) + y^k(s)\right)\right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a\left[zq^{jk} - \left(y^j(s) + y^k(s)\right)\right] \right\} dF_H(s).$$

Since $q^{jk} = q^j + q^k$, when the liquid securities are combined agent O's payoff is unchanged.

Next consider the illiquid case. Once again ignoring the irrelevant terms, Agent O's payoff when the two securities are separate is:

$$\lambda \int \left\{ a\left[zq^j - y^j(s)\right] + a\left[zq^k - y^k(s)\right] \right\} dF_L(s) + (1-\lambda) \int \left\{ ay^j(s) + ay^k(s) \right\} dF_H(s)$$

and when they are combined is:

$$\lambda \int \left\{ a\left[zq^{jk} - \left(y^j(s) + y^k(s)\right)\right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a\left(y^j(s) + y^k(s)\right) \right\} dF_H(s).$$

Once again, when the illiquid securities are combined agent O's payoff is unchanged. □

A.2 Proof of Proposition 2

Proof. First note that the feasible set is compact, convex and nonempty so the optimization problem must have a solution. Moreover, since the objective function is bounded above, the solution must be finite. The Lagrangian of the optimization problem is

$$\mathcal{L}(x;\gamma,\mu,\mu_x) = \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds + \int_{s_L}^{s_H} \gamma(s)\left[s - s_L - \int_{s_L}^{s} x(j)dj\right] ds$$
$$+ \mu\left\{\int_{s_L}^{s_H}\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]x(s)ds + (1-\zeta)\phi\right\} + \int_{s_L}^{s_H} \mu_x(s)x(s)ds.$$

Note that for any feasible x and for $\gamma \geq 0$, $\mu \geq 0$ and $\mu_x \geq 0$ we have

$$\mathcal{L}(x;\gamma,\mu,\mu_x) \geq \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds.$$

Let $\mathcal{L}(\gamma,\mu,\mu_x) = \max_x \mathcal{L}(x;\gamma,\mu,\mu_x)$. Let $\mathcal{L}^* = \min_{\gamma\geq 0,\mu\geq 0,\mu_x\geq 0} \mathcal{L}(\gamma,\mu,\mu_x)$. Note that $\mathcal{L}^*$ is the value of the original optimization problem. We can rewrite $\mathcal{L}(x;\gamma,\mu,\mu_x)$ as

$$\mathcal{L}(x;\gamma,\mu,\mu_x) = \int_{s_L}^{s_H}\left\{\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] - \int_s^{s_H}\gamma(j)dj + \mu_x(s)\right\}x(s)ds$$
$$+\mu(1-\zeta)\phi + \int_{s_L}^{s_H}\left(\int_s^{s_H}\gamma(j)dj\right)ds.$$

Let $\eta(s) = \int_s^{s_H}\gamma(j)dj$. We can rewrite the problem as:

$$\mathcal{L}(x;\eta,\mu,\mu_x) = \int_{s_L}^{s_H}\left\{\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] - \eta(s) + \mu_x(s)\right\}x(s)ds$$
$$+\mu(1-\zeta)\phi + \int_{s_L}^{s_H}\eta(s)\,ds.$$

Now note that the quantity inside the curly brackets must be zero or otherwise the value of the optimization problem would be infinite. Consider the following dual problem of the optimization problem,

$$\min_{\mu\geq 0}\ \min_{\eta\geq 0,\mu_x\geq 0}\ \mu(1-\zeta)\phi + \int_{s_L}^{s_H}\eta(s)\,ds$$
$$s.t.\quad \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] - \eta(s) + \mu_x(s) = 0.$$

Note that the value of this problem is $\mathcal{L}^*$. Let $H_\mu(s) = \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]$. We can rewrite the above problem one more time as:

$$\min_{\mu\geq 0}\ \min_{y\geq 0}\ \mu(1-\zeta)\phi + \int_{s_L}^{s_H}\eta(s)\,ds$$
$$s.t.\quad \eta(s) \geq H_\mu(s),$$

and the constraint that $\eta(s)$ is a decreasing function in s. Note, $h_\mu(s) \equiv \frac{\partial H_\mu(s)}{\partial s} = -f_H(s)\left[1 + \mu\left(\frac{f_L(s)}{f_H(s)} - \zeta\right)\right]$. Clearly $H_\mu(s_L) > 0$ and $H_\mu(s_H) = 0$. Since $\mu > 0$ we must have $h_\mu(s_L) < 0$. To see this suppose

$h_\mu(s_L) \geq 0$. Then it must be the case that $1+\mu\left(\frac{f_L(s)}{f_H(s)}-\zeta\right) \leq 0$. Since $\frac{f_L(s)}{f_H(s)}$ is decreasing, this implies that $h_\mu(s) > 0$ for all $s \in (s_L, s_H]$ contradicting that $H_\mu(s_H) = 0$.

Since $\frac{f_L(s)}{f_H(s)}$ is decreasing in s one of the following must be true:

(i) There exists a unique cutoff $\widehat{s}_\mu \in (s_L, s_H)$ such that $h_\mu(s) < 0$ for $s < \widehat{s}_\mu$ and $h_\mu(s) > 0$ for $s > \widehat{s}_\mu$,

(ii) $h_\mu(s) < 0$ for all $s \in (s_L, s_H)$.

In case (i) the function $H_\mu(s)$ crosses from positive to negative once, eventually increasing to zero at s_H. In case (ii) $H_\mu(s)$ decreases to zero at s_H. Let $s^*_\mu \in (s_L, s_H)$ be the unique s for which $H_\mu(s) = 0$ if it exists, otherwise let $s^*_\mu = s_H$.

Note that for given $\mu \geq 0$ optimal η_μ is given by:

$$\eta_\mu(s) = \begin{cases} H_\mu(s) & \text{if} \quad s \leq s^*_\mu \\ 0 & \text{if} \quad s > s^*_\mu \end{cases}.$$

Plugging this into the minimization problem we get:

$$\min_{\mu \geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{s^*_\mu} \left(\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]\right) ds.$$

The first order condition for this problem is:

$$(1-\zeta)\phi + \int_{s_L}^{s^*_\mu} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds + \frac{\partial s^*_\mu}{\partial \mu} H_\mu\left(s^*_\mu\right) \geq 0$$

Because $H_\mu\left(s^*_\mu\right) = 0$,

$$(1-\zeta)\phi + \int_{s_L}^{s^*_\mu} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds \geq 0$$

with complementary slackness.

Let $s^* \in (s_L, s_H]$ be the unique s for which

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds = 0$$

if it exists. If

$$(1-\zeta)\phi + \int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds > 0$$

for all $s \in [s_L, s_H]$, then $s^* = s_H$.

If $s^* < s_H$ then $\mu > 0$, $s^*_\mu = s^*$, and

$$\mathcal{L}^* = \mu(1-\zeta)\phi + \int_{s_L}^{s^*} \left(\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]\right) ds = \int_{s_L}^{s^*} \widetilde{F}_H(s) ds.$$

If $s^* = s_H$ then $\mu = 0$, $s^*_\mu = s_H$, and

$$\mathcal{L}^* = \int_{s_L}^{s_H} \widetilde{F}_H(s) ds.$$

To complete the proof note that $x(s) = 1$ for $s \in [s_L, s^*]$ and $x(s) = 0$ for $s \in [s^*, s_H]$ achieves the value $\mathcal{L}^*$ and it is feasible, and must be optimal for the original problem. □

A.3 Proof of Proposition 3

Proof. Observe that to maximize (23) agent O must set δ as large as possible subject to satisfying the constraint (24). We first show that either there is a unique δ that satisfies (24) with equality, or (24) is not binding. Let

$$\begin{aligned}\mathcal{T}(x) &\equiv (z-1)\left[\phi + s_L + \lambda \int_{s_L}^{x} \widetilde{F}_L(s)ds + (1-\lambda)\int_{s_L}^{x} \widetilde{F}_H(s)ds\right] - \lambda \int_{s_L}^{x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds \\ &= (z-1)\left[\phi + s_L + \int_{s_L}^{x} \widetilde{F}_H(s)ds\right] - z\lambda \int_{s_L}^{x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds.\end{aligned}$$

Observe that,

$$\mathcal{T}(s_L) = (z-1)(\phi + s_L) > 0, \qquad \mathcal{T}'(x) = (z-1)\widetilde{F}_H(x) - z\lambda\left[\widetilde{F}_H(x) - \widetilde{F}_L(x)\right],$$

$$\mathcal{T}'(s_L) = z - 1 > 0, \qquad \mathcal{T}'(s_H) = 0,$$

$$\mathcal{T}''(x) = -(z-1)f_H(x) + z\lambda\left[f_H(x) - f_L(x)\right] = f_H(x)\left[z(\lambda-1) + 1 - z\lambda\frac{f_L(x)}{f_H(x)}\right].$$

When $\frac{f_L(x)}{f_H(x)}$ is monotonically decreasing in s, $\mathcal{T}(x)$ is quasi-concave with $\mathcal{T}(s_L) > 0$. So, there is either a unique δ that satisfies $\mathcal{T}(\delta) = 0$ or $\mathcal{T}(x) > 0$ for all $x \in [s_L, s_H]$.

Case (i): Constraint (24) is binding. In this case the face value of the debt contract that solves the security design problem is given by:

$$\phi = \frac{z}{z-1}\lambda \int_{s_L}^{\delta} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{\delta} \widetilde{F}_H(s)ds - s_L. \tag{A.1}$$

In addition, the asset price ϕ satisfies (25). Substituting for q_D and q_E we rewrite (25) as:

$$\phi = \frac{\beta}{1-\beta z}\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{\delta}^{s_H} \widetilde{F}_H(s)ds\right\}. \tag{A.2}$$

Substituting ϕ in (A.1) using (A.2), the equilibrium can be solved by a single equation of δ, $\Gamma(\delta) = 0$, where

$$\begin{aligned}\Gamma(\delta) = &\frac{\beta}{1-\beta z}\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{\delta}^{s_H} \widetilde{F}_H(s)ds\right\} \\ &- \frac{z}{z-1}\lambda \int_{s_L}^{\delta} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds + \int_{s_L}^{\delta} \widetilde{F}_H(s)ds + s_L\end{aligned}$$

Observe that:

$$\begin{aligned}
\Gamma'(\delta) &= \frac{\beta}{1-\beta z}(1-\lambda)(z-1)\widetilde{F}_H(\delta) - \frac{z}{z-1}\lambda\left[\widetilde{F}_H(\delta) - \widetilde{F}_L(\delta)\right] + \widetilde{F}_H(\delta) \\
&= \left[\frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right]\widetilde{F}_H(\delta) + \frac{z}{z-1}\lambda\widetilde{F}_L(\delta). \\
\Gamma''(\delta) &= -\left[\frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right] f_H(\delta) - \frac{z}{z-1}\lambda f_L(\delta) \\
&= f_H(\delta)\left\{\frac{z}{z-1}\lambda\left[1 - \frac{f_L(\delta)}{f_H(\delta)}\right] - \frac{\beta}{1-\beta z}(1-\lambda)(z-1) - 1\right\} \\
\Gamma(s_L) &= s_L\left[1 + \frac{\beta}{1-\beta z}(1-\lambda)(z-1)\right] + \frac{\beta}{1-\beta z}\left[z\lambda E_L s + (1-\lambda)E_H s\right] > 0 \\
\Gamma'(s_L) &= \frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 > 0 \\
\Gamma'(s_H) &= 0.
\end{aligned}$$

Once again $\Gamma(s)$ is quasi-concave if $\frac{f_L(\delta)}{f_H(\delta)}$ is monotonically decreasing in D. Because $\Gamma(s_L) > 0$, there is a unique equilibrium. The constraint (24) is binding iff $\Gamma(s_H) < 0$. We rewrite $\Gamma(s_H)$ as:

$$\begin{aligned}
\Gamma(s_H) &= \frac{\beta z}{1-\beta z}\left[\lambda E_L s + (1-\lambda)E_H s\right] - \frac{z}{z-1}\lambda\int_{s_L}^{s_H}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right]ds + \int_{s_L}^{s_H}\widetilde{F}_H(s)ds + s_L \\
&= \frac{E_H s}{(1-\beta z)(z-1)}\left[\lambda z(1-\beta)\left(\frac{E_L s}{E_H s} - 1\right) + z - 1\right].
\end{aligned}$$

Hence, $\Gamma(s_H) < 0$ iff

$$\frac{E_L s}{E_H s} < 1 - \frac{z-1}{z\lambda(1-\beta)}.$$

Case (ii): Constraint (24) is not binding. □

A.4 Proof of Proposition 4

Claim 1. Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. The optimal securities are

$$\begin{aligned}
y_{1t}(s) &= \phi + s_L + (s - s_L)\mathbb{I}(s \le \delta), \\
y_{2t}(s) &= (s - s_L)\mathbb{I}(s > \delta).
\end{aligned}$$

for some $\delta \in (s_L, s_H]$.

Proof. The maximization

$$\arg\max_{x,m} \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds, \tag{A.3}$$

$$s.t. \int_{s_L}^{s} x(j)dj \leq s - s_L, \forall s \in [s_L, s_H], \tag{A.4}$$

$$\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi \geq 0, \tag{A.5}$$

$$\int_{s_L}^{s} x(j)dj \geq 0, \forall s \in [s_L, s_H] \tag{A.6}$$

First note that the feasible set is compact, convex and nonempty so the optimization problem must have a solution. Moreover, since the objective function is bounded above, the solution must be finite. The Lagrangian of the optimization problem is

$$\mathcal{L}(x;\gamma,\mu,\nu) = \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds + \int_{s_L}^{s_H} \gamma(s)\left[s - s_L - \int_{s_L}^{s} x(j)dj\right] ds$$
$$+ \mu\left\{\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi\right\} + \int_{s_L}^{s_H} \nu(s)\left[\int_{s_L}^{s} x(j)dj\right] ds.$$

Note that for any feasible x and for $\gamma \geq 0$, $\mu \geq 0$ and $\nu \geq 0$ we have

$$\mathcal{L}(x;\gamma,\mu,\nu) \geq \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds.$$

Let $\mathcal{L}(\gamma,\mu,\nu) = \max_x \mathcal{L}(x;\gamma,\mu,\nu)$. Let $\mathcal{L}^* = \min_{\gamma\geq 0,\mu\geq 0,\mu_x\geq 0} \mathcal{L}(\gamma,\mu,\nu)$. Note that $\mathcal{L}^*$ is the value of the original optimization problem. We can rewrite $\mathcal{L}(x;\gamma,\mu,\nu)$ as

$$\mathcal{L} = \int_{s_L}^{s_H} \left\{\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] + \int_{s}^{s_H} [\nu(j) - \gamma(j)]\, dj\right\} x(s)ds$$
$$+ \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \left(\int_{s}^{s_H} \gamma(j)dj\right) ds$$

Now note that the quantity inside the curly brackets must be zero or otherwise the value of the optimization problem would be infinite. Consider the following dual problem of the optimization problem,

$$\min_{\mu\geq 0} \quad \min_{\gamma\geq 0,\nu\geq 0} \quad \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \left(\int_{s}^{s_H} \gamma(j)dj\right) ds$$
$$s.t. \qquad \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] + \int_{s}^{s_H} [\nu(j) - \gamma(j)]\, dj = 0.$$

Note that the value of this problem is $\mathcal{L}^*$. Let $H_\mu(s) = \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]$. Let $\eta(s) = \int_s^{s_H} \gamma(j)dj$, $\xi(s) = \int_s^{s_H} \nu(j)dj$. We can rewrite the above problem one more time as:

$$\min_{\mu\geq 0} \quad \min_{\eta,\xi\geq 0} \quad \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \eta(s)ds$$
$$s.t. \qquad H_\mu(s) + \xi(s) - \eta(s) = 0.$$

and the constraints that $\eta(s)$ and $\xi(s)$ are decreasing functions in s.

Note, $h_\mu(s) \equiv \frac{\partial H_\mu(s)}{\partial s} = -f_H(s)\left[1+\mu\left(\frac{f_L(s)}{f_H(s)}-\zeta\right)\right]$. Clearly $H_\mu(s_L) > 0$ and $H_\mu(s_H) = 0$. Since $\mu > 0$ we must have $h_\mu(s_L) < 0$. To see this suppose $h_\mu(s_L) \geq 0$. Then it must be the case that $1+\mu\left(\frac{f_L(s)}{f_H(s)}-\zeta\right) \leq 0$. Since $\frac{f_L(s)}{f_H(s)}$ is decreasing, this implies that $h_\mu(s) > 0$ for all $s \in (s_L, s_H]$ contradicting that $H_\mu(s_H) = 0$.

Since $\frac{f_L(s)}{f_H(s)}$ is decreasing in s one of the following must be true:

(i) There exists a unique cutoff $\widehat{s}_\mu \in (s_L, s_H)$ such that $h_\mu(s) < 0$ for $s < \widehat{s}_\mu$ and $h_\mu(s) > 0$ for $s > \widehat{s}_\mu$,

(ii) $h_\mu(s) < 0$ for all $s \in (s_L, s_H)$.

In case (i) the function $H_\mu(s)$ crosses from positive to negative once, eventually increasing to zero at s_H. In case (ii) $H_\mu(s)$ decreases to zero at s_H.

Note that for given $\mu \geq 0$ optimal η_μ and ξ_μ are given by:

$$\xi_\mu(s) = \begin{cases} -H_\mu(\hat{s}_\mu) & \text{if} \quad s \leq \hat{s}_\mu, \\ -H_\mu(s) & \text{if} \quad s > \hat{s}_\mu. \end{cases}$$

$$\eta_\mu(s) = \begin{cases} H_\mu(s) - H_\mu(\hat{s}_\mu) & \text{if} \quad s \leq \hat{s}_\mu, \\ 0 & \text{if} \quad s > \hat{s}_\mu. \end{cases}$$

This is because ξ_μ and η_μ must be decreasing in s. When $s > \hat{s}_\mu$, $H_\mu(s)$ is increasing. So it is feasible to let $\eta_\mu(s) = 0$ and $\xi_\mu(s) = -H_\mu(s)$ in this region. When $s < \hat{s}_\mu$, $H_\mu(s)$ is decreasing in s. The optimal η and ξ would be $\xi_\mu(s) = -H_\mu(\hat{s}_\mu)$ and $\eta_\mu(s) = H_\mu(s) - H_\mu(\hat{s}_\mu)$. Plugging this into the minimization problem we get:

$$\begin{aligned} &\min_{\mu\geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left(H_\mu(s) - H_\mu(\hat{s}_\mu)\right) ds \\ =&\min_{\mu\geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left\{\widetilde{F}_H(s) - \widetilde{F}_H(\hat{s}_\mu) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(\hat{s}_\mu) - \zeta\widetilde{F}_H(\hat{s}_\mu)\right)\right]\right\} ds \end{aligned}$$

The first order condition for this problem is:

$$\Gamma(\hat{s}_\mu) \equiv (1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(\hat{s}_\mu) - \zeta\widetilde{F}_H(\hat{s}_\mu)\right)\right] ds \geq 0$$

with complementary slackness.

$$\frac{\partial\Gamma(s^*)}{\partial s^*} = (s^* - s_L) f_H(s^*)\left(\frac{f_L(s^*)}{f_H(s^*)} - \zeta\right).$$

By definition of $\hat{s}_\mu$, $\frac{f_L(s^*)}{f_H(s^*)} - \zeta = -\frac{1}{\mu}$. So, $\frac{\partial \Gamma(s^*)}{\partial s^*} = -(s^* - s_L)\frac{f_H(s^*)}{\mu} < 0$. And $\Gamma(s_L) = (1-\zeta)\phi > 0$. Then, if there exists a solution for $\Gamma(s^*) = 0$, the solution is unique and it satisfies

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(s^*) - \zeta\widetilde{F}_H(s^*)\right)\right] ds = 0$$

Otherwise,

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(s^*) - \zeta\widetilde{F}_H(s^*)\right)\right] ds > 0$$

for all $s \in [s_L, s_H]$ and $s^* = s_H$.

If $s^* < s_H$ then $\mu > 0$, $\hat{s}_\mu = s^*$, and

$$\mathcal{L}^* = \int_{s_L}^{s^*} \left[\widetilde{F}_H(s) - \widetilde{F}_H(s^*)\right] ds.$$

If $s^* = s_H$ then $\mu = 0$, $s^*_\mu = s_H$, and

$$\mathcal{L}^* = \int_{s_L}^{s_H} \widetilde{F}_H(s) ds.$$

To complete the proof note that $\int_{s_L}^{s} x(j)dj = s - s_L$ for $s \in [s_L, s^*]$ and $\int_{s_L}^{s} x(j)dj = 0$ for $s \in [s^*, s_H]$ achieves the value $\mathcal{L}^*$ and it is feasible, and must be optimal for the original problem. □

Proof. Given Claim 1 that the optimal security design under Arrow securities has two tranches, the liquid tranche $y_{1t}(s)$ and illiquid tranche $y_{2t}(s)$.

$$y_{1t}(s) = \phi + s_L + (s - s_L)\mathbb{I}(s \le \delta),$$
$$y_{2t}(s) = (s - s_L)\mathbb{I}(s > \delta).$$

The equilibrium is solved by the following two equations, representing the incentive constraint of an owner with high quality collateral and the Euler equation for the asset price. In the incentive constraint,

$$z\left[\phi + s_L + \int_{s_L}^{\delta} (s - s_L)\, dF_\lambda(s)\right] - \left[\phi + s_L + \int_{s_L}^{\delta} (s - s_L)\, dF_H(s)\right] \ge 0$$

One can easily verify that the left hand side of the incentive constraint is decreasing in δ as long as the monotone likelihood ratio assumption holds. This confirms the conjecture of the optimal security design.

The Euler equation for the asset price is

$$\phi = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s - s_L) dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{\delta}^{s_H} (s - s_L)\, dF_H(s)\right]$$

The equilibrium value of δ is determined by

$$0 = \Gamma(\delta) = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s - s_L) dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{\delta}^{s_H} (s - s_L)\, dF_H(s)\right]$$
$$- \frac{s_L + \int_{s_L}^{\delta} (s - s_L) dF_H(s) - z\left[s_L + \int_{s_L}^{\delta} (s - s_L) dF_\lambda(s)\right]}{z-1}$$

$$\Gamma(s_L) = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s-s_L)dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{s_L}^{s_H}(s-s_L)\,dF_H(s)\right] + s_L > 0$$

$$\begin{aligned}
\Gamma'(\delta) &= \frac{\beta(z-1)(1-\lambda)\delta f_H(\delta)}{1-\beta z} - \frac{\delta f_H(\delta) - z\delta\left[\lambda f_L(\delta) + (1-\lambda)f_H(\delta)\right]}{z-1} \\
&= \delta f_H(\delta)\left\{\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1-z\left[\lambda f_L(\delta)/f_H(\delta) + (1-\lambda)\right]}{z-1}\right\} \\
&= \delta f_H(\delta)\left\{\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1-z(1-\lambda)}{z-1} + \frac{z\lambda f_L(\delta)/f_H(\delta)}{z-1}\right\}
\end{aligned}$$

If $\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1-z(1-\lambda)}{z-1} < 0$, there exists a unique δ^* such that $\Gamma'(\delta) > 0$ if and only if $\delta < \delta^*$. There exists a most one solution for the equation $\Gamma(\delta) = 0$.

$$\begin{aligned}
\Gamma(s_H) &= \frac{\beta z}{1-\beta z}\left[(1-\lambda)\mathbb{E}_H s + \lambda \mathbb{E}_L s\right] - \frac{\left[1-(1-\lambda)z\right]\mathbb{E}_H s - z\lambda\mathbb{E}_L s}{z-1} \\
&= \mathbb{E}_H s\left[\frac{\beta z}{1-\beta z}(1-\lambda) - \frac{1-(1-\lambda)z}{z-1} + \left(\frac{\beta z}{1-\beta z}\lambda + \frac{z\lambda}{z-1}\right)\frac{\mathbb{E}_L s}{\mathbb{E}_H s}\right].
\end{aligned}$$

The condition for there to be a unique $\delta \in (s_L, s_H)$ in equilibrium is

$$\begin{aligned}
\frac{\mathbb{E}_L s}{\mathbb{E}_H s} &\leq -\frac{\left(\frac{\beta z}{1-\beta z} + \frac{z}{z-1}\right)(1-\lambda) - \frac{1}{z-1}}{\left(\frac{\beta z}{1-\beta z} + \frac{z}{z-1}\right)\lambda} \\
&= 1 - \frac{z-1}{\lambda z\,(1-\beta)}.
\end{aligned}$$

□

A.5 Proof of Proposition 5

Proof. To be completed. □