Choosing to Disagree in Financial Markets

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February 19, 2019

Abstract

The rational expectations paradigm restricts the subjective beliefs of investors to align with the objective distribution. We relax this constraint and analyze how investors optimally choose their subjective beliefs about the information contained in their private signals and in prices. We show that investors systematically choose to deviate from rational expectations. In any symmetric equilibrium, investors optimally exhibit overconfidence in their private information but dismiss the information in prices. However, when aggregate risk aversion is sufficiently low, symmetric equilibria do not exist. Instead, there arises an asymmetric equilibrium in which investors endogenously separate into (i) fundamental investors, who also ignore the information in prices, and (ii) "technical" traders, who overweight the information in prices. Relative to the corresponding rational expectations equilibrium, these equilibria feature higher (i) return volatility, (ii) price informativeness, (iii) trading volume, and (iv) return predictability. Finally, such deviations by informed investors improve the welfare of liquidity traders under the objective distribution.

JEL Classification: D8, G1

Keywords: difference of opinions, optimal expectations, overconfidence.

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1 Introduction

The traditional approach to modeling beliefs in economics and finance is to assume rational expectations i.e., an agent's subjective beliefs are forced to coincide with the objective distribution. However, a growing body of empirical evidence establishes that people do not behave this way.[1] For example, a robust finding in both the economics and psychology literatures is that people tend to be overconfident about their own views, but dismissive of others. Moreover, models that incorporate differences of opinions or behavioral biases (e.g., overconfidence, dismissiveness, cursedness) have been useful in understanding several stylized facts about financial markets that are difficult to reconcile with the standard rational expectations framework. However, such models typically provide little guidance in understanding when such deviations arise because agents are constrained to exhibit such behavior by assumption.

In this paper, we ask whether investors would exhibit rational expectations if they could choose their subjective beliefs. If they choose to deviate from rational expectations, do empirically-observed behavioral biases like overconfidence and dimissiveness arise endogenously? If so, under what conditions? To explore the answers to these questions, we extend an otherwise standard model in which a continuum of symmetric, privately informed investors trade a single risky asset against noise traders in a perfectly competitive market (à la Hellwig (1980)). However, instead of assuming that investors exhibit rational expectations or manifest a specific behavioral bias, we allow investors to optimally choose how to interpret the information in their private signals and in prices. Specifically, given the actions of all other investors, each agent chooses her perceived precision of her private signal and her perceived precision of the information contained in the price.

An investor's interpretation of the informativeness of these signals determines her subjective beliefs about fundamentals and the potential speculative gains from trade. These, in turn, determine her anticipated utility, i.e., the expected future utility she receives from optimally trading the risky asset given her subjective beliefs. Each investor optimally chooses her subjective beliefs to maximize her anticipated utility net of the costs of deviating from rational expectations.[2] As in the "optimal expectations" framework of Brunnermeier and Parker (2005), we focus on the implicit costs introduced under the objective distribution when agents use their subjective beliefs to choose their position in the risky asset. By deviating from rational expectations, an investor can increase her *anticipated* utility, for instance, by believing that her signal is more informative than it actually is. Such a deviation, how-

[1] See Bénabou and Tirole (2016) and Barberis (2018) for recent surveys of the literatures in economics and finance, respectively.

[2] One can view the benchmark rational expectations approach as a special case of our model in which the cost of deviations in beliefs is infinite.

ever, carries a cost: the investor's position will be suboptimal from the perspective of the objective distribution and hence the average *ex-post* utility will be lower.[3] For robustness,we also consider a more general approach which imposes a direct utility cost of deviating from the rational expectations benchmark.

We find that investors optimally choose to deviate from rational expectations in systematic ways. In any symmetric equilibrium, we show that investors endogenously choose to be overconfident about their private signals but, in addition, also choose to underreact to the information in prices (consistent with models of differences of opinions or cursedness).[4] The key to understanding this result is that, in a speculative setting, each investor prefers not only to be better informed, but also to have information that is not shared by others, i.e., she *chooses to disagree* with other investors. Possessing more precise information about the asset's payoff implies that the investor faces less uncertainty about the trading opportunity and, consequently, can trade more aggressively. Together, this implies that interpreting a signal as being more precise (than it actually is) increases an investor's anticipated utility — we refer to this as the **information channel**. This effect applies to both private signals and the information contained in the price. However, a price that is more informative necessarily implies a price which tracks fundamentals more closely: this reduces the potential gains from speculating against other investors.[5] As a result, believing that the price is more informative also reduces anticipated utility — we term this the **speculative channel**. We show that while the information channel leads investors to overweight their private information the offsetting effects of the two channels leads investors to optimally underweight the information contained in prices.

We also show that allowing investors to choose their beliefs can also change the very nature of the observed equilibrium. When investors choose their beliefs under optimal expectations, we find that a unique, symmetric equilibrium exists when investor risk-aversion is sufficiently large. In this symmetric equilibrium, investors fully ignore the information in prices: the speculative channel dominates the information channel.[6] However, when risk

[3]More concretely, an investor's anticipated utility is her expected utility from her (distorted) trading strategy, under the subjective distritbution, while her average ex-post utility is her expected utility from the same (distorted) trading strategy, under the objective distribution. The investor's optimal choice of subjective beliefs trades off these two measures of expected utility.

[4]An overconfident investor, who believes her signal is more informative than it is, necessarily puts relatively less weight on the information in the price when updating her beliefs. We show that, in addition to such an effect, the investor optimally chooses to believe that the price is less informative than it objectively is, which further amplifies this distortion in her updating process.

[5]In the limit, if the price is perfectly informative, there are no speculative gains from trading. There may still be a gain from the risk premia earned by holding the risky asset, which reflects the compensation for bearing aggregate risk.

[6]In our benchmark model, agents exhibit behavior akin to the "fully cursed" agents of Eyster et al. (2018) or that found in a pure "differences of opinion" framework (e.g., found in Banerjee et al. (2009), Banerjee

aversion is sufficiently low, we also show that a symmetric equilibrium cannot exist. Instead, there can arise an asymmetric equilibrium in which a majority of investors objectively interpret their private signals and ignore the price, but a minority overweight *both* their private signals and the information in the price. In such cases, the price is sufficiently informative so that for a small subset of investors, the information channel dominates the speculative channel: the benefit from the reduction in posterior variance outweighs the cost from lower trading gains, especially since most other investors choose to ignore the information in prices. Such asymmetric equilibria help us understand why ex-ante symmetric investors can endogenously choose to separate into fundamental investors (who also ignore the information in prices) and "technical" traders, who overweight the information in prices. More generally, such endogenous investor heterogeneity does not commonly arise in standard models, because investors are simply assumed (constrained) to exhibit rational expectations or specific behavioral biases.[7] As such, our analysis may help shed light on why some systematic deviations from rational expectations are prevalent in certain settings but not others, and why we sometimes observe seemingly opposing biases in the same market.

The implications of subjective belief choice for common return-volume characteristics are broadly consistent with the empirical evidence. For instance, we show that across both the symmetric and asymmetric equilibria, our model generates higher return predictability, higher trading volume, and higher price informativeness than the rational expectations benchmark.[8] Intuitively, these results are driven by the fact that investors (in the aggregate) under-react to the price and over-react to their private information. We also show that return volatility is higher than in the rational expectations equilibrium when price informativeness is sufficiently high, but lower otherwise.

Finally, we characterize investor welfare in our setting and compare it to the rational expectations benchmark. Note that the expected utility of investors depends on the reference beliefs.[9] We take a conservative stance on welfare by (i) evaluating expected utility under

(2011)). Agents do not distort their beliefs in their private signal but fail to condition on the information found in prices. Under alternative frameworks, we show that overconfidence arises along with dismissiveness ("cursedness").

[7] In Brunnermeier and Parker (2005), general equilibria are analyzed in which some investors choose to be optimistic about an asset's payoff. If such behavior distorts the price sufficiently upward, some agents choose to be pessimistic about the asset's payoff which is equivalent to be being *optimistic* about the potential trading gains. Effectively, beliefs are distorted for the same reason (though in different directions). In our asymmetric equilibrium, ex-ante well-being is the same across both types, but the channels through which this arises are distinct across the two types of investor.

[8] The higher return predictability in our setting arises due to an underreaction to price information, i.e., the aggregation of investors' private signals.

[9] Given that they choose to deviate from rational expectations, the investors' anticipated utility net of costs must be higher than under the rational expectations equilibrium (according to their subjective beliefs). However, from the perspective of a social planner who is restricted to objective beliefs, investors' expected

objective beliefs and (ii) focusing on the objective disutility that investors incur due to their subjective choices. Using the objective measure, informed investors are always worse off than in a rational expectations equilibrium. However, we find that the noise traders are always better off under optimal expectations because they incur a lower price impact on their trades relative to the rational expectations equilibrium. Surprisingly, we find that overall welfare (where we sum the objective expected utility across informed and liquidity traders) can be higher under optimal expectations. We show that this depends on the aggregate risk aversion: welfare is lower under the rational expectations equilibrium when risk aversion is sufficiently high, but higher otherwise.

Our analysis suggests that focusing on the rational expectations benchmark is restrictive: when given a choice, investors optimally deviate from holding such objective beliefs. Moreover, our results suggests that allowing for such deviations may be helpful in understanding observable patterns in returns and trading volume. Our analysis also highlights a potential advantage of allowing for belief choice over the standard approach in the behavioral economics literature which assumes the existence of such deviations (e.g., over-confidence, dismissiveness, cursedness) and then studies their impact in isolation. The equilibria which arise in our setting suggest that, in a richer framework, such deviations can arise together and interact with each other in novel ways, as in our symmetric equilibria. As importantly, even in settings where investors are ex-ante identical, the analyzed asymmetric equilibria highlights that allowing for belief choice can lead to endogenous heterogeneity in investor behavior in equilibrium where subsets of investors exhibit different systematic deviations in the same economy.

We conclude this section by reviewing the related literature. Section 2 introduces the model setup and financial market equilibrium, given investor beliefs. Section 3 studies the tradeoffs associated with deviating from rational expectations. Section 4 studies a setting in which the investor chooses beliefs only about her private signal while in Section 5, we solve the more general setting and explore the models' implications. Section 6 analyzes the impact on welfare implicationsand Section 7 concludes. Proofs and extensions can be found in the Appendix.

Related literature

Bénabou and Tirole (2016) survey the now extensive literature on belief choice (e.g., Akerlof and Dickens (1982), Loewenstein (1987), Caplin and Leahy (2001), Eliaz and Spiegler (2006)). They emphasize that subjective beliefs can fill both psychological and functional

utility is lower when they deviate from rational expectations because their demand for the risky asset is suboptimal.

needs for individuals. Our paper is related to the former channel, in which beliefs directly enter the agent's utility. We focus on the special class of models which utilize anticipatory utility: agents optimally manipulate their beliefs in order to improve their perception of the future, trading off such benefits against the costly mistakes they induce. Our model is most closely related to Brunnermeier and Parker (2005) and Caplin and Leahy (2018).

Brunnermeier and Parker (2005)'s influential work proposes a model of optimal expectations in which agents maximize their expected wellbeing. In contrast to expected utility, agents also derive utility from their expectations of future utility flows, i.e., anticipatory utility. In particular, agents are happier (sadder) if they overestimate (underestimate) the probabilities of states of the world in which their investments pay off. Our paper focuses on such a setting: investors choose their beliefs about the quality of their information sources to trade off higher anticipated utility against the utility loss of deviating from the objective distribution. This paper (along with Brunnermeier et al. (2007)) apply this framework to understand risk-taking, preference for skewness, portfolio under-diversification and consumption/savings patterns. In contrast, we extend this behavior to a setting with asymmetric information (similar to Hellwig (1980)) where investors form beliefs not only about exogenous variables (fundamentals, signals) but also endogenous objects (equilibrium prices). Caplin and Leahy (2018) also consider a setting in which agents get utility from their subjective beliefs. However, the cost of choosing distorted beliefs depends upon the distance between the subjective and objective distributions. They show that subjective belief choice can help explain a number of behavioral biases, including as optimism, procrastination, confirmation bias, polarization and the endowment effect. Similarly, we consider general cost functions and show how endogenous belief choice can give rise to both over-confidence and under-reaction to price information (e.g., cursedness or dismissiveness).

Our paper contributes to two strands of the literature studying the financial market impact of deviations from rational expectations. The first strand focuses on differences of opinion, whereby investors "agree to disagree" about the joint distribution of payoffs and signals and therefore, incorrectly condition on the information in prices (e.g., Harrison and Kreps (1978), Kandel and Pearson (1995), Banerjee et al. (2009) and Banerjee (2011)). Other approaches that lead investors to discount the information in prices include models that feature cursedness (e.g., Eyster et al. (2018)) and costly learning from prices (e.g., Vives and Yang (2017))).[10] The second, related strand focuses on the impact of overconfidence: specifically, settings in which agents believe their private information is more precise than it

[10]While Eyster et al. (2018) show that cursedness can generate distinct predictions from a model of differences of opinions (which they term dismissiveness) when there is imperfect competition and no noise trading, our setting features perfectly competitive markets and noise in prices, and so cursedness and differences of opinions are effectively isomorphic.

objectively is (e.g., Odean (1998a); Daniel et al. (1998, 2001)). These two strands highlight how such deviations can explain a number of stylized facts about financial markets that are difficult to reconcile in the rational expectations framework, including excess trading volume and return predictability. In contrast to the existing literature, however, we do not assume that investors exhibit differences of opinions or overconfidence. Instead, investors are allowed to choose their beliefs, and importantly, exhibiting rational expectations is within their choice set. We show that, given the choice, investors optimally choose to deviate from rational expectations, and we characterize conditions under which they endogenously choose to exhibit both overconfidence and differences of opinions.

2 Model setup

Asset payoffs. There are two securities. The gross return on the risk-free security is normalized to one. The terminal payoff (fundamental value) of the risky security is F, which is normally distributed with mean m and prior precision τ i.e.,

$$F \sim \mathcal{N}\left(m, \frac{1}{\tau}\right). \tag{1}$$

We denote the market-determined price of the risky security by P, and the aggregate supply of the risky asset by z, where

$$z \sim \mathcal{N}\left(0, \frac{1}{\tau_z}\right). \tag{2}$$

Information. There is a continuum of investors, indexed by $i \in [0,1]$. Before trading, each investor is endowed with a private signal s_i, where

$$s_i = F + \varepsilon_i \qquad \varepsilon_i \sim N\left(0, \frac{1}{\tau_e}\right) \tag{3}$$

and ε_i is independent and identically distributed across investors so that $\int \varepsilon_i \, di = 0$. Moreover, investors can update their beliefs about F by conditioning on the information in the price P.

Beliefs and preferences. Each investor i is endowed with initial wealth W_0 and zero shares of the risky security, and exhibits CARA utility with coefficient of absolute risk aversion γ over terminal wealth W_i:

$$W_i = W_0 + x_i (F - P), \tag{4}$$

where x_i denotes her demand for the risky security. We allow each investor to interpret the quality of the information in both her private signals and the price subjectively. Specifically, we assume that investor i believes that the noise in her private signal is given by[11]

$$\varepsilon_i \sim_i \mathcal{N}\left(0, \frac{1}{\delta_{e,i}\tau_e}\right), \tag{5}$$

and believes the distribution of the asset's aggregate supply, which as we show below is proportional to the noise in the signal investors can extract from the price, is given by

$$z \sim_i \mathcal{N}\left(0, \frac{1}{\delta_{z,i}\tau_z}\right). \tag{6}$$

In what follows, we will denote the expectation and variance of random variable X under investor i's beliefs about the information environment, i.e., $\delta_{e,i}$ and $\delta_{z,i}$ by $\mathbb{E}_i[X]$ and $\mathrm{var}_i[X]$, respectively. As is standard, we will denote the expectation and variance of X under the true (objective) distribution by $\mathbb{E}[X]$ and $\mathrm{var}[X]$, respectively.

The parameters $\delta_{e,i}, \delta_{z,i} \in [0, \infty]$ reflect the degree to which investor i over- or under-estimates the precision of the private signal and aggregate noise, respectively. When $\delta_{e,i} = \delta_{z,i} = 1$, investor i's beliefs satisfy **rational expectations**: her beliefs coincide with the objective distribution of the underlying shocks. On the other hand, when $\delta_{e,i} > 1$, investor i is overconfident about her private signal: she believes her private signal is more informative than it objectively is and she overweights it in forming her beliefs. The opposite is true when $\delta_{e,i}$ is less than one. Similarly, when $\delta_{z,i} > 1$ ($\delta_{z,i} < 1$), investor i believes the price to be more informative (less informative, respectively) about fundamentals.[12] We assume that

[11] Here $\sim_i$ denotes "distributed as, according to investor i's beliefs."

[12] One interpretation of our modeling choice is that we allow for subjective beliefs about the volatility of noise trading in the economy. However, the more general feature we capture through $\delta_{z,i}$ is the subjective interpretation of the precision of the price signal. An alternative interpretation is that $\delta_{z,i}$ translates investor i's beliefs about the quality of other investors' private information into her beliefs about the informativeness of the price. To see how this would work, consider a specification in which investor i's beliefs about investor j's signals are of the form:

$$\varepsilon_j =_i \alpha F + \sqrt{1-\alpha^2} u_j, \text{ where } \alpha \in (-1, 1), \text{ and } u_j \sim_i \mathcal{N}\left(0, \frac{1}{\tau_e}\right). \tag{7}$$

In this case, one can show that from the perspective of investor i's beliefs, $\delta_{z,i} = (1+\alpha)^2$ — when $\alpha > 0$ ($\alpha < 0$), the price is perceived to be more (less) informative about fundamentals than it actually is.

such deviations from the objective distribution impose a utility cost, denoted by $C(\delta_{e,i}, \delta_{z,i})$.

Given her choice of subjective beliefs, each investor optimally chooses her position in the risky security. Thus, optimally chosen subjective beliefs maximize her anticipated utility, net of cost $C(\cdot)$. Formally, denote investor i's optimal demand, given her beliefs, by:

$$x_i^*(\delta_{e,i}, \delta_{z,i}) = \arg\max_{x_i} \mathbb{E}_i\left[-\exp\left\{-\gamma x_i (F-P) - \gamma W_0\right\} \middle| s_i, P\right]. \tag{8}$$

and denote investor i's **anticipated utility** by

$$AU_i(\delta_{e,i}, \delta_{z,i}) \equiv \mathbb{E}_i\left[\mathbb{E}_i\left[-\exp\left\{-\gamma x_i^* (F-P) - \gamma W_0\right\} \middle| s_i, P\right]\right]. \tag{9}$$

Then, investor i optimally chooses subjective beliefs $\delta_{e,i}$ and $\delta_{z,i}$ to maximize:

$$\max_{\delta_{e,i},\delta_{z,i}} AU_i(\delta_{e,i}, \delta_{z,i}) - C(\delta_{e,i}, \delta_{z,i}). \tag{10}$$

We assume that the cost function $C(\cdot)$ penalizes deviations from the objective distribution (i.e., from $\delta_{e,i} = \delta_{z,i} = 1$) and is well-behaved as defined below.

Definition 1. A cost function $C(\delta_{e,i}, \delta_{z,i})$ is **well-behaved** if $C(1,1) = \frac{\partial C}{\delta_{e,i}}(1,1) = \frac{\partial C}{\delta_{z,i}}(1,1) = 0$, and C is strictly convex (i.e., its global minimum is at $(1,1)$).

While many of our results apply to general cost functions, our main analysis focuses on a closely related setting: the "optimal expectations" framework introduced in Brunnermeier and Parker (2005). In this setting, the cost each investor incurs by distorting her subjective beliefs is the reduction in expected utility (under the *objective* distribution) when her position in the risky asset, $x_i^*(\delta_{e,i}, \delta_{z,i})$, is determined by the chosen *subjective* distribution. As is well-established, any deviation from the rational expectations benchmark ($\delta_{e,i} = \delta_{z,i} = 1$) is objectively inefficient: the investor is over- or under-weighting the information she receives.

Definition 2. Investor i exhibits **optimal expectations** if her beliefs maximize:

$$\max_{\delta_{e,i},\delta_{z,i}} AU_i(\delta_{e,i}, \delta_{z,i}) + \mathbb{E}\left[-\gamma \exp\left\{-\gamma x_i^*(\delta_{e,i}, \delta_{z,i}) \times (F-P) - \gamma W_0\right\}\right]. \tag{11}$$

As we show in the appendix, the cost of forming optimal expectations ($C_{obj}(\delta_{e,i}, \delta_{z,i})$) is well-behaved, where we define

$$C_{obj}(\delta_{e,i}, \delta_{z,i}) \equiv \begin{array}{c} \mathbb{E}\left[-\gamma \exp\left\{-\gamma x_i^*(\delta_{e,i}, \delta_{z,i}) \times (F-P) - \gamma W_0\right\}\right] \\ -\mathbb{E}\left[-\gamma \exp\left\{-\gamma x_i^*(1,1) \times (F-P) - \gamma W_0\right\}\right] \end{array}. \tag{12}$$

2.1 Financial market equilibrium

We first solve for the financial market equilibrium, taking investors' chosen subjective beliefs as given. We consider equilibria in which the price P is a linear combination of fundamentals F and noise trading z, and conjecture that observing the price is equivalent to observing a signal of the form:

$$s_p = F + \beta z. \tag{13}$$

The variance of this signal is $\tau_p^{-1} = \beta^2/\tau_z$, and β is a constant determined in equilibrium. Given investor i's subjective beliefs $\delta_{e,i}$ and $\delta_{z,i}$, and conditional on her observed signals, s_i and s_p, investor i's posterior subjective beliefs are given by:

$$F|s_i, s_p \sim_i \mathcal{N}\left(\mu_i, \frac{1}{\omega_i}\right), \text{ where} \tag{14}$$

$$\mu_i \equiv \mathbb{E}_i\left[F|s_i, s_p\right] = m + A_i\left(s_i - m\right) + B_i\left(s_p - m\right), \tag{15}$$

$$\omega_i \equiv \frac{1}{\text{var}_i\left[F|s_i, s_p\right]} = \frac{\tau}{1 - A_i - B_i}, \text{ and} \tag{16}$$

$$A_i \equiv \frac{\delta_{e,i}\tau_e}{\tau + \delta_{e,i}\tau_e + \delta_{z,i}\tau_p}, \text{ and } B_i \equiv \frac{\delta_{z,i}\tau_p}{\tau + \delta_{e,i}\tau_e + \delta_{z,i}\tau_p}. \tag{17}$$

The optimal demand for investor i, given her subjective beliefs, is given by

$$x_i^* = \frac{\mathbb{E}_i\left[F|s_i, P\right] - P}{\gamma \text{var}_i\left[F|s_i, P\right]} = \frac{\omega_i}{\gamma}\left(\mu_i - P\right). \tag{18}$$

Equilibrium prices are determined by market clearing:

$$\int_i x_i^* di = z, \tag{19}$$

which implies:

$$P = \frac{\int_i \omega_i \left\{m + A_i\left(F - m\right) + B_i\left(s_p - m\right)\right\} di}{\int_i \omega_i di} - \frac{\gamma}{\int_i \omega_i di} z \tag{20}$$

This verifies our conjecture for functional form of the price and we can write

$$\beta = \frac{-\gamma}{\int_i \omega_i A_i di} = -\frac{\gamma}{\tau_e \int_i \delta_{e,i} di}. \tag{21}$$

The above results are summarized in the following lemma.

Lemma 1. *Given investor i's subjective beliefs $\delta_{e,i}$ and $\delta_{z,i}$, there always exists a unique,* *linear, financial market equilibrium in which*

$$P = m + \Lambda\left(s_p - m\right), \text{ where } \Lambda = \frac{\int_i \delta_{e,i}\tau_e + \delta_{z,i}\tau_p di}{\int_i \tau + \delta_{e,i}\tau_e + \delta_{z,i}\tau_p di}, \tag{22}$$

$s_p = F + \beta z$, $\tau_p = \tau_z/\beta^2$, *and* $\beta = -\frac{\gamma}{\tau_e \int_i \delta_{e,i} di}$. *When subjective belief choices are symmetric (i.e.,* $\delta_{e,i} = \delta_e$ *and* $\delta_{z,i} = \delta_z$ *for all i), then the price is given by:*

$$P = m + \Lambda\left(s_p - m\right), \text{ where } s_p = \left(F - \tfrac{\gamma}{\tau_e \delta_e} z\right), \Lambda = \frac{\delta_e\tau_e + \delta_z\tau_p}{\tau + \delta_e\tau_e + \delta_z\tau_p}, \text{ and } \tau_p = \frac{\tau_z\tau_e^2\delta_e^2}{\gamma^2}. \tag{23}$$

As the above lemma makes apparent, the choice of investor beliefs affect equilibrium prices through two channels. First, an increase in the perceived precision of private signals (higher $\delta_{e,i}$) increases the signal to noise ratio of the signal s_p (since $|\beta|$ is decreasing in $\delta_{e,i}$). Investors trade more aggressively on their private information which is then reflected in the objective quality of the information in the price. Second, an increase in the perceived precision of either private signals (i.e., higher $\delta_{e,i}$) or price information (i.e., higher $\delta_{z,i}$) increases the sensitivity of the price to fundamentals (F) through Λ. These channels interact to affect a number of empirically observable features of the financial market equilibrium, which we characterize next.

2.2 Return and volume characteristics

Given the financial market equilibrium in Lemma 1, we characterize how the underlying parameters of the model, in combination with investors' choice of beliefs, affect observable return and volume characteristics and the degree to which prices reflect information. Since the risk-free security is the numeraire, the (net) return on it is zero. Consequently, the (dollar) return on the risky security is given by

$$R \equiv F - P. \tag{24}$$

Furthermore, because the risky security is in zero net supply, the unconditional expected return is zero i.e., $\mathbb{E}[R] = 0$. However, conditional on the price, the expected return can be expressed as:

$$\mathbb{E}\left[F - P \middle| P\right] = m + \theta\left(P - m\right), \text{ where } \theta \equiv \frac{\text{cov}\left(F - P, P\right)}{\text{var}\left(P\right)}. \tag{25}$$

Here θ reflects the degree to which the returns are predictable and, as such, we refer to it as the return predictability coefficient. The unconditional variance in returns is given by

$$\sigma_R^2 = \text{var}\left(F - P\right), \tag{26}$$

while the conditional variance in returns is characterized by

$$\text{var}\left(F - P|P\right) = \tau_p^{-1}. \tag{27}$$

Note that the conditional variance in returns is inversely related to one measure of price informativeness, as τ_p reflects how precise the price signal is about fundamentals F. Finally, since investors start without an endowment of the risky security, trading volume in our economy can be characterized as

$$\mathcal{V} \equiv \int_i |x_i^*|\, di. \tag{28}$$

Given investor beliefs, the following result describes how these return-volume characteristics depend on the underlying parameters.

Lemma 2. *Consider the unique financial market equilibrium described in Lemma 1. Then,*

(ii) the unconditional variance in returns is $\sigma_R^2 = \frac{(1-\Lambda)^2}{\tau} + \frac{\Lambda^2\beta^2}{\tau_z}$

(iii) the return predictability coefficient is $\theta = \frac{1}{\Lambda}\left(\frac{1/\tau}{\beta^2/\tau_z + 1/\tau} - \Lambda\right)$,

(iv) price informativeness is $\tau_p = \tau_z/\beta^2$, *and*

(v) expected trading volume is

$$\mathbb{E}\left[\mathcal{V}\right] = \int_i \frac{\omega_i}{\gamma}|\mu_i - P|\, di = \int_i \frac{\omega_i}{\gamma}\sqrt{\frac{2}{\pi}\left(\frac{A_i^2}{\tau_e} + \frac{\left(B_i - \Lambda\right)^2\beta^2}{\tau_z} + \frac{\left(A_i + B_i - \Lambda\right)^2}{\tau}\right)}di, \tag{29}$$

where ω_i, A_i,B_i, β *and* Λ *are defined in (16)-(17) and Lemma 1.*

To provide intuition for the dependence of these equilibrium characteristics on the underlying parameters, we make use of the signal to noise ratio (or Kalman gain) for the price signal s_p, which can be written as

$$\kappa \equiv \frac{\text{var}\left(F\right)}{\text{var}\left(s_p\right)} = \frac{1/\tau}{\beta^2/\tau_z + 1/\tau} = \frac{\tau_p}{\tau + \tau_p}. \tag{30}$$

First, note that an increase in the price sensitivity Λ has two offsetting effects on return volatility. On the one hand, when the price is more sensitive to s_p, it reflects fundamentals more closely, and this decreases volatility (through the $\frac{(1-\Lambda)^2}{\tau}$ term). On the other hand, a higher Λ also implies that the price is more sensitive to shocks to the asset supply, which increases volatility (through the $\frac{\Lambda^2\beta^2}{\tau_z}$ term). The first effect dominates when the price sensitivity Λ is lower than the signal to noise ratio κ (i.e., $\Lambda < \kappa$), while the latter effect dominates when price sensitivity is higher.

Second, note that the return predictability coefficient is positive when the signal to noise ratio is higher than the price sensitivity (i.e., $\theta > 0$). In this case, prices exhibit drift — a higher price today predicts higher future returns. On the other hand, when the signal to noise ratio is lower than Λ, prices exhibit reversals. Comparing the expression of κ above to Λ in (22), prices cannot exhibit drift unless investors under-react to price information (i.e., $\delta_{z,i} < 1$). Conversely, when investors correctly interpret the precision of price information, the prices exhibit reversals (i.e., $\delta_{z,i} = 1$ implies $\Lambda > \kappa$).[13] In particular, prices exhibit reversals when investors exhibit rational expectations. Such reversals arise because the aggregate supply of the asset is subject to transitory shocks and investors are risk-averse. Moreover, we note that holding fixed the signal to noise ratio κ, an increase in price sensitivity Λ decreases return predictability: even though an increase in Λ increases the covariance of F and P, this effect is swamped by the increase in the variance of the price.

Third, price informativeness, τ_p, naturally decreases in the magnitude of β — when investors have less private information, the price is more sensitive to aggregate supply shocks.. Finally, note that volume reflects the cross-sectional variation across investor valuations (i.e., μ_i), scaled by their posterior variance (i.e., ω_i^{-1}). The variation is driven by three channels: (i) the weight each investor's beliefs place on the noise in their private signals (i.e., the $\frac{1}{\tau_e}$ term), (ii) the relative weight on the noise in prices (i.e., the $\frac{\beta^2}{\tau_z}$ term) and (iii) the relative weight on the true fundamental value (i.e., the $\frac{1}{\tau}$ term). Note that the last term is absent in symmetric equilibria, since $A_i + B_i = \Lambda$ in this case. However, in asymmetric equilibria, this final term reflects the variation in valuations due to asymmetric reaction to private signals and the information in prices (see Section 5.2).

3 Anticipated utility and subjective beliefs

With the financial market equilibrium established, we can now characterize the optimal subjective beliefs of an investor. Importantly, we assume that each investor takes as given the subjective belief distortion chosen by other investors: she does not assume that other investors hold rational expectations.

[13]This is because, as we show below, in equilibrium investors never choose to set $\delta_{e,i} = 0$.

Given the optimal demand for the risky asset (18), anticipated utility is given by

$$AU_i(\delta_{e,i}, \delta_{z,i}) = \mathbb{E}_i \left[-\exp\left\{ -\frac{(\mathbb{E}_i[F|s_i, P] - P)^2}{2\text{var}_i[F|s_i, P]} \right\} \right]. \tag{31}$$

Moreover, given the characterization of the equilibrium price in Lemma 1, investor i's beliefs about the conditional return are given by:

$$\mathbb{E}_i[\mathbb{E}_i[F|s_i, P] - P] = m - m = 0, \text{ and} \tag{32}$$

$$\text{var}_i[\mathbb{E}_i[F|s_i, P] - P] = \text{var}_i[F - P] - \text{var}_i[F|s_i, P], \tag{33}$$

where the first equality follows from the law of iterated expectations and the second equality follows from the law of total variance.[14] With this in mind, the above expectation reduces to

$$AU_i(\delta_{e,i}, \delta_{z,i}) = -\sqrt{\frac{\text{var}_i[F|s_i, P]}{\text{var}_i[F - P]}}. \tag{35}$$

From this, we derive the following result.

Lemma 3. *Anticipated utility is always (weakly) increasing in $\delta_{e,i}$; it is strictly increasing as long as $\delta_{z,i} > 0$. Anticipated utility is non-monotonic in $\delta_{z,i}$: there exists some $\bar{\delta}$ such that for all $\delta_{z,i} < \bar{\delta}$ anticipated utility is decreasing in $\delta_{z,i}$ while for all $\delta_{z,i} > \bar{\delta}$ it is increasing.*

To gain some intuition, we note that anticipated utility is simply a monotonic transformation of

$$\frac{\text{var}_i[F - P]}{\text{var}_i[F|s_i, P]} = \text{var}_i\left(\frac{\mathbb{E}_i[F - P|s_i, P]}{\sqrt{\text{var}_i[F - P|s_i, P]}}\right) \equiv \text{var}_i(SR_i) \tag{36}$$

where

$$SR_i \equiv \frac{\mathbb{E}_i[F - P|s_i, P]}{\sqrt{\text{var}_i[F - P|s_i, P]}} \tag{37}$$

is investor i's conditional Sharpe ratio, given her beliefs. When the variance of the conditional Sharpe ratio is higher, the investor expects to observe both (i) more profitable trading opportunities and (ii) the opportunity to trade more aggressively. Of course, she also faces an increased likelihood of facing the opposite scenario, but the benefit on the upside always outweighs the reduction in expected profits on the downside.[15]As a result, anticipated utility

[14]The law of total variance implies

$$\text{var}_i[F - P] = \mathbb{E}_i[\text{var}_i[F - P|s_i, P]] + \text{var}_i[\mathbb{E}_i[F - P|s_i, P]], \tag{34}$$

which in turn, implies the above expression.

[15]This arises because the trading opportunity and the investor's position act as complements - effectively,

is higher when the variance in the conditional Sharpe ratio is higher.

Intuitively, reducing the perceived uncertainty (i.e., $\text{var}_i [F - P|s_i, P]$) about the trading opportunity is valuable - if the investor has better information about the value of the asset this increases her utility. Increasing the perceived precision of the private signal (i.e., increasing $\delta_{e,i}$) has this effect and so anticipated utility increases when the investor inflates her perception of the quality of the private signal.

On the other hand, increasing the perceived precision of the price signal (i.e., increasing $\delta_{z,i}$) has two off-setting effects. First, the **information effect** of learning from prices reduces the conditional variance $\text{var}_i [F - P|s_i, P]$: the investor has better information about the asset's value which increases anticipated utility. This information effect reduces the volatility of the perceived return on the risky security, a benefit in and of itself, but it also allows the investor to scale up her trading position. Second, the **speculative effect** of believing prices are more informative decreases the perceived variance of the conditional expected return (i.e., $\text{var}_i (\mathbb{E}_i [F|s_i, P] - P)$), which lowers anticipated utility. Intuitively, when the price is more informative, it tracks fundamentals more closely and, as a result, the trading opportunity is less profitable. The overall effect of changing the perceived precision of the price signal depends on the relative magnitude of these two effects. As we show in the proof of Lemma 3, the latter effect dominates when $\delta_{z,i}$ is low, while the former effect dominates when $\delta_{z,i}$ is sufficiently high, which is what drives the non-monotonicity in $\delta_{z,i}$.

This is the key distinction between learning from private signals and learning from price information and it drives our equilibrium results below. Learning from either source is informative about fundamentals which naturally increases utility. However, learning from prices also reduces the investor's perception of the potential trading opportunity. We explore how this distinction leads to differences in investors' subjective interpretation of private signals and the information in the price in the next two sections.

4 Benchmark: Belief choice about private signals

We begin with a benchmark in which investors are forced to have objective beliefs about the price signal (i.e., we assume $\delta_{z,i} = 1$ for all i) but can choose their beliefs about the precision of their private signals. Unsurprisingly, given the intuition laid out above, we find that all investors choose to exhibit over-confidence about their private information in equilibrium.

Proposition 1. *Suppose investors have objective beliefs about the price signal i.e., $\delta_{z,i} = 1$ for all i, and the cost function $C(\delta_{e,i}, \delta_{z,i})$ is well-behaved. Then, there exists a unique*

the utility is convex in the trading opportunity (as captured by the conditional Sharpe ratio), and so an increase in the perceived variance is beneficial.

symmetric equilibrium in which all agents are over-confident about their private signal.

With objective beliefs about the informativeness of the price, Lemma 3 implies that an investor's anticipated utility strictly increases in the perceived precision of her private signal. Since the cost of setting $\delta_{e,i} = 1$ is zero (i.e., $C(1,1) = 0$) and the marginal cost of increasing $\delta_{e,i}$ at $\delta_{e,i} = \delta_{z,i} = 1$ is also zero, investors prefer to optimally choose $\delta_{e,i} > 1$ i.e., they optimally choose to be over-confident about her private signal.

Proposition 2. *Suppose investors have objective beliefs about the price signal i.e., $\delta_{z,i} = 1$ for all i, and exhibit optimal expectations, i.e., they solve (11). Then, there exists a unique equilibrium in which the optimal choice of $\delta_{e,i} = \delta_e$ satisfies:*

$$\frac{\left(\tau + \tau_p + \tau_e \delta_e \left(2 - \delta_e\right)\right)^{\frac{3}{2}}}{\left(\tau + \tau_p + \tau_e \delta_e\right)^{\frac{3}{2}}} = 2\left(\delta_e - 1\right). \tag{38}$$

Moreover, the equilibrium overconfidence parameter, δ_e, (i) increases with τ and τ_z, (ii) decreases with risk-aversion γ, and (iii) is U-shaped in τ_e.

Consistent with the intuition laid out above, equation (38) shows that in a symmetric equilibrium, $\delta_{e,i} > 1$ for all agents.[16] What drives the degree of overconfidence? As prior uncertainty falls ($\uparrow \tau$) and as the quality of the information in prices rises ($\uparrow \tau_z, \downarrow \gamma$), both the benefit and cost of being overconfident falls: overconfidence is less distortive of the investor's perceived information advantage. Interestingly, as overconfidence grows, the cost falls more quickly, and so when investors have access to better outside information, overconfidence is higher.[17] While similar logic applies with respect to the quality of the investor's private signal, increasing overconfidence directly distorts how this information is utilized. As a result, for low values of τ_e the benefit of increased overconfidence falls more quickly, which introduces a non-monotonicity in δ_e as τ_e increases.

Corollary 1. *Relative to the rational expectations equilibrium (i.e., when $\delta_{z,i} = \delta_{e,i} = 1$ for all i), the equilibrium characterized in Proposition 1 features: (i) lower return volatility, (ii) a less negative predictability coefficient, (iii) higher price informativeness, and (iv) higher expected volume.*

These implications follow naturally from the expressions in 2 and noting the fact that because $\delta_e > 1$, the price sensitivity Λ in this equilibrium is higher than in the rational

[16]The LHS of (38) is always positive, which indicates that $\delta_e > 1$.

[17]This can be seen by evaluating the numerator of equation (38).

expectations equilibrium and both are higher than the signal to noise ratio i.e.,

$$\Lambda = \frac{\delta_e \tau_e + \tau_p}{\tau + \delta_e \tau_e + \tau_p} > \Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p} > \kappa. \tag{39}$$

Our result on return predictability is consistent with Daniel et al. (1998), who argue that investor overconfidence results in price reversals. Moreover, overconfidence induces investors to trade more aggressively based on their signals. This results in more informative prices, which is consistent with empirical evidence in Hirshleifer et al. (1994); Kyle and Wang (1997); Odean (1998b); Hirshleifer and Luo (2001). Finally, consistent with the large literature on overconfidence, our model suggests that such behavior by investors can help explain the relatively high trading volume that has been extensively documented empirically.

5 Belief choices and equilibrium characterization

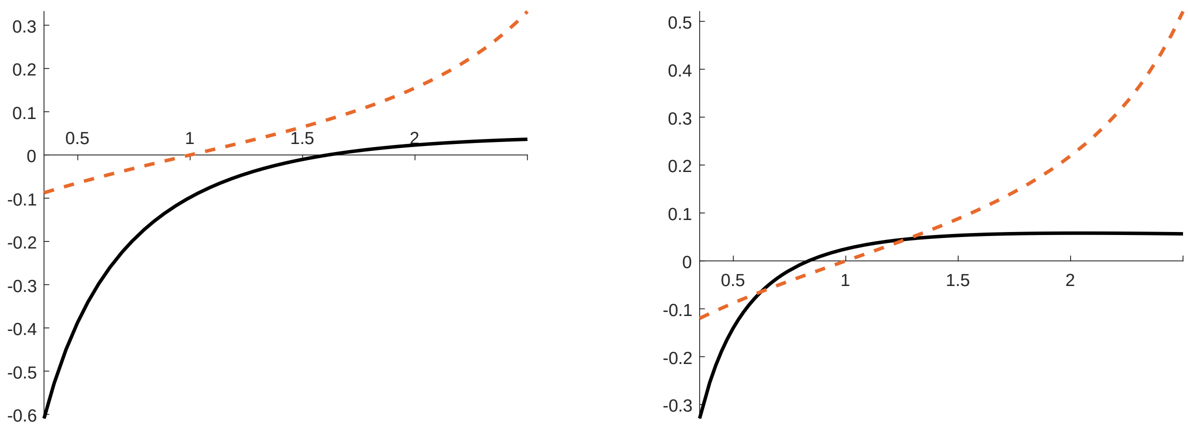
We now turn to the more general setting in which investors can optimally choose their beliefs about both the quality of their private signal as well as the information contained in prices. We begin by characterizing the characteristics of any feasible symmetric equilibrium.

Proposition 3. *Suppose the cost function $C(\delta_{e,i}, \delta_{z,i})$ is well-behaved. In any symmetric equilibrium, all investors are (weakly) over-confident about their private signal (i.e., $\delta_{e,i} \geq 1$ for all i) but choose to under-react to the information in prices (i.e., $\delta_{z,i} < 1$ for all i).*

Thus, in any symmetric equilibrium, investors always choose to over-confident about their private information (as above) but under-react to the information in prices. This is a robust outcome in our setting. Consider the choices $\delta_{e,i}$ and $\delta_{z,i}$ of investor i in a symmetric equilibrium where all other investors choose δ_e and δ_z, respectively. Recall that for a well-behaved cost function, deviations away from rational expectations (i.e., $\delta_{e,i} = 1$ and $\delta_{z,i} = 1$) are penalized i.e., the cost function is decreasing below one and increasing above one. Since anticipated utility is always (weakly) increasing in $\delta_{e,i}$, this immediately implies any symmetric equilibrium features (weak) over-confidence about private information (i.e., $\delta_{e,i} \geq 1$). Intuitively, increasing the perceived precision of private information always increases anticipated utility, and so it is natural that investors choose to be over-confident about their private signals.

However, as Lemma 3 establishes, anticipated utility is U-shaped in $\delta_{z,i}$. Moreover, as we show in the proof of Proposition 3, when other investors choose δ_z, anticipated utility is decreasing in $\delta_{z,i}$ at $\delta_{z,i} = \delta_z$. This implies the equilibrium choice of $\delta_{z,i}$ cannot be higher than one, since if this were the case, investor i could increase her anticipated utility and

Figure 1: Marginal Anticipated Utility vs. Marginal Cost for Optimal Expectations
The figure shows marginal anticipated utility (solid black line) and marginal cost function (dashed orange line) as a function of δ_{zi}. The marginal cost function is under the assumption that investors exhibit optimal expectations (i.e., their beliefs satisfy (11)). Other parameters are: $\tau = \tau_e = \tau_z = \delta_e = \delta_{e,i} = 1$, $\delta_z = 0.5$.



$\gamma = 0.8$ (b) $\gamma = 0.3$

decreases her costs by reducing $\delta_{z,i}$, an unambiguously better outcome. Intuitively, in a symmetric equilibrium, investor i has an incentive to decrease the perceived precision of price information relative to the choice of others because by doing so, she improves her ability to speculate on her private information by decreasing the correlation between her conditional valuation (μ_i) and those of other investors (i.e., $\int_j \mu_j dj$), which is reflected in the equilibrium price.

Given the above characterization for arbitrary cost functions, the next subsections characterize conditions for the existence of symmetric equilibria in an optimal expectations setting.

5.1 Symmetric equilibrium and under-reaction to prices

We begin with a sufficient condition for the existence and uniqueness of symmetric equilibria.

Proposition 4. *Suppose investors exhibit optimal expectations i.e., their beliefs satisfy (11). There exists a $\bar{\gamma}$ such that for all $\gamma \geq \bar{\gamma}$, there exists a unique, symmetric equilibrium in which all investors ignore the information in prices (i.e., $\delta_{z,i} = 0$ for all i), and correctly interpret their private signals (i.e., $\delta_{e,i} = 1$ for all i) .*

The plot in Figure 1, panel (a), provides a numerical illustration. The figure plots the marginal anticipated utility (solid) and the marginal cost function (dashed) for an investor

i as a function of her choice $\delta_{z,i}$. Recall that deviations away from $\delta_{z,i}$ are costly — as a result, the marginal cost for $\delta_{z,i} < 1$ is negative. Moreover, note that Lemma 3 implies that the marginal anticipated utility is negative below a threshold $\bar{\delta}$ (which is a little above 1.5 in the plot). Finally, note that while the marginal anticipated utility when $\delta_{z,i} = 0$ is $-\infty$, the marginal cost in an optimal expectations framework is always negative but *finite*. At any alternative symmetric equilibrium the marginal benefit and marginal cost must intersect.[18] A sufficiently high γ ensures that (i) the marginal anticipated utility curve intersects zero at a point to the right of $\delta_{z,i} = 1$, and (ii) the marginal cost curve is sufficiently flat between $\delta_{z,i} = 0$ and $\delta_{z,i} = 1$. This in turn ensures that the two curves never intersect, and the symmetric equilibrium is a corner solution at $\delta_{z,i} = 0$.

Intuitively, when γ is high, price informativeness τ_p is relatively low. In this case, the speculative effect of learning from prices dominates the information effect, and investors prefer to under-weight the information in prices. When γ is sufficiently high, the price is sufficiently uninformative, and investors optimally choose to ignore the information in prices. Since the marginal anticipated utility does not change with $\delta_{e,i}$ when $\delta_{z,i} = 0$, investors optimally choose to correctly interpret their private information (i.e., $\delta_{e,i} = 1$).

Given the above equilibrium, the following result compares return-volume characteristics to the rational expectations benchmark.

Corollary 2. *Relative to the rational expectations equilibrium (i.e., when $\delta_{z,i} = \delta_{e,i} = 1$ for all i), the equilibrium characterized in Proposition 4 features: (i) higher predictability coefficient, (ii) equal price informativeness, and (iii) equal expected volume. Return volatility is higher than in the rational expectations equilibrium iff price informativeness is sufficiently high (i.e., $\tau_p \geq \frac{\sqrt{\tau^2+8\tau\tau_e+8\tau_e^2}}{2} - \frac{\tau}{2}$).*

Since $\delta_e = 1$ in the symmetric equilibrium, the signal to noise ratio κ is the same as in the rational expectations equilibrium (since $\beta = -\gamma/\tau_e$). However, the price sensitivity in the symmetric equilibrium, $\Lambda_{SE} = \frac{\tau_e}{\tau+\tau_e}$, is lower than that in the rational expectations equilibrium (i.e., $\Lambda_{RE} = \frac{\tau_e+\tau_p}{\tau+\tau_e+\tau_p}$). Following the discussion of Lemma 2, this immediately implies the results on the predictability coefficient and price informativeness. The volume remains the same across the two symmetric equilibria since the investors weight their private signals correctly (i.e., $\delta_{ei} = 1$) in either case. This implies that the cross-sectional variation in valuations, scaled by their posterior variance, remains unaffected by whether or not they condition on prices.[19] Finally, we characterize conditions under which return volatility is higher under the optimal expectations equilibrium. Recall that return volatility is decreasing

[18] Specifically, any other potential maximum lies at every second intersection of the two curves.

[19] This is apparent in the limit when there is no private information i.e., $\tau_e = 0$. In this case, volume is zero in any symmetric equilibrium, irrespective of whether investors condition on prices or not.

in Λ when the signal to noise ratio $\kappa = \frac{\tau_p}{\tau+\tau_p}$ is sufficiently high (relative to Λ). The condition on τ_p above ensures that the decrease in Λ (from Λ_{RE} to Λ_{SE}) leads to an increase in volatility.

5.2 Asymmetric equilibrium and specialization of beliefs

The next result establishes sufficient conditions to rule out the existence of a symmetric equilibrium.

Proposition 5. *Suppose investors exhibit optimal expectations i.e., their beliefs satisfy (11). There exists a $\underline{\gamma}$ such that for all $\gamma < \underline{\gamma}$, there cannot exist a symmetric equilibrium.*

Again, consider the numerical example plotted in Figure 1, panel (b). When γ is sufficiently low, the marginal anticipated utility curve crosses zero below $\delta_{z,i} = 1$. This implies that there is always a (local) maximum, corresponding to the second intersection of the solid and dotted lines. Investor i might prefer to deviate from the corner ($\delta_{z,i} = 0$) to this interior maximum, if her expected anticipated utility, net of cost, is higher.

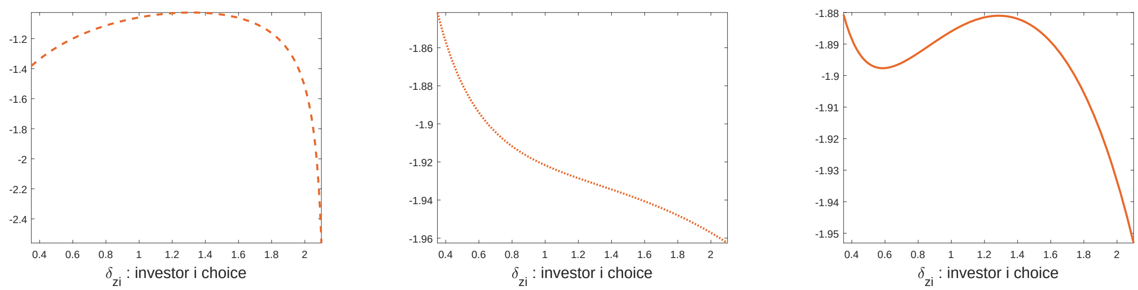
Intuitively, this can occur when γ is sufficiently low because price informativeness is sufficiently high (investors trade more aggressively on their information). Moreover, in any symmetric equilibrium, investors under-react to the information in prices. Together, these imply that an individual investor may have an incentive to deviate and condition *more* aggressively on the information in prices — in such a case, the speculative effect is dominated by the information effect. But such profitable deviations rule out a symmetric equilibrium.

The plots in Figure 2 provide a numerical example. The panels show investor i's anticipated utility, net of costs, as a function of $\delta_{z,i}$, given the behavior of others. In panel (a), all other investors choose $\delta_z = 0$. In this case, investor i has an incentive to deviate by over-weighting the information in prices (i.e., by setting $\delta_{zi} \approx 1.3$). Even though the price is objectively very informative (large information effect), because other investors are ignoring it ($\delta_z = 0$), the speculative effect of overweighting the price is relatively small. In panel (b), we consider an alternative symmetric equilibrium in which all other investors choose $\delta_z > 0$. Now, the speculative effect dominates and investor i strictly prefers to ignore the information in prices. In both cases, a symmetric equilibrium is ruled out because an individual investor has an incentive to deviate from the equilibrium behavior.

Given the non-existence of symmetric equilibria, we numerically explore the existence of asymmetric equilibria in which investors mix between two sets of beliefs. We assume a fraction λ optimally chooses $\delta_{e,i} = 1$ and $\delta_{z,i} = 0$, while the remaining fraction $1-\lambda$ optimally chooses $\delta_{e,i} = \delta_e$ and $\delta_{z,i} = \delta_z$. The following result characterizes the existence of such an equilibrium.

Figure 2: Anticipated utility net of costs versus $\delta_{z,i}$

The figure plots the anticipated utility net of costs for investor i as a function of her choice $\delta_{z,i}$. Other parameters are: $\tau = \tau_e = \tau_z = \delta_e = \delta_{e,i} = 1$, $\gamma = 0.3$



(a) $\delta_z = 0$ (b) $\delta_z = 1$ (c) Mixed eqm.

Lemma 4. *An asymmetric equilibrium is characterized by the triple* $(\lambda, \delta_e, \delta_z)$ *which solve a system of three equations (specified in the Appendix). The equilibrium price is given by:* $P = m + \Lambda_{AE}(s_p - m)$, *where* $s_p = F - \frac{\gamma}{\bar{\delta}_e \tau_e} z$,

$$\Lambda_{AE} \equiv \frac{\bar{\delta}_e \tau_e + \bar{\delta}_z \tau_{p,AE}}{\tau + \bar{\delta}_e \tau_e + \bar{\delta}_z \tau_{p,AE}}, \qquad \tau_{p,AE} = \frac{\tau_z \tau_e^2 \bar{\delta}_e^2}{\gamma^2}, \tag{40}$$

$\bar{\delta}_e = (\lambda + (1-\lambda)\delta_e)$ *and* $\bar{\delta}_z = (1-\lambda)\delta_z$. *Moreover,* $\delta_e, \delta_z \leq 3/2$, *and* $\Lambda_{AE} < \Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p}$ *under some conditions reported in the appendix.*

Panel (c) of Figure 2 illustrates an instance of the asymmetric equilibrium. In this case, each investor is indifferent between two (sets of) beliefs. In equilibrium, one set of investors (a fraction $\lambda = 0.95$) ignore the information in prices while the remaining fraction $1-\lambda = 0.05$) overweight the information in prices.

Corollary 3. *Relative to the rational expectations equilibrium (i.e., when* $\delta_{z,i} = \delta_{e,i} = 1$ *for all* i*), the equilibrium characterized in Proposition 5 features: (i) higher predictability coefficient, (ii) higher price informativeness, and (iii) higher expected volume.*

The asymmetric equilibrium has three main implications. First, the return predictability coefficient is higher than in an rational expectations equilibrium and can even be positive. In the traditional noisy-rational expectations setting with exogenous, transient noise trading (e.g., Hellwig (1980)), returns exhibit reversals. Intuitively, an aggregate demand (supply) shock temporarily pushes the current price up (down, respectively), but since the shock is not persistent, prices revert in the future. In our model, because some investors underweight the information in prices, prices do not adjust to their informationally efficient levels and there is residual (positive) predictability in returns. This mechanism is similar to the one found in Hong and Stein (1999) and Banerjee et al. (2009).

Second, we show that price informativeness is higher in this equilibrium compared to a standard REE model. In our model, price informativeness increases with the weight investors place on private signals. Since in an asymmetric equilibrium, a fraction of agents place strictly higher weight than in a rational expectations model (and all others objectively weight their private information), the model features higher price informativeness than if all investors held rational expectations. This result is consistent with empirical evidence in Hirshleifer et al. (1994); Hirshleifer and Luo (2001).

Third, volume in an asymmetric equilibrium is higher than in the corresponding REE model. In an asymmetric equilibrium, in addition to volume generated by cross-sectional variation in private signals, there is additional trade between the different groups of investors. This latter source of additional volume is absent in both the rational expectations and the symmetric, optimal expectations equilibria.

Figure 3 provides a numerical illustration of these results. Specifically, the figure plots price sensitivity, predictability, volume and volatility for the rational expectations (dashed) and optimal expectations equilibria (solid) as a function of risk aversion γ. Moreover, the kink in the solid lines corresponds to the value of γ at which the optimal expectations equilibrium switches from the asymmetric equilibrium (low γ) to the symmetric equilibrium (higher γ). Consistent with the predictions of Corollaries 2 and 3, predictability and volume are (weakly) higher under optimal expectations than under rational expectations. Moreover, volatility is higher under optimal expectations when risk aversion is sufficiently low (i.e., τ_p is sufficiently high), but is higher otherwise. Finally, in this parameter region, the price sensitivity Λ is always lower for the optimal expectations equilibrium.

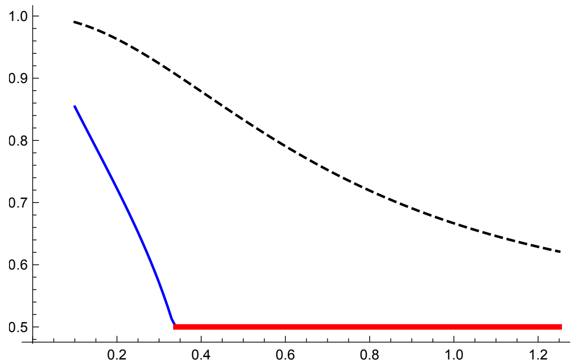
6 Welfare

In this section, we explore the welfare implications of allowing investors to choose their beliefs optimally. We begin by noting that welfare for the informed investors depends on the reference beliefs used. From the perspective of the investors' subjective beliefs, expected utility is higher when they deviate from rational expectations. However, from the perspective of a social planner who is restricted to hold objective beliefs, expected utility for informed investors is strictly lower when they deviate from rational expectations - their demand for the risky asset is suboptimal given their information sets.

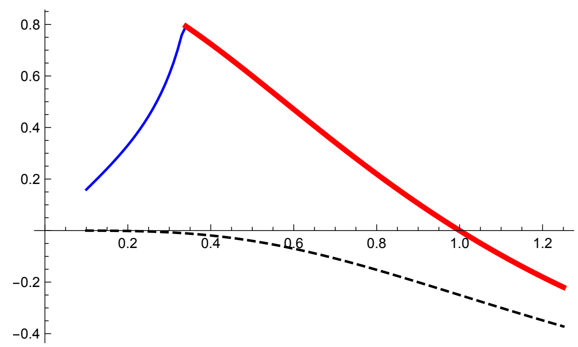
For our welfare analysis below, we use the objective distribution as the reference beliefs

Figure 3: Comparison of return and volume characteristics

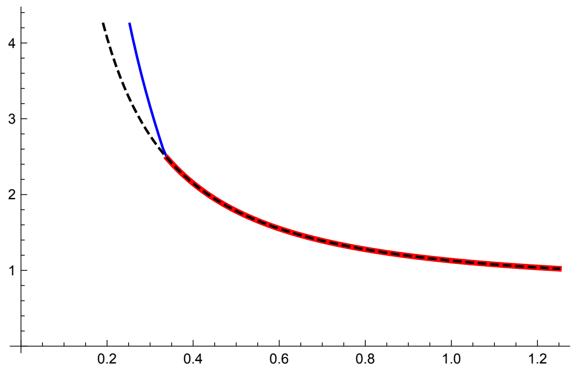
The figure plots price sensitivity, return predictability, trading volume and return volatility (variance) as a function of risk aversion for optimal expectations (solid line) and rational expectations (dotted line). Other parameters are set to $\tau = \tau_e = \tau_z = 1$. The thin (blue) part of the solid line corresponds to the asymmetric equilibrium, while the thick (red) part corresponds to the symmetric equilibrium.



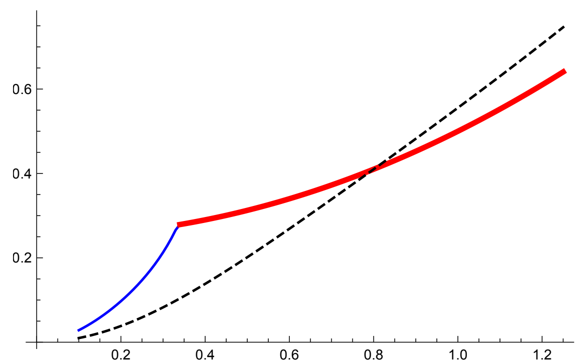
(a) Price sensitivity Λ vs. γ



(b) Predictability θ vs. γ



(c) Volume vs. γ



(d) Volatility vs. γ

and define expected utility for an informed investor as

$$U_i \equiv \mathbb{E}\left[-\exp\left\{-\gamma x_i^*\left(\delta_{e,i}^*, \delta_{z,i}^*\right) \times (F - P) - \gamma W_0\right\}\right], \tag{41}$$

where $x_i^*\left(\delta_{e,i}^*, \delta_{z,i}^*\right)$ is her optimal demand under her optimally chosen beliefs $\delta_{e,i}^*, \delta_{z,i}^*$. Note that this is a conservative measure of expected utility as it only accounts for the costs of deviating from rational expectations and does not include any of the gains from anticipated utility.

We can also consider the effect of informed investors' deviations from rational expectations on the welfare of liquidity (or noise) traders. Recall that the aggregate supply, z, is noisy. Suppose this reflects the sale of the risky asset by a liquidity trader, who has CARA utility with risk aversion γ_z and is endowed with initial wealth W_0. Then, her expected utility is given by

$$U_z \equiv \mathbb{E}\left[-\exp\left\{-\gamma_z(-z)\times(F-P)-\gamma_z W_0\right\}\right]. \tag{42}$$

The following result compares the expected utility of liquidity traders and the overall welfare in the constrained and unconstrained equilibria.

Proposition 6. *In equilibrium, the expected utility of a liquidity trader is given by:*

$$U_z = -\sqrt{\frac{\tau_z}{\tau_z+2\gamma_z\left(\beta\Lambda-\frac{1}{2\tau}\gamma_z(1-\Lambda)^2\right)}}\exp\left\{-\gamma_z W_0\right\} \tag{43}$$

Suppose $\gamma_z \leq \gamma$. Then:

(i) Liquidity traders have higher expected utility in the symmetric equilibrium than in the rational expectations equilibrium.

(ii) In any asymmetric equilibrium in which $\Lambda_{AE} < \Lambda_{RE}$, liquidity traders have higher expected utility in the asymmetric equilibrium than in the rational expectations equilibrium.

(iii) There exists $\underline{\gamma} \geq 0$ such that for all $\gamma \geq \underline{\gamma}$, total welfare is higher under the optimal expectations equilibrium than under the rational expectations equilibrium.

Expected utility for a liquidity trader depends on the equilibrium parameters through a term

$$\beta\Lambda - \tfrac{1}{2\tau}\gamma_z\left(1-\Lambda\right)^2. \tag{44}$$

To gain some intuition for this expression, it is illustrative to consider the (conditional) expected utility of the liquidity trader if she had mean-variance preferences.[20] Selling z units gives her utility $u(z)$, where

$$u(z) = -z\mathbb{E}\left[F-P|z\right] - \tfrac{1}{2}\gamma_z z^2 \text{var}\left(F-P|z\right) \tag{46}$$

$$= \beta\Lambda z^2 - \tfrac{1}{2}\gamma_z z^2\left(1-\Lambda\right)^2\tfrac{1}{\tau}, \tag{47}$$

where $\beta = -\frac{\gamma}{\tau_e\delta_e} < 0$. Therefore, a liquidity trader's utility is driven by two components. The first component ($\beta\Lambda z^2$) reflects her disutility from price impact — for instance, a larger sale (higher z) pushes prices downward, which reduces her proceeds. The second term ($-\frac{1}{2}\gamma z_z^2\left(1-\Lambda\right)^2\frac{1}{\tau}$) reflects a standard risk-aversion channel — when prices are less informative about fundamentals, the liquidity trader faces more uncertainty about her payoff, which

[20]Note that by the Law of Iterated Expectations, we have:

$$U_z \propto \mathbb{E}\left[\mathbb{E}\left[-\gamma\exp\left\{-\gamma\left(-x\left(F-P\right)\right)\right\}\middle|z\right]\right] = \mathbb{E}\left[-\exp\left\{-\gamma u(z)\right\}\right], \tag{45}$$

so considering mean-variance preferences is qualitatively without loss of generality.

reduces utility. It is important to note that expected utility is finite only when

$$\tau_z + 2\gamma_z \left(\beta\Lambda - \tfrac{1}{2\tau}\gamma_z \left(1 - \Lambda\right)^2\right) > 0. \tag{48}$$

Intuitively, if the combined disutility from the price impact and risk aversion terms are too large, the liquidity trader's expected utility from being forced to trade z units approaches negative infinity — she would rather exit the market and not trade if she could.[21]

The proposition characterizes conditions under which liquidity traders are better off when informed investors choose their beliefs. As discussed earlier, price sensitivity, Λ, is higher when investors exhibit rational expectations: $\Lambda_{RE} > \Lambda_{SE}, \Lambda_{AE}$. This has offsetting effects on the liquidity trader's utility. On the one hand, a higher Λ implies that the price is more sensitive to her trade and so utility falls through the price impact channel. On the other hand, a higher Λ implies prices track fundamentals more closely which reduces the risk in the liquidity trader's payoff. As we show in the proof of Proposition 6, the price impact effect always dominates the risk-aversion effect if the risk aversion of the investors is weakly higher than that of the liquidity traders (i.e., $\gamma_z \leq \gamma$). In this case, liquidity traders are always better off when informed investors choose to deviate from rational expectations.

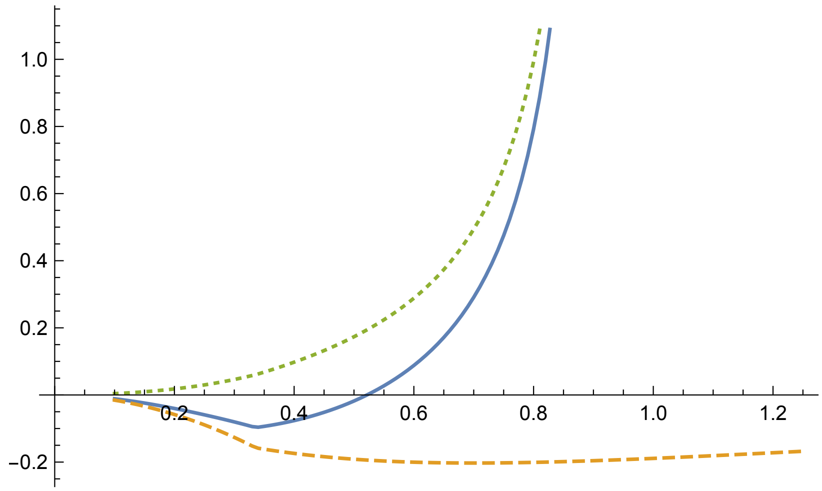
Note that $\gamma_z \leq \gamma$ is a sufficient condition, but it is not necessary for liquidity traders to be better off under the optimal expectations equilibrium. Figure 4 plots the difference in expected utility between the optimal expectations equilibrium and the rational expectations equilibrium as a function of investor risk aversion γ for each group separately, and for both groups as a whole. The plot illustrates that for this set of parameters, liquidity traders are always better off under optimal expectations — the dotted line is always above zero — irrespective of whether informed investors are more or less risk averse than them. In particular, note that noise trader risk aversion γ_z is fixed at 0.75, but informed investor risk aversion γ ranges from 0.1 to 1.2. Moreover, under the objective distribution, the informed investors are worse off under the optimal expectations equilibrium — the dashed line is always below zero.

The solid line in Figure 4 illustrates the aggregate welfare ranking in Proposition 6. Specifically, aggregate welfare appears to be higher in the rational expectations equilibrium when informed investor risk aversion is low, but higher under optimal expectations when risk aversion is high. First, when risk aversion is low, price impact is low, and so the relative benefit to noise traders in the optimal expectations equilibrium is low. Second, when risk aversion is low, informed investors trade more aggressively on their distorted beliefs, and

[21] This highlights a limitation of assuming that liquidity traders submit price insensitive orders. An alternative approach would be to model liquidity shocks as hedging demands for the informed investors. However, this makes the analysis less tractable and the intuition for the results in the rest of the paper less clear.

Figure 4: Difference in expected utility $U_{OE} - U_{RE}$ versus γ

The figure plots the difference in utility across the optimal expectations and rational expectations equilibria. The dashed line plots the difference in expected utility of informed investors (under the objective distribution) (i.e., $U_{i,OE} - U_{i,RE}$), the dotted line plots the difference in expected utility for the noise traders (i.e., $U_{z,OE} - U_{z,RE}$), and the solid line plots the difference in utility across both groups (i.e., $U_{i,OE} + U_{z,OE} - (U_{i,RE} + U_{z,RE})$). Other parameters are set to $\tau = \tau_e = \tau_z = 1$, and $\gamma_z = 0.75$.



ΔU for investors (dashed), noise traders (dotted), both (solid)

this decreases their expected utility more.

In contrast, when risk aversion is sufficiently high, the benefit to noise traders is larger (due to higher price impact). Moreover, when risk aversion is sufficiently high, price informativeness is low and investors trade less aggressively, which imply that the relative cost of distorted beliefs in the optimal expectations equilibrium is lower (the dashed line is eventually increasing in γ). As a result, when risk aversion is high, welfare is higher under optimal expectations.

Our results suggest that while deviations from rational expectations are arguably costly for informed investors, they may make liquidity traders better off. Moreover, our welfare results do not account for changes in the real (allocative) efficiency. Since price informativeness is higher when informed investors deviate from rational expectations, the real efficiency in the economy can also be higher under such deviations if investment / allocative decisions are made on the information in prices.

7 Extensions and concluding remarks

In this paper, we analyze how investors rationally distort their beliefs about their informational environment in the context of an otherwise standard competitive trading environment. We show that in any symmetric equilibrium, investors are always (i) weakly overconfident in the quality of their private signal (so that their perceived private information advantage is preserved or amplified), and (ii) discount the quality of the information in prices (so that their perceived trading opportunity is maximized).

We have also shown that similar results arise in related settings. In a setting without aggregate noise, investors still choose to under-weight the information in prices but overweight their private information.[22] In this case, the price provides a perfectly revealing signal about the fundamental F;however, we show that investors still choose to partially dismiss this information in any symmetric equilibrium.

In Appendix B, we explore how, in addition to the choices made in the benchmarket model, investors would choose to interpret the informativeness of a public signal. We show that, like the price, an increase in the perceived precision of the public signal increases anticipated utility through the information channel, but reduces it through the speculative channel. However, in contrast to the information in prices, we show that the informational channel dominates the speculative channel in any symmetric equilibrium. As a result, we find that investors tend to over-weight the information in public signals.

Our analysis suggests that allowing for belief choice is likely to be a fruitful approach to understanding individual behavior in strategic settings. In a financial market setting, we show that common behavioral biases such as overconfidence and dismissiveness arise naturally as outcomes of belief choice. Moreover, as highlighted by our asymmetric equilibrium, this approach can give rise to endogenous heterogeneity in investor behavior. While a financial market is characterized by strategic substitutability, other strategic settings feature complementarity (e.g., beauty contest games, coordination games). In future work, we hope to explore the implications of subjective belief choice in these settings. Taken together, these results may help us better understand why different biases arise in different environments.

[22] Specifically, we consider a setting in which the true distribution of signals is given by (3), but investor i believes that the error in investor j's signal is given by:

$$\varepsilon_j =_i \frac{u}{\sqrt{\delta_p}} + \sqrt{1 - \tfrac{1}{\delta_p}}u_i, \quad u, u_i \sim N\left(0, \frac{1}{\tau_e}\right). \tag{49}$$

Intuitively, we allow each investor to believe that the error in others' private information is correlated. We also assume the aggregate supply of the asset $z = 0$ and that this is known by all investors when they choose their beliefs.

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8 Appendix A - Proofs

8.1 Proof of Lemma 3

Lemma 1 implies that the price is of the form: $P = m + \Lambda(s_p - m)$. This implies anticipated utility is given by

$$AU(\delta_{e,i}, \delta_{z,i}) = -\sqrt{\frac{\frac{1}{\tau + \delta_{e,i}\tau_e + \delta_{z,i}\tau_p}}{(1-\Lambda)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{z,i}\tau_p}}}. \tag{50}$$

Note that given other investors' choices, investor i's marginal anticipated utility is

$$\frac{\partial}{\partial \delta_{e,i}} AU = \frac{\tau_e}{2(\tau + \delta_{e,i}\tau_e + \delta_{z,i}\tau_p)} \times \sqrt{\frac{\frac{1}{\tau + \delta_{e,i}\tau_e + \delta_{z,i}\tau_p}}{(1-\Lambda)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{z,i}\tau_p}}} \geq 0 \tag{51}$$

$$\frac{\partial}{\partial \delta_{z,i}} AU = \frac{(1-\Lambda)^2 \delta_{z,i}^2 \tau_p^2 - \Lambda^2 \tau (\delta_{e,i}\tau_e + \tau)}{2\delta_{z,i}(\Lambda^2 \tau + (1-\Lambda)^2 \delta_{z,i}\tau_p)(\delta_{e,i}\tau_e + \delta_{z,i}\tau_p + \tau)} \times \sqrt{\frac{\frac{1}{\tau + \delta_{e,i}\tau_e + \delta_{z,i}\tau_p}}{(1-\Lambda)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{z,i}\tau_p}}} \tag{52}$$

This implies anticipated utility is always increasing in $\delta_{e,i}$, and increasing in $\delta_{z,i}$ when

$$\frac{\delta_{z,i}^2}{\delta_{e,i}\tau_e + \tau} > \frac{\Lambda^2}{(1-\Lambda)^2} \frac{\tau}{\tau_p^2}, \tag{53}$$

i.e., it is initially decreasing and then increasing in $\delta_{z,i}$. Moreover, note that

$$\lim_{\delta_{z,i} \to 0} \frac{\partial}{\partial \delta_{z,i}} AU = -\infty, \quad \lim_{\delta_{z,i} \to 0} \frac{\partial}{\partial \delta_{z,i}} AU = 0 \tag{54}$$

and $\frac{\partial}{\partial \delta_{z,i}} AU$ equals zero at:

$$\delta_{z,i}^* = \frac{1}{\tau_p} \left(\frac{\Lambda}{1-\Lambda} \right) \sqrt{\tau (\delta_{e,i}\tau_e + \tau)} \tag{55}$$

□

8.2 Proof of Proposition 1

The objective of investor i given by

$$\max_{\delta_{ei}} AU_i(\delta_{ei}) - C(\delta_{ei}).$$

Lemma 3 implies that anticipated utility increases with overconfidence parameter δ_{ei}. So, investor tries to balance the benefit of increasing δ_{ei} with the cost of increasing δ_{ei}. The FOC with respect to δ_{ei} is

$$\frac{\tau_e}{2\left(\frac{(1-\Lambda)^2}{\tau_0}+\frac{\Lambda^2}{\tau_p}\right)^{\frac{1}{2}}\left(\tau_e\delta_{ei}+\tau_p+\tau_0\right)^{\frac{3}{2}}}=\frac{\partial C}{\partial \delta_{ei}} \tag{56}$$

and the second order condition is

$$-\frac{3\tau_e^2}{4\left(\frac{(1-\Lambda)^2}{\tau_0}+\frac{\Lambda^2}{\tau_p}\right)^{\frac{1}{2}}\left(\tau_e\delta_{ei}+\tau_p+\tau_0\right)^{\frac{5}{2}}}-\frac{\partial^2 C}{\partial \delta_{ei}^2}<0.$$

Condition (56) implies that the optimal overconfidence parameter is always greater than one i.e., $\delta_e^* \geq 1$. □

8.3 Proof of Proposition 2

The cost function in the case of optimal expectations is given by

$$C\left(\delta_{ei}\right)=\frac{1}{\sqrt{\left(\frac{(1-\Lambda)^2}{\tau_0}+\frac{\Lambda^2}{\tau_p}\right)\left(\tau_0+\tau_p+\tau_e\delta_{ei}\left(2-\delta_{ei}\right)\right)}}$$

The FOC in the case of optimal expectations is given by

$$\frac{\tau_e}{2\left(\frac{(1-\Lambda)^2}{\tau_0}+\frac{\Lambda^2}{\tau_p}\right)^{\frac{1}{2}}\left(\tau_e\delta_{ei}+\tau_p+\tau_0\right)^{\frac{3}{2}}}=\frac{\tau_e\left(\delta_{ei}-1\right)}{\left(\frac{(1-\Lambda)^2}{\tau_0}+\frac{\Lambda^2}{\tau_p}\right)^{\frac{1}{2}}\left[\left(\tau_0+\tau_p+\tau_e\delta_{ei}\left(2-\delta_{ei}\right)\right)\right]^{\frac{3}{2}}} \tag{57}$$

which simplifies to

$$\frac{\left(\tau_0+\tau_p+\tau_e\delta_{ei}\left(2-\delta_{ei}\right)\right)^{\frac{3}{2}}}{\left(\tau_p+\tau_0+\tau_e\delta_{ei}\right)^{\frac{3}{2}}}=2\left(\delta_{ei}-1\right) \tag{58}$$

which establishes the result. □

8.4 Proof of Corollary 1

Taking derivatives of the return-volume characteristics in Lemma (2) with respect to δ_e gives:

$$\frac{\partial \sigma_R^2}{\partial \delta_e} = -\frac{2\gamma^2\tau_e\left(\gamma^6+\delta_e\tau_e\tau_z\left(3\gamma^4+\gamma^2\tau_z\left(3\delta_e\tau_e+\tau\right)+\delta_e^2\tau_e^2\tau_z^2\right)\right)}{\tau_z\left(\gamma^2\tau+\delta_e\tau_e\left(\gamma^2+\delta_e\tau_e\tau_z\right)\right)^3} < 0 \tag{59}$$

$$\frac{\partial \theta}{\partial \delta_e} = \frac{\gamma^4\tau\tau_e\tau_z\left(\gamma^2\left(2\delta_e\tau_e+\tau\right)+3\delta_e^2\tau_e^2\tau_z\right)}{\left(\gamma^2+\delta_e\tau_e\tau_z\right)^2\left(\gamma^2\tau+\delta_e^2\tau_e^2\tau_z\right)^2} > 0 \tag{60}$$

$$\frac{\partial \tau_p}{\partial \delta_e} = \frac{2\delta_e\tau_e^2\tau_z}{\gamma^2} > 0 \tag{61}$$

$$\frac{\partial \mathbb{E}\left[\mathcal{V}\right]}{\partial \delta_e} = \frac{\sqrt{\frac{2}{\pi}}\delta_e\tau_e}{\gamma\sqrt{\frac{\gamma^2+\delta_e^2\tau_e\tau_z}{\tau_z}}} > 0 \tag{62}$$

which establishes the result. □

8.5 Proof of Proposition 3

Equation (51) shows that marginal anticipated utility is weakly increasing in $\delta_{e,i}$. As long as $\frac{\partial C(1,\delta_z)}{\partial \delta_{e,i}} = 0$ (which holds under any well-behaved cost function), then the first-order condition in a symmetric equilibrium

$$\frac{\tau_e}{2\left(\frac{(1-\Lambda)^2}{\tau}+\frac{\Lambda^2}{\delta_z\tau_p}\right)^{\frac{1}{2}}\left(\tau_e\delta_e+\delta_z\tau_p+\tau\right)^{\frac{3}{2}}} = \frac{\partial C(1,\delta_z)}{\partial \delta_{ei}} \tag{63}$$

implies that $\delta_e \geq 1$ with $\delta_e > 1$ if $\delta_z \neq 0$. This proves the first half of the proposition.

Lemma 1 implies that in any symmetric equilibrium, we have $\Lambda = \frac{\delta_e\tau_e+\delta_z\tau_p}{\tau+\delta_e\tau_e+\delta_z\tau_p}$. Moreover, note that $\frac{\partial}{\partial \delta_{z,i}}AU = 0$ at

$$\bar{\delta}_{z,i} = \frac{1}{\tau_p}\left(\frac{\Lambda}{1-\Lambda}\right)\sqrt{\tau\left(\delta_e\tau_e+\tau\right)} \tag{64}$$

$$= \sqrt{1+\frac{\tau_e}{\tau}}\left(\delta_z+\frac{\tau_e}{\tau_p}\delta_s\right) > \delta_z \tag{65}$$

But this implies $\frac{\partial}{\partial \delta_{z,i}}AU\left(\delta_{z,i}=\delta_z\right) < 0$ since $\frac{\partial AU}{\partial \delta_{z,i}} < (>)0$for all $\delta_{z,i} < (>)\bar{\delta}_{z,i}$. Next, note that if $\delta_{z,i} = \delta_z \geq 1$, then $C'(\delta_{z,i}) > 0$. Taken together, this proves that at any proposed symmetric equilibrium where $\delta_z > 1$, investor i has an incentive to deviate. Thus, the only possible symmetric equilibrium is one in which each investor chooses $\delta_{z,i} < 1$. This proves the second half of the proposition.

□

8.6 Proof of Propositions 4 and 5

For an investor exhibiting optimal expectations, choosing $(\delta_{ei}, \delta_{zi})$ yields anticipated utility and cost given by:

$$AU\left(\delta_{e,i}, \delta_{z,i}\right) = -\sqrt{\frac{\frac{1}{\tau+\delta_{e,i}\tau_e+\delta_{z,i}\tau_p}}{\left(1-\Lambda\right)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{z,i}\tau_p}}} \tag{66}$$

$$C\left(\delta_{ei}, \delta_{zi}\right) = \frac{1}{\sqrt{\Lambda^2 \left(\delta_{zi}-1\right)^2 + \left(\frac{(1-\Lambda)^2}{\tau} + \frac{\Lambda^2}{\tau_p}\right)\left(\tau + \tau_e\delta_{ei}\left(2-\delta_{ei}\right) + \tau_p\delta_{zi}\left(2-\delta_{zi}\right)\right)}}. \tag{67}$$

Let $\kappa \equiv \left(\frac{\Lambda}{1-\Lambda}\right)^2 \frac{\tau}{\tau_p}$. Then,

$$AU\left(\delta_{e,i}, \delta_{z,i}\right) = -\sqrt{\frac{\tau}{\left(1-\Lambda\right)^2}}\sqrt{\frac{1}{\left(1+\frac{\kappa}{\delta_{zi}}\right)\left(\tau+\delta_{e,i}\tau_e+\delta_{z,i}\tau_p\right)}} \tag{68}$$

$$C\left(\delta_{ei}, \delta_{zi}\right) = \sqrt{\frac{\tau}{\left(1-\Lambda\right)^2}}\sqrt{\frac{1}{\left(1-\delta_{zi}\right)^2 \kappa\tau_p + \left(1+\kappa\right)\left(\tau+\tau_e\delta_{ei}\left(2-\delta_{ei}\right)+\tau_p\delta_{zi}\left(2-\delta_{zi}\right)\right)}} \tag{69}$$

Suppose all others are playing $\bar{\delta}_e, \bar{\delta}_z$. Then, $\Lambda = \frac{\tau_e\bar{\delta}_e+\tau_p\bar{\delta}_z}{\tau+\tau_e\bar{\delta}_e+\tau_p\bar{\delta}_z}$ and so

$$\kappa = \left(\frac{\Lambda}{1-\Lambda}\right)^2 \frac{\tau}{\tau_p} = \frac{\gamma^2\left(\tau_e\bar{\delta}_e+\tau_p\bar{\delta}_z\right)^2}{\tau\tau_e^2\tau_z\bar{\delta}_e^2}. \tag{70}$$

Then, $(1, 0)$ is a symmetric equilibrium iff

$$AU\left(1,0\right) - C\left(1,0\right) > AU\left(\delta_{e,i}, \delta_{z,i}\right) - C\left(\delta_{e,i}, \delta_{z,i}\right) \tag{71}$$

for all $\delta_{e,i}, \delta_{z,i}$, or equivalently,_

$$H \equiv 1 + R - L > 0 \tag{72}$$

where

$$R \equiv \frac{AU\left(\delta_{e,i},\delta_{z,i}\right)}{-C\left(\delta_{ei},\delta_{zi}\right)} \tag{73}$$

$$= \sqrt{\frac{\gamma^4\tau\delta_z\tau_z\left(\tau_e\left(\frac{\tau_e\tau_z\bar{\delta}_e^2\left(\bar{\delta}_z^2-(\delta_z-2)\delta_z\right)}{\gamma^2}+2\bar{\delta}_e\bar{\delta}_z-(\delta_e-2)\delta_e\right)+\frac{\tau_e\left(\tau_e\tau_z\bar{\delta}_e^2-\gamma^2(\delta_e-2)\delta_e\right)\left(\tau_e\tau_z\bar{\delta}_e\bar{\delta}_z+\gamma^2\right)^2}{\gamma^4\tau\tau_z}+\tau+\frac{\gamma^2}{\tau_z}\right)}{\left(\tau_e^2\delta_z\tau_z\bar{\delta}_e^2+\gamma^2(\delta_e\tau_e+\tau)\right)\left(\left(\tau_e\tau_z\bar{\delta}_e\bar{\delta}_z+\gamma^2\right)^2+\gamma^2\tau\delta_z\tau_z\right)}} \tag{74}$$

$$L \equiv \frac{C\left(0,1\right)}{C\left(\delta_{ei},\delta_{zi}\right)} \tag{75}$$

$$= \sqrt{\frac{\tau_e\left(\frac{\tau_e\tau_z\bar{\delta}_e^2\left(\bar{\delta}_z^2-(\delta_z-2)\delta_z\right)}{\gamma^2}+2\bar{\delta}_e\bar{\delta}_z-(\delta_e-2)\delta_e\right)+\frac{\tau_e\left(\tau_e\tau_z\bar{\delta}_e^2-\gamma^2(\delta_e-2)\delta_e\right)\left(\tau_e\tau_z\bar{\delta}_e\bar{\delta}_z+\gamma^2\right)^2}{\gamma^4\tau\tau_z}+\tau+\frac{\gamma^2}{\tau_z}}{(\tau_e+\tau)\left(\frac{(\tau_e\tau_z\bar{\delta}_e\bar{\delta}_z+\gamma^2)^2}{\gamma^2\tau\tau_z}+1\right)+\frac{\tau_e^2\bar{\delta}_e^2(\tau_e\tau_z\bar{\delta}_e\bar{\delta}_z+\gamma^2)^2}{\gamma^4\tau}}} \tag{76}$$

Note that

$$\lim_{\gamma\to\infty} R = \sqrt{\frac{\left(\left(2-\delta_e\right)\delta_e\tau_e+\tau\right)\delta_z}{\delta_e\tau_e+\tau}}, \quad \lim_{\gamma\to\infty} L = \sqrt{\frac{\left(2-\delta_e\right)\delta_e\tau_e+\tau}{\tau_e+\tau}} \tag{77}$$

$$\lim_{\gamma\to\infty} H = 1+\sqrt{\frac{\left(\left(2-\delta_e\right)\delta_e\tau_e+\tau\right)\delta_z}{\delta_e\tau_e+\tau}}-\sqrt{\frac{\left(2-\delta_e\right)\delta_e\tau_e+\tau}{\tau_e+\tau}} \tag{78}$$

$$\geq 1+\sqrt{\frac{\left(\left(2-\delta_e\right)\delta_e\tau_e+\tau\right)\delta_z}{\delta_e\tau_e+\tau}}-\sqrt{\frac{\tau_e+\tau}{\tau_e+\tau}} \geq 0 \tag{79}$$

which implies $(1,0)$ is an equilibrium for γ sufficiently high.

Next, note that,

$$\lim_{\gamma\to 0} R = \lim_{\gamma\to 0} L = \sqrt{\frac{\frac{\tau_e^4\tau_z^2\bar{\delta}_e^4\bar{\delta}_z^2}{\tau}}{\frac{\tau_e^4\tau_z^2\bar{\delta}_e^4\bar{\delta}_z^2}{\tau}}} = 1, \tag{80}$$

so that

$$\lim_{\gamma\to 0} H = 1+R-L > 0 \tag{81}$$

which implies that for sufficiently low γ, an investor prefers to deviate to $(1,0)$ for any

$\bar{\delta}_e, \bar{\delta}_z \neq 0$. Finally, consider an equilibrium in which $\bar{\delta}_e = 1, \bar{\delta}_z = 0$. In this case,

$$\lim_{\gamma \to 0} R = \lim_{\gamma \to 0} \sqrt{2 - \delta_z} \tag{82}$$

$$\lim_{\gamma \to 0} L = \lim_{\gamma \to 0} \tfrac{1}{\gamma} \sqrt{(2 - \delta_z)} = \infty \tag{83}$$

which suggests that

$$\lim_{\gamma \to 0} H < 0 \tag{84}$$

and so $(1, 0)$ cannot be an equilibrium for γ sufficiently low. However,

$$\lim_{\gamma \to \infty} R = \sqrt{\frac{((2 - \delta_e)\, \delta_e \tau_e + \tau)\, \delta_z}{\delta_e \tau_e + \tau}}, \quad \lim_{\gamma \to \infty} L = \sqrt{\frac{(2 - \delta_e)\, \delta_e \tau_e + \tau}{\tau_e + \tau}} \tag{85}$$

which implies $\lim_{\gamma \to \infty} H \geq 0$ as before, and so $(1, 0)$ is an equilibrium for γ sufficiently high. □

8.7 Proof of Corollary 2

Denote the return characteristics in the rational expectations equilibrium (symmetric equilibrium) by RE (SE, respectively). Note that

$$\theta_{RE} - \theta_{OE} = -\frac{\tau \tau_e^2 \tau_z^2}{(\gamma^2 + \tau_e \tau_z)\,(\gamma^2 \tau + \tau_e^2 \tau_z)} < 0 \tag{86}$$

$$\tau_{p,RE} - \tau_{p,OE} = 0 \tag{87}$$

$$\mathbb{E}[\mathcal{V}_{RE}] - \mathbb{E}[\mathcal{V}_{OE}] = \frac{\sqrt{\frac{2}{\pi}} \left(\sqrt{\frac{\gamma^2 + \tau_e \tau_z}{\tau_z (\gamma^2 (\tau_e + \tau) + \tau_e^2 \tau_z)^2}}\, (\gamma^2 (\tau_e + \tau) + \tau_e^2 \tau_z) - \gamma^2 (\tau_e + \tau) \sqrt{\frac{\gamma^2 + \tau_e \tau_z}{\gamma^4 (\tau_e + \tau)^2 \tau_z}} \right)}{\gamma} = 0 \tag{88}$$

Finally, note that

$$\sigma_{R,RE}^2 - \sigma_{R,OE}^2 = \frac{\tau \left(\tau_p^2 + \tau \tau_p - 2\tau_e (\tau_e + \tau) \right)}{(\tau_e + \tau)^2 (\tau_e + \tau_p + \tau)^2}, \tag{89}$$

which is positive iff $\tau_p > \frac{1}{2} \sqrt{8 \tau \tau_e + 8 \tau_e^2 + \tau^2} - \frac{\tau}{2}$. □

8.8 Proof of Lemma 4

Suppose λ fraction of investors chose $(\delta_{z1}, \delta_{e1})$ and $(1-\lambda)$ investors chose $(\delta_{z2}, \delta_{e2})$. This implies that price is given by

$$P = m + \Lambda(s_p - m) \ \text{, where } \Lambda = \frac{(\lambda\delta_{e1} + (1-\lambda)\delta_{e2})\tau_e + (\lambda\delta_{z1} + (1-\lambda)\delta_{z2})\tau_p}{\tau + (\lambda\delta_{e1} + (1-\lambda)\delta_{e2})\tau_e + (\lambda\delta_{z1} + (1-\lambda)\delta_{z2})\tau_p}.$$

Assume that risk aversion is not sufficiently high, this implies that investor's objective function has a local interior maxima. Investor then evaluates his objective at this interior maxima and the boundary $\delta_{zi} = 0$ and chooses the one where his objective is highest. For the mixed equilibrium to sustain, we need $\delta_{z1} = 0$ (which implies $\delta_{e1} = 1$) and $\delta_{z2} = \delta_z^* \geq 1$ (and let $\delta_{e2} = \delta_e^*$). For this mixed equilibrium, investor has to be indifferent between the two points, which implies that the following conditions have to hold:

$$\frac{\partial AU}{\partial \delta_{ei}}|_{\{\delta_{zi}=\delta_z^*, \delta_{ei}=\delta_e^*\}} = C'(\delta_e^*)$$

$$\frac{\partial AU}{\partial \delta_{zi}}|_{\{\delta_{zi}=\delta_z^*, \delta_{ei}=\delta_e^*\}} = C'(\delta_z^*)$$

$$AU(0,1) - C(0,1) = AU(\delta_z^*, \delta_e^*) - C(\delta_z^*, \delta_e^*). \tag{90}$$

The first two conditions are the FOCs for local maxima (δ_z^*, δ_e^*) and the third condition says that investors are indifferent between the local maxima and the corner solution (0,1). These three equations will help us solve for 3 unknowns δ_z^*, δ_e^* and λ. Suppose $\bar{\delta}_e = \bar{\delta}_e = (\lambda + (1-\lambda)\delta_e)$ and $\bar{\delta}_z = (1-\lambda)\delta_z$ denote the average action of investors. The FOCs can be rewritten as

$$R^3 = \frac{2(\delta_z^* - 1)}{1 - (\delta_e^*\tau_e + \tau)\frac{\left(\bar{\delta}_e\tau_e + \bar{\delta}_z\tau_p\right)^2}{\tau_p^2\delta_z^{*2}\tau}} \tag{91}$$

$$R^3 = \frac{2(\delta_e^* - 1)}{1 + \frac{\Lambda^2}{\tau_p \mathbf{var}(F-P)}\left(\frac{1}{\delta_z^*} - 1\right)}, \tag{92}$$

where

$$R^2 = \frac{\frac{\left(\bar{\delta}_e\tau_e + \bar{\delta}_z\tau_p\right)^2}{\tau^2} + \left(\frac{1}{\tau} + \frac{\left(\bar{\delta}_e\tau_e + \bar{\delta}_z\tau_p\right)^2}{\tau^2\tau_p}\right)(\tau + \tau_e\delta_e^*(2 - \delta_e^*)) + \frac{\tau_p}{\tau}\delta_z^*(2 - \delta_z^*)}{\left(\frac{1}{\tau} + \frac{\left(\bar{\delta}_e\tau_e + \bar{\delta}_z\tau_p\right)^2}{\delta_z^*\tau_p\tau^2}\right)(\tau + \delta_e^*\tau_e + \delta_z^*\tau_p)}.$$

Since any deviations from rational expectations generate higher anticipated utility and lower true utility,$R < 1$. Using this inequality in equation 91 gives us that $\delta_z^* < \frac{3}{2}$. Similarly, using

$R < 1$ in equation 92 gives us that $1 < \delta_e^* < \frac{3}{2}$.

Moreover $\Lambda_{RE} > \lambda_{AE} \iff \tau_e + \frac{\tau_p}{\bar{\delta}_e^2} > \bar{\delta}_e \tau_e + \bar{\delta}_z \tau_p$. Let x denote the ratio of these two i.e.,

$$x = \frac{\bar{\delta}_e \tau_e + \bar{\delta}_z \tau_p}{\tau_e + \frac{\tau_p}{\bar{\delta}_e^2}}.$$

We need to prove that $x < 1$.Take the limit as $\gamma \to 0$,

$$\lim_{\gamma \to 0} R^2 = \lim_{\tau_p \to \infty} \frac{\frac{\left(\tau_e + \frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2} + \left(\frac{1}{\tau} + \frac{\left(\tau_e + \frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2 \tau_p}\right)\left(\tau + \tau_e \delta_e^* \left(2 - \delta_e^*\right)\right) + \frac{\tau_p}{\tau}\delta_z^* \left(2 - \delta_z^*\right)}{\left(\frac{1}{\tau} + \frac{\left(\tau_e + \frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\delta_z^* \tau_p \tau^2}\right)\left(\tau + \delta_e^* \tau_e + \delta_z^* \tau_p\right)}$$

$$= \lim_{\tau_p \to \infty} \frac{\frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2} + \frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2 \tau_p}\left(\tau + \tau_e \delta_e^* \left(2 - \delta_e^*\right)\right) + \frac{\tau_p}{\tau}\delta_z^* \left(2 - \delta_z^*\right)}{\left(\frac{\delta_z^* \tau_p}{\tau} + \frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2}\right)}$$

There are 2 cases to consider, Case 1: $\tau_p x \to \infty$, Case 2:$\tau_p x \to \mathit{constant}$. In case 2, it is immediate that $x < 1$ for sufficiently large τ_p i.e., for sufficiently large γ. Next, I will prove that case 1 is not possible in our setting.

Suppose, for now, case 1 is possible. In this case, $\tau_p x \to \infty$ as $\gamma \to 0$. In this case,

$$\lim_{\gamma \to 0} R^2 = \lim_{\tau_p \to \infty} \frac{\frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2} + \frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2 \tau_p}\left(\tau + \tau_e \delta_e^* \left(2 - \delta_e^*\right)\right) + \frac{\tau_p}{\tau}\delta_z^* \left(2 - \delta_z^*\right)}{\left(\frac{\delta_z^* \tau_p}{\tau} + \frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2}\right)}$$

$$= 1$$

and the indifference condition of the investor becomes

$$\lim_{\tau_p = \infty} \frac{\sqrt{\frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2} + \left(\frac{1}{\tau} + \frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2 \tau_p}\right)\left(\tau + \tau_e \delta_e^* \left(2 - \delta_e^*\right)\right) + \frac{\tau_p}{\tau}\delta_z^* \left(2 - \delta_z^*\right)}}{\sqrt{\frac{\left(\frac{\tau_p}{\bar{\delta}_e^2}\right)^2 x^2}{\tau^2}}} = 2$$

The LHS of above expression is 1 and hence indifference condition cannot be satisfied. This implies that case 1 is not possible. This implies that $\tau_p x$ tends to a finite constant. This immediately implies that for γ sufficiently low, $x < 1$.

□

8.9 Proof of Corollary 3

Denote the return characteristics in the rational expectations equilibrium (optimal expectations asymmetric equilibrium) by RE (AE, respectively). Let $\bar{\delta}_e = \lambda + (1-\lambda)\,\delta_e^*$ and $\bar{\delta}_z = (1-\lambda)\,\delta_z^*$ denote the average beliefs about the precision of private signals and price signal. Note that

$$\tau_{p,AE} - \tau_{p,RE} = \frac{\tau_z \tau_e^2}{\gamma^2}\left(\bar{\delta}_e^2 - 1\right) > 0.$$

$$\begin{aligned}
\theta_{AE} - \theta_{RE} &= \frac{\tau_z}{\Lambda_{AE}\left(\beta_{AE}^2 \tau + \tau_z\right)} - \frac{\tau_z}{\Lambda_{RE}\left(\beta_{RE}^2 \tau + \tau_z\right)} \\
&= \frac{\tau_z}{\Lambda_{AE}\left(\frac{\beta^2}{\bar{\delta}_e^2}\tau + \tau_z\right)}\left(1 - \frac{\Lambda_{AE}}{\Lambda_{RE}}\frac{\beta^2\tau + \bar{\delta}_e^2\tau_z}{\bar{\delta}_e^2\beta^2\tau + \bar{\delta}_e^2\tau_z}\right)
\end{aligned}$$

Since $\bar{\delta}_e > 1$, we have $\frac{\beta^2\tau + \bar{\delta}_e^2\tau_z}{\bar{\delta}_e^2\beta^2\tau + \bar{\delta}_e^2\tau_z} < 1$. Moreover, since $\frac{\Lambda_{AE}}{\Lambda_{RE}} < 1$ by Lemma 4, we have $\theta_{AE} > \theta_{RE}$.

$$\mathbb{E}\left[\mathcal{V}\right] = \int_i \frac{\omega_i}{\gamma}\sqrt{\frac{2}{\pi}\left(\frac{\left(A_i + B_i - \Lambda\right)^2}{\tau} + \frac{A_i^2}{\tau_e} + \frac{\left(B_i - \Lambda\right)^2}{\tau_p}\right)}di \tag{93}$$

$$\mathbb{E}\left[\mathcal{V}_{AE}\right] = \lambda V_1 + (1-\lambda)\,V_2, \text{where} \tag{94}$$

$$V_1 \equiv \frac{(\tau_e + \tau)}{\gamma}\sqrt{\frac{2}{\pi}\left(\frac{1}{\tau}\left(\frac{\tau_e}{\tau + \tau_e} - \Lambda_{AE}\right)^2 + \frac{1}{\tau_e}\left(\frac{\tau_e}{\tau + \tau_e}\right)^2 + \frac{1}{\tau_{p,AE}}\Lambda_{AE}^2\right)} \tag{95}$$

$$V_2 \equiv \frac{\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE}}{\gamma}\sqrt{\frac{2}{\pi}\left(\begin{array}{c}\frac{1}{\tau}\left(\frac{\delta_e^*\tau_e + \delta_z^*\tau_{p,AE}}{\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE}} - \Lambda_{AE}\right)^2 + \frac{1}{\tau_e}\left(\frac{\delta_e^*\tau_e}{\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE}}\right)^2 \\ + \frac{1}{\tau_{p,AE}}\left(\frac{\delta_z^*\tau_{p,AE}}{\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE}} - \Lambda_{AE}\right)^2\end{array}\right)} \tag{96}$$

Let

$$A(x) = \frac{x\tau_e + (1-x)\,\delta_e^*\tau_e}{x(\tau+\tau_e) + (1-x)(\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE})} \tag{97}$$

$$B(x) = \frac{x0 + (1-x)(\delta_z^*\tau_p)}{x(\tau+\tau_e) + (1-x)(\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE})} \tag{98}$$

$$\omega(x) = x(\tau_e+\tau) + (1-x)(\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE}) \tag{99}$$

$$V(x) = \frac{\omega(x)}{\gamma}\sqrt{\frac{2}{\pi}\left(\tfrac{1}{\tau}(A(x)+B(x)-\Lambda)^2 + \tfrac{1}{\tau_e}A(x)^2 + \tfrac{1}{\tau_p}(B(x)-\Lambda)^2\right)} \tag{100}$$

and

$$\mathbb{E}[\mathcal{V}_{AE}] = \lambda V(1) + (1-\lambda)V(0) \tag{101}$$

Note that

$$V(\lambda) = \frac{\omega(\lambda)}{\gamma}\sqrt{\frac{2}{\pi}\left(\tfrac{1}{\tau_e}A(\lambda)^2 + \tfrac{1}{\tau_{p,RE}}(B(\lambda)-\Lambda)^2\right)} \tag{102}$$

$$= \frac{1}{\gamma}\sqrt{\frac{2}{\pi}\left(\tfrac{(\lambda\tau_e+(1-\lambda)\delta_e^*\tau_e)^2}{\tau_e} + \tfrac{((1-\lambda)(\delta_z^*\tau_p)-\lambda(\tau_e+0\tau_p)-(1-\lambda)(\delta_e^*\tau_e+\delta_z^*\tau_p))^2}{\tau_{p,RE}}\right)} \tag{103}$$

$$= \frac{1}{\gamma}\sqrt{\frac{2}{\pi}\left(\tfrac{(\lambda\tau_e+(1-\lambda)\delta_e^*\tau_e)^2}{\tau_e} + \tfrac{(\lambda\tau_e+(1-\lambda)\delta_e^*\tau_e)^2}{\tau_{p,RE}}\right)} \tag{104}$$

$$= \frac{1}{\gamma}\sqrt{\frac{2}{\pi}\left(\bar{\delta}_e^2\tau_e + \frac{\tau_e^2}{\tau_p}\right)} \geq \frac{1}{\gamma}\sqrt{\frac{2}{\pi}\left(\tau_e + \frac{\tau_e^2}{\tau_p}\right)} = \mathbb{E}[\mathcal{V}_{RE}] \tag{105}$$

where $\delta_e \equiv \lambda + (1-\lambda)\,\delta_e^*$. It remains to be shown that:

$$\lambda V(1) + (1-\lambda)V(0) \geq V(\lambda) \tag{106}$$

Note that

$$V(x) = \frac{1}{\gamma}\sqrt{\frac{2}{\pi}\left(\tfrac{1}{\tau}(\alpha(x)+\beta(x)-\Lambda\omega(x))^2 + \tfrac{1}{\tau_e}\alpha(x)^2 + \tfrac{1}{\tau_p}(\beta(x)-\Lambda\omega(x))^2\right)} \tag{107}$$

where

$$\alpha(x) = x\tau_e + (1-x)\,\delta_e^*\tau_e \equiv a_0 + a_1 x \tag{108}$$

$$\beta(x) = x0 + (1-x)(\delta_z^*\tau_p) \equiv b_0 + b_1 x \tag{109}$$

$$\omega(x) = x(\tau_e+\tau) + (1-x)(\tau + \delta_e^*\tau_e + \delta_z^*\tau_{p,AE}) \equiv w_0 + w_1 x \tag{110}$$

$$\frac{V_{xx}}{V^3} = 4\tfrac{\tau+\tau_e+\tau_p}{\pi^2\gamma^4\tau\tau_e\tau_p}(-a_0b_1 + a_0\Lambda w_1 + a_1b_0 - a_1\Lambda w_0)^2 > 0 \tag{111}$$

which implies $V(x)$ is convex, which implies:

$$\mathbb{E}[\mathcal{V}_{AE}] = \lambda V(1) + (1-\lambda) V(0) \geq V(\lambda) \geq \mathbb{E}[\mathcal{V}_{RE}] \tag{112}$$

8.10 Proof of Proposition 6

The utility of noise traders is

$$\begin{aligned} U_z &= -E\left(\gamma_z \exp\{+\gamma_z z (F-P)\}\right) \\ &= -E\left(\gamma_z \exp\{\gamma_z z F (1-\Lambda) - \gamma \Lambda \beta z^2\}\right) \\ &= -E\left(\gamma_z \exp\left\{\left(\frac{\gamma_z^2 (1-\Lambda)^2}{2\tau} - \gamma_z \Lambda \beta\right) z^2\right\}\right) \\ &= -\gamma_z \frac{1}{\sqrt{1 - 2\frac{1}{\tau_z}\left(\left(\frac{\gamma_z^2(1-\Lambda)^2}{2\tau} - \gamma_z \Lambda \beta\right)\right)}} \\ &= -\gamma_z \sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda)^2}{\tau} + 2\gamma_z \Lambda \beta}} \end{aligned}$$

where we used the fact that $E\left(e^{a\epsilon^2}\right) = \frac{1}{\sqrt{1-2a\sigma_\epsilon^2}}$. This implies that utility of noise traders is monotonically decreasing in $\frac{\gamma_z(1-\Lambda)^2}{2\tau} - \Lambda\beta$.

(i) Rational expectations vs. Symmetric equilibrium: In this case, $\bar{\delta}_e = 1$ and

$$\Lambda_{SE} = \frac{\tau_e}{\tau + \tau_e}, \quad \Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p} \tag{113}$$

so

$$U_{SE} - U_{RE} > 0 \tag{114}$$

$$\Leftrightarrow \frac{\gamma \tau \tau_p}{\tau_e(\tau_e+\tau)(\tau_e+\tau_p+\tau)} > \frac{\tau\gamma_z}{2} \frac{\tau_p(2\tau_e+\tau_p+2\tau)}{(\tau_e+\tau)^2(\tau_e+\tau_p+\tau)^2} \tag{115}$$

$$\Leftrightarrow \frac{\gamma}{\gamma_z} > \frac{\tau_e}{\tau+\tau_e} \frac{2(\tau_e+\tau)+\tau_p}{2(\tau_e+\tau_p+\tau)} \tag{116}$$

which implies if $\gamma \geq \gamma_z$, then $U_{SE} > U_{RE}$.

(ii) Rational Expectations vs. Asymmetric Equilibrium: In this case, $\bar{\delta}_e \geq 1$ and

$$\Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p}, \quad \Lambda_{AE} = \frac{\delta_e \tau_e + \delta_z \tau_p}{\tau + \delta_e \tau_e + \delta_z \tau_p} \tag{117}$$

so that

$$U_{AE} - U_{RE} > 0 \tag{118}$$

$$\Leftrightarrow \gamma\left(\frac{\Lambda_{RE}}{\tau_e} - \frac{\Lambda_{AE}}{\tau_e\delta_e}\right) > \gamma_z\left(\frac{(1-\Lambda_{AE})^2}{2\tau} - \frac{(1-\Lambda_{RE})^2}{2\tau}\right) \tag{119}$$

$$\Leftrightarrow \frac{\gamma}{\tau_e}\left(\Lambda_{RE} - \frac{\Lambda_{AE}}{\delta_e}\right) > \frac{\gamma_z}{2\tau}\left(\Lambda_{RE} - \Lambda_{AE}\right)\left(2 - \left(\Lambda_{AE} + \Lambda_{RE}\right)\right) \tag{120}$$

Note that $\delta_e \geq 1$, so it is sufficient to establish:

$$\frac{\gamma}{\tau_e}\left(\Lambda_{RE} - \Lambda_{AE}\right) > \frac{\gamma_z}{2\tau}\left(\Lambda_{RE} - \Lambda_{AE}\right)\left(2 - \left(\Lambda_{AE} + \Lambda_{RE}\right)\right) \tag{121}$$

When $\Lambda_{RE} > \Lambda_{AE}$, the above is equivalent to:

$$\frac{\gamma}{\gamma_z} > \frac{\tau_e}{2}\left(\frac{1}{\tau+\tau_e+\tau_p} + \frac{1}{\tau+\delta_e\tau_e+\delta_z\tau_p}\right) \tag{122}$$

which is always true if $\gamma \geq \gamma_z$.

(iii) Total welfare is given by:

$$W\left(\delta_e, \delta_z\right) = -\frac{1}{\sqrt{\Lambda^2(\delta_z-1)^2 + \left(\frac{(1-\Lambda)^2}{\tau} + \frac{\Lambda^2}{\tau_p}\right)\left(\tau+\tau_e\delta_e(2-\delta_e)+\tau_p\delta_z(2-\delta_z)\right)}} - \sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda)^2}{\tau} + 2\gamma_z\Lambda\beta}} \tag{123}$$

Moreover, for the rational expectations equilibrium, we have $\delta_e = \delta_z = 1$, so that

$$W_{RE} = -\frac{1}{\sqrt{\left(\frac{(1-\Lambda_{RE})^2}{\tau} + \frac{\Lambda^2}{\tau_p}\right)(\tau+\tau_e+\tau_p)}} - \sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda_{RE})^2}{\tau} + 2\gamma_z\Lambda_{RE}\beta_{RE}}} \tag{124}$$

This implies that the difference in welfare is:

$$W_{OE} - W_{RE} = \begin{array}{c} \frac{1}{\sqrt{\left(\frac{(1-\Lambda_{RE})^2}{\tau} + \frac{\Lambda^2}{\tau_p}\right)(\tau+\tau_e+\tau_p)}} + \sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda_{RE})^2}{\tau} + 2\gamma_z\Lambda_{RE}\beta_{RE}}} \\ -\frac{1}{\sqrt{\Lambda_{OE}^2(\delta_z-1)^2 + \left(\frac{(1-\Lambda_{OE})^2}{\tau} + \frac{\Lambda_{OE}^2}{\tau_p}\right)\left(\tau+\tau_e\delta_e(2-\delta_e)+\tau_p\delta_z(2-\delta_z)\right)}} \\ -\sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda_{OE})^2}{\tau} + 2\gamma_z\Lambda_{OE}\beta_{OE}}} \end{array} \tag{125}$$

Above, we have established that when $\gamma_z \leq \gamma$ and $\Lambda_{OE} < \Lambda_{RE}$, we have:

$$U_{z,OE} = -\sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma^2(1-\Lambda_{OE})^2}{\tau} + 2\gamma\Lambda_{OE}\beta_{OE}}} > -\sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda_{RE})^2}{\tau} + 2\gamma_z\Lambda_{RE}\beta_{RE}}} = U_{z,RE} \tag{126}$$

$$\Leftrightarrow \tau_z - \frac{\gamma_z^2\left(1-\Lambda_{OE}\right)^2}{\tau} + 2\gamma_z\Lambda_{OE}\beta_{OE} \geq \tau_z - \frac{\gamma_z^2\left(1-\Lambda_{RE}\right)^2}{\tau} + 2\gamma_z\Lambda_{RE}\beta > 0 \tag{127}$$

Let

$$\bar{\gamma} \equiv \frac{\tau_z - \frac{\gamma_z^2(1-\Lambda_{RE})^2}{\tau}}{\frac{2\gamma_z\Lambda_{RE}}{\tau_e}} = \frac{\tau_e\left(\tau_z(\tau_e+\tau_p+\tau)^2 - \tau\gamma_z^2\right)}{2\gamma_z(\tau_e+\tau_p)(\tau_e+\tau_p+\tau)}. \tag{128}$$

Note that

$$\lim_{\gamma\uparrow\bar{\gamma}} \sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda_{RE})^2}{\tau} + 2\gamma_z\Lambda_{RE}\beta_{RE}}} = \infty, \tag{129}$$

but

$$\lim_{\gamma\uparrow\bar{\gamma}} \begin{array}{c} \frac{1}{\sqrt{\left(\frac{(1-\Lambda_{RE})^2}{\tau} + \frac{\Lambda^2}{\tau_p}\right)(\tau+\tau_e+\tau_p)}} - \sqrt{\frac{\tau_z}{\tau_z - \frac{\gamma_z^2(1-\Lambda_{OE})^2}{\tau} + 2\gamma_z\Lambda_{OE}\beta_{OE}}} \\ -\frac{1}{\sqrt{\Lambda_{OE}^2(\delta_z-1)^2 + \left(\frac{(1-\Lambda_{OE})^2}{\tau} + \frac{\Lambda_{OE}^2}{\tau_p}\right)(\tau+\tau_e\delta_e(2-\delta_e)+\tau_p\delta_z(2-\delta_z))}} \end{array} \geq -c \tag{130}$$

for some $c \leq \infty$. This implies

$$\lim_{\gamma\to\bar{\gamma}} W_{OE} - W_{RE} > 0, \tag{131}$$

or equivalently, $\exists\underline{\gamma} \leq \bar{\gamma}$, such that for all $\gamma > \underline{\gamma}$, $W_{OE} > W_{ER}$. □

9 Appendix B - Belief choice about public signals

In this section, we introduce a public signal $s_q = F + \eta$, where $\eta \sim N\left(0, \tau_\eta^{-1}\right)$ and is independent of all other random variables. We allow each investor to choose how to interpret the quality of the information in the public signal. Specifically, we assume that investor i believes that the noise in the public signal is given by

$$\eta \sim_i \mathcal{N}\left(0, \frac{1}{\delta_{\eta,i}\tau_\eta}\right). \tag{132}$$

Given investor i's subjective beliefs $\delta_{e,i}$, $\delta_{z,i}$, and $\delta_{\eta,i}$ and conditional on her observed signals, s_i,s_p and s_q, investor i's posterior subjective beliefs are given by:

$$F|s_i, s_p \sim_i \mathcal{N}\left(\mu_i, \frac{1}{\omega_i}\right), \text{ where} \tag{133}$$

$$\mu_i \equiv \mathbb{E}_i\left[F|s_i, s_p\right] = m + A_i\left(s_i - m\right) + B_i\left(s_p - m\right) + C_i\left(s_q - m\right), \tag{134}$$

$$\omega_i \equiv \frac{1}{\text{var}_i\left[F|s_i, s_p\right]} = \frac{\tau}{1 - A_i - B_i - C_i}, \text{ and} \tag{135}$$

$$A_i \equiv \frac{\delta_{e,i}\tau_e}{\tau+\delta_{e,i}\tau_e+\delta_{z,i}\tau_p+\delta_{\eta,i}\tau_\eta}, B_i \equiv \frac{\delta_{z,i}\tau_p}{\tau+\delta_{e,i}\tau_e+\delta_{z,i}\tau_p+\delta_{\eta,i}\tau_\eta}\ C_i \equiv \frac{\delta_{\eta,i}\tau_\eta}{\tau+\delta_{e,i}\tau_e+\delta_{z,i}\tau_p+\delta_{\eta,i}\tau_\eta} \tag{136}$$

Similar to the benchmark model, the price can be written

$$P = m + \Lambda\left(s_p - m\right) + C\left(s_q - m\right),$$
$$\text{where } \Lambda = \frac{\int_i \delta_{e,i}\tau_e + \delta_{z,i}\tau_p di}{\int_i \tau+\delta_{e,i}\tau_e+\delta_{z,i}\tau_p+\delta_{\eta,i}\tau_\eta di}, \quad C = \frac{\int_i \delta_{\eta,i}\tau_\eta di}{\int_i \tau+\delta_{e,i}\tau_e+\delta_{z,i}\tau_p+\delta_{\eta,i}\tau_\eta di},$$

and where $s_p = F + \beta z$, $\tau_p = \tau_z/\beta^2$, and $\beta = -\frac{\gamma}{\tau_e \int_i \delta_{e,i} di}$. Given this equilibrium price and investor i's subjective beliefs ($\delta_{e,i}$, $\delta_{z,i}$, and $\delta_{\eta,i}$), her anticipated utility is

$$AU\left(\delta_{e,i}, \delta_{z,i}, \delta_{\eta,i}\right) = -\sqrt{\frac{\text{var}_i\left[F|s_i, P, s_q\right]}{\text{var}_i\left[F - P\right]}}. \tag{137}$$

This expression closely matches the expression in the benchmark model, found in (35). The numerator captures the information channel, updated to reflect the investor's beliefs about the quality of the public signal. The denominator captures the speculative channel. Using this, we can update Lemma 3 to reflect the inclusion of the public signal.

Lemma 5. *Anticipated utility is always*

(i) (weakly) increasing in $\delta_{e,i}$: it is strictly increasing as long as $\delta_{z,i} > 0$, and

(ii) non-monotonic in $\delta_{z,i}$: there exists some $\bar{\delta}_z$ such that for all $\delta_{z,i} < \bar{\delta}_z$ anticipated utility is decreasing in $\delta_{z,i}$ while for all $\delta_{z,i} > \bar{\delta}_z$ it is increasing.

When all other investors ignore public information (i.e., $\delta_{\eta,-i} = 0$), then investor i's anticipated utility is strictly increasing in $\delta_{\eta,i}$: Otherwise (when $\delta_{\eta,-i} > 0$), anticipated utility is non-monotonic in $\delta_{\eta,i}$: there exists some $\bar{\delta}_\eta$ such that for all $\delta_{\eta,i} < \bar{\delta}_\eta$ anticipated utility is decreasing in $\delta_{\eta,i}$ while for all $\delta_{\eta,i} > \bar{\delta}_\eta$ it is increasing.

As with beliefs about price information (i.e., $\delta_{z,i}$), anticipated utility is generically non-monotonic in the investor's perception of its informativeness (i.e., $\delta_{\eta,i}$). Moreover, this non-monotonicity is driven by the same channels. There is an (i) information channel, in which learning from the public signal reduces the conditional variance $\text{var}_i\left[F - P|s_i, P, s_q\right]$ and a (ii) speculative channel, in which a more informative public signal increases the precision of other investors' beliefs, lowering potential speculative trading gains. Note, however, that when all other investors ignore the information in a public signal, it becomes effectively private for investor i — in this case, the information effect dominates because the speculative effect is zero, and anticipated utility is strictly increasing in $\delta_{\eta,i}$. [23]

Finally, we can characterize the equilbrium optimal beliefs which arise with any well-behaved cost function.

Proposition 7. *Suppose the cost function $C(\delta_{e,i}, \delta_{z,i}, \delta_{\eta,i})$ is well-behaved. In any symmetric equilibrium, all investors are (weakly) over-confident about their private signal (i.e., $\delta_{e,i} \geq 1$ for all i), choose to under-react to the information in prices (i.e., $\delta_{z,i} < 1$ for all i), but choose to over-react to the information in the public signal (i.e., $\delta_{\eta,i} > 1$ for all i).*

Though all investors observe the price and the public signal, they respond to the information in each source very differently: in any symmetric equilibrium, investors underreact to information in price, but overreact to the public signal. At any proposed symmetric equilibrium, we show that an investor's anticipated utility increases when she believes that the public signal is more informative. As a result, the equilibrium choice of $\delta_{\eta,i}$ cannot be lower than one: for any proposed equilibrium with $\delta_{\eta,i} < 1$, an investor can increase her anticipated utility and lower her costs by believing the price is more informative. This logic is similar to that which follows Proposition 3 explaining underreaction to prices, but with the relative effect of the two channels flipped. Intuitively, believing that the price is more informative has a direct impact on the speculative opportunity, while believing that the public signal is more informative alters the investor's perceived trading gains indirectly through the actions of other investors. This indirect channel is always dominated by the information channel, and so investors choose to overreact to the public signal but not to prices.

[23] This is similar to the effect found in Myatt and Wallace (2012), whereby acquiring more information from a public source is private information unless all other investors' condition on the same "public" information.