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Designing Stress Scenarios

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Abstract

We study the optimal design of stress scenarios. A principal manages the unknown risk exposures of agents by asking them to report losses under hypothetical scenarios before taking remedial actions. We apply a Kalman filter to solve the learning problem and we relate the optimal design to the risk environment, the principal's preferences and available interventions. Applied to banking stress tests, optimal capital requirements cover losses under an adverse scenario while targeted interventions depend on covariances among residual exposures and systematic risks. Our calibration reveals that information is particularly valuable for targeted interventions as opposed to broad capital requirements.

JEL Classification: G2, D82, D83

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Introduction

Stress tests are ubiquitous in risk management and financial supervision. Risk officers use stress tests to set and monitor risk limits within their organizations, and financial regulators around the world use stress tests to assess the health of financial institutions. For example, financial firms use stress tests to complement their statistical risk management tools (e.g., Value at Risk); asset managers stress test their portfolios; trading venues stress tests their counter-party exposures; regulators mandate large scale stress tests for banks and insurance companies and use the results to enforce capital requirements and validate dividend policies.[1]

Despite the growing importance of stress testing, and the amount of resources devoted to them, there is little theoretical guidance on exactly how one should design stress scenarios. A theoretical literature has focused on the trade-offs involved in the *disclosure* of supervisory information (see Goldstein and Sapra, 2014 for a review), which range from concerns about the reputation of the regulator (Shapiro and Skeie, 2015) to the importance of having a fiscal backstop (Faria-e-Castro et al., 2017). These papers provide insights into disclosure and regulatory actions but they are silent about the design of forward-looking hypothetical scenarios. In that sense, existing models are models of asset quality reviews (and their disclosure) rather than models of *stress testing.*

The goal of our paper is to start filling this void. Risk management is a two-tier process involving risk discovery (learning) and risk mitigation (intervention). Stress tests belong to the risk discovery phase but one cannot analyze the design of a test without understanding the remedial actions that can be taken once the results are known. We therefore model both the risk discovery stage and the risk mitigation stage.

We consider a principal and a potentially large number of agents. The agents can be traders within a financial firm, or they can be financial firms within a financial system. The principal can be a regulator designing supervisory tests, or a risk officer running an internal stress test. For concreteness we will use the supervisory stress testing analogy in much of the paper. The key issue in all cases is that the principal does not directly observe the exposures of agents to systematic risk factors.

The regulator is risk averse and worries about the financial system experiencing large losses in

[1] Central banks in the United States, Europe, England, Brazil, Chile, Singapore, China, Australia, and New Zealand, as well as the International Monetary Fund in Japan, have recently used stress tests to evaluate the banking sector's solvency and guide banking regulation.

some states of the world. The regulator then designs a set of hypothetical scenarios and asks the banks to report their losses under these scenarios. The regulator uses reported losses across all banks and scenarios to extract information about underlying exposures. Based on this information, the regulator decides how to intervene, i.e., she can ask a set of banks to change their exposures to some factors. Interventions are costly, either directly – by drawing on limited regulatory resources, creating disruptions – or indirectly – by preventing banks from engaging in valuable activities.

Our model allows for two types of interventions: broad capital requirements and targeted risk reductions. We can interpret the targeted interventions as limits on specific asset classes (e.g., LTV ratios on particular mortgages) or as matters requiring attention (MRA) in the language of supervision. With regards to capital requirements, banking stress tests often rely on a particular scenario – usually called the adverse scenario – to deliver "pass/fail" grades. The mechanical link between the scenario and the grade conflates learning and intervening, which are two conceptually separate issues.[2] In our baseline model we assume instead that regulators choose optimal actions conditional on the results of the test. This gives them complete freedom to design the most informative scenarios.

Our first main insight comes from writing the learning problem as a Kalman filter. The filter gives us a mapping from prior beliefs and test results into posterior beliefs about exposures. The precision of the mapping depends on the scenarios in the stress test. We can then formulate the regulator's problem as an information acquisition problem in which the regulator chooses the precision of her signals about risk exposures. Formally, we map the primitive parameters of the model, such as the priors of the regulator regarding the banks' exposures, to the endogenous feasible set of posteriors beliefs, usually taken as exogenous in the literature on information acquisition. If, for instance, the regulator is worried about a particular risk factor, we can derive the stress test that maximizes learning about exposures to that factor.

Will the regulator focus a particular risk factor? Or will she try and learn about several factors at the same time? We show how the answers depend on her prior beliefs about the banks' risk exposures and on the information-sensitivity of her interventions. The regulator can always mandate a broad risk reduction, such as an increase in overall capital requirements, which does not require much information but is likely to involve unnecessary changes and disruptions. With more

[2] For instance, imagine that a bank needs the same level of ex-ante equity to satisfy a 9% capital requirement after scenario 1 or a 7% requirement after scenario 2 (presumably because scenario 2 embodies a higher degree of stress). As far as ex-ante capital adequacy is concerned, these two regulations are equivalent.

accurate information the regulator can better target her interventions and reduce the associated costs. The regulator therefore values information insofar as it enables accurate and parsimonious interventions.Whether or not the optimal scenario implies specialization in learning depends on the sensitivity of targeted interventions to stress test information, and on the trade off between noise and information quality along different different dimensions of risk. The reduction in overall information quality depends on the prior distribution of the risk exposures through the Kalman filter.

More generally, the *prior beliefs* of the regulator are central in determining the optimal scenario design.The regulator's priors about average exposures – holding constant her uncertainty – also have two effects on the optimal stress scenario. A higher expected exposure increases the likelihood of intervention, which makes accurate information more valuable. This effect pushes the regulator to learn about factors with high expected risk exposures. On the other hand, when the regulator's prior mean is high, her posterior mean is likely to remain high and thus her action is less responsive to new information, which discourages learning along that dimension. This second effect dominates when the expected risk exposure is high. Hence, the weight of a factor in the stress scenario is hump-shaped with respect to the regulator's prior. With uncorrelated factors, we find optimal scenarios with zero weight on factors with high expected risk exposures. The regulator's prior uncertainty about risk exposures or risk factors and the intervention costs also shape the optimal stress scenario design through the sensitivity of the intervention policy to new information.

Correlated risk exposures, within or across banks, play an important role in the optimal scenario choice, as well. When exposures are correlated, learning about one provides information about the others. The regulator therefore stresses more the factors with correlated exposures. This is true for correlated exposures within a bank as well as correlated exposures across banks. Correlated factors are more systemic and our model predicts that they play an outsized role in scenario design. The regulator may focus mostly on these factors if the correlation is high enough, but, due to the convexity of information sets, specialization is usually incomplete and the design tends to put some weight on all factors.

The optimal design approach in our paper allows us to shed new light on stress tests in practice. First, as a matter of implementation, we can always recast our model in terms of pass/fail outcomes based on pre-specified rules since optimal interventions are predictable *functions* of stress test results. Second, and more importantly, we can use our framework to quantify the value of well designed stress tests for the regulator.

In Section (5), we calibrate our model to U.S. stress tests using quarterly bank-level and macroeconomic data. We illustrate how to apply our framework in the context of credit losses, measured as total net charge offs, and a representative bank. Our calibrated model predicts reasonable scenarios that range between one and two additional standard deviations of the risk factors in times of distress. More interestingly, given our calibration, we find that endogenous new information from stress testing adds little value to the choice of capital requirements compared to a simple rule based on the average adverse scenario. On the other hand, information can be valuable when the planner uses targeted interventions. Quantitatively, stress tests with optimally designed scenarios with targeted interventions can achieve an increase in welfare of the same order of magnitude as a 10% decrease in the cost of bank capital.

Our calibration results speak to a recurring debate about the use of stress tests. Are they simply a way of implementing Basel-style requirements? Or are they meant to uncover hidden risks in the financial system? Our results suggests that stress tests are just an implementation rule when it comes to capital requirements. However, when it comes to targeted interventions information is valuable and scenarios should be designed to elicit new information by deviating from the average bad state.

Literature Review

Most of the literature on stress tests focuses on banking. Several recent papers study specifically the trade-offs involved in disclosing stress test results. Goldstein and Leitner (2018) focus on the Hirshleifer (1971) effect: revealing too much information destroys risk-sharing opportunities between risk neutral investors and (effectively) risk averse bankers. These risk-sharing arrangements also play an important role in Allen and Gale (2000). Shapiro and Skeie (2015) study the reputation concerns of a regulator when there is a trade-off between moral hazard and runs. Faria-e-Castro et al. (2017) study a model of optimal disclosure where the government trades off Lemon market costs with bank run costs, and show that a fiscal backstop allows government to run more informative stress tests. Schuermann (2014) analyzes the design and governance (scenario design, models and projection, and disclosure) for more effective stress test exercises. Schuermann (2016) particularly determines how stress testing in crisis times can be adapted to normal times in order to insure adequate lending capacity and other key financial services. Orlov et al. (2017) looks at the optimal disclosure policy when it is jointly determined with capital requirements, while Gick and Pausch (2014), Inostroza and Pavan (2017), and Williams (2017) do so in the context of Bayesian persuasion. Our model's predictions are consistent with the results in Orlov et al. (2017) that the optimal sequential capital requirements involve a precautionary recapitalization of banks followed by a recapitalization contingent on stress test results. Huang (2021) studies the optimal disclosure in banking networks with potential spillovers and contagion among banks. As argued by Goldstein and Leitner (2020), stress test design and disclosure policy are connected. We complement this strand of papers by explicitly modeling the stress scenario design, which allows us to study the kind of information in the optimal stress test–the relative weight of each factor in the optimal scenario–and not only on how much information it contains.

While most of the existing literature on stress testing, theoretical and empirical, analyzes the disclosure of stress test results, some papers have focused on the risk modeling part of stress testing. For example, Leitner and Williams (2018) focus on the disclosure of the regulator's risk modeling. The paper examines the trade-offs involved in disclosing the model the regulator uses to perform the stress test to banks. Relatedly, Cambou and Filipovic (2017) focus on how scenarios translate into losses when the regulator and the banks face model uncertainty. On the banks' risk modeling side, Colliard (2019) and Leitner and Yilmaz (2019) focus on how a regulator should elicit information from banks when banks can lie or hide information. However, none of these papers consider the optimal scenario design, which is the focus of our paper.

Most empirical papers on stress tests focus on the information content at the time of disclosure, using an event study methodology to determine whether stress tests provide valuable information to investors. Petrella and Resti (2013) assess the impact of the 2011 European stress test exercise. For the 51 banks with publicly traded equity, they find that the publication of the detailed results provided valuable information to market participants. Similarly, Donald et al. (2014) evaluate the 2009 U.S. stress test conducted on 19 bank holding companies and find significant abnormal stock returns for banks with capital shortfalls. Candelon and Sy (2015), Bird et al. (2015), and Fernandes et al. (2015) also find significant average cumulative abnormal returns for stress tested BHCs around many of the stress test disclosure dates. Flannery et al. (2017) find that U.S. stress tests contain significant new information about assessed BHCs. Using a sample of large banks with publicly traded equity, the authors find significant average abnormal returns around many of the stress test disclosures dates. They also find that stress tests provide relatively more information about riskier and more highly leveraged bank holding companies. Glasserman and

Tangirala (2016) evaluate one aspect of the relevance of scenario choices. They show that the results of U.S. stress tests are somewhat predictable, in the sense that rankings according to projected stress losses in 2013 and 2014 are correlated. Similarly, the rankings across scenarios in a given year are also correlated. They argue that regulators should experiment with more diverse scenarios, so that it is not always the same banks that project the higher losses. Acharya et al. (2014) compare the capital shortfalls from stress tests with the capital shortfalls predicted using the systemic risk model of Acharya et al. (2016) based on equity market data. Camara et al. (2016) study the quality of the 2014 EBA stress tests using the actual micro data from the tests.

Finally, our paper is related to the large theoretical literature on information acquisition following Verrecchia (1982), Kyle (1989), and especially Van Nieuwerburgh and Veldkamp (2010). In this class of models, the cost of acquiring information pins down the set of feasible precisions and determines whether the signals are complement or substitutes. Vives (2008) and Veldkamp (Veldkamp) provide a comprehensive review of this literature. These papers take the information processing constraint on the signal precisions as given. In contrast, our paper focuses on the design of the signals that the regulator receives and endogenizes the information processing constraint.

The rest of the paper is organized as follows. Section 1 describes the environment. Section 2 describes how the regulator learns from stress test. Sections 3 and 4 characterize the optimal intervention policy and the optimal stress scenarios, respectively. Section 6 discusses the practical implications of our analysis and concludes.

1 Technology and Preferences

We consider the problem of a principal who wants to manage the risk exposures of a set of agents. The model has several natural interpretations. The principal could be a chief risk officer and the agents could be traders in her firm. The remedial actions could be hedging or downsizing the traders' positions. Alternatively, the principal could be a regulator and the agents could be a set of banks. The remedial actions could be hedging, reducing new deal flows, selling non-performing assets, or raising capital.

To be concrete we use the regulator/banks metaphor when describing the model. The regulator elicits information from the banks in the form of stress tests. In our model, a stress test is a technology used by regulators to ask questions about profits and losses under hypothetical scenarios. The banks cannot evade the questions and have to answer to the best of their abilities.

Banks in our model can only lie by omission: they do not have to volunteer information, but they have to provide estimates of their losses under various scenarios.

1.1 Banks and Risks

There is one regulator overseeing N banks indexed by $i \in [1, .., N]$ exposed to systematic risks. The state of the economy is given by the random vector

$$\mathrm{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_J \end{bmatrix},$$

where $\{s_j\}$ are the J macroeconomic risk factors. To simplify the analysis, we normalize the unconditional expectation of s to zero $\mathbb{E}[\mathrm{s}] = 0$. Therefore, the realization of the state s should be interpreted as a deviation from the baseline.[3] The net worth of bank i in state s is given by

$$w_i(\mathrm{s}) = \overline{w}_i - y_i(\mathrm{s}), \tag{1}$$

where $\overline{w}_i$ is the mean level of net worth and $y_i(\mathrm{s})$ represents the bank's cumulative losses if state s is realized. Bank i's losses in state s are given by

$$y_i(\mathrm{s}) = \sum_{j=1}^{J} x_{i,j} s_j, \tag{2}$$

[3]Regulators specify stress scenarios in terms of traditional macroeconomic variables such as GDP, unemployment, and house prices. In a DSGE model these macro variables would themselves be functions of underlying structural shocks such as productivity, beliefs, risk aversion, etc. Formally, let ϵ^s be the structural shocks and H the solution matrix of the DSGE model, so that $\mathrm{s} = H\epsilon^s$. In a fully specified model, banks' losses would also be functions of the structural shocks: $y_i(\epsilon^{\mathrm{s}}) = \tilde{\mathrm{x}}_{\mathrm{i}}'\epsilon^{\mathrm{s}} + \eta_i$, where $\tilde{\mathrm{x}}_i$ are structural exposures. This equation is equivalent to (2) when H is invertible. In that case we can write $\epsilon^s = H^{-1}\mathrm{s}$ and define $\mathrm{x}_i = H'^{-1}\tilde{\mathrm{x}}_i$, and we obtain $y_i(\mathrm{s}) = \mathrm{x}_i'\mathrm{s} + \eta_i$. In theory the regulator could supply the structural shocks ϵ^s and ask for estimated losses. In practice regulators supply directly the macro variables s. This reflects the fundamental issue of model ambiguity. Even if H is invertible, models for H would likely differ across banks as well as between banks and regulators. By contrast, a handful of macro-economic variables (GDP, credit spreads, house and stock prices, etc.) are well-understood by all participants and capture much of the macro-economic dynamics that matter for expected losses. This is why stress tests are written in terms of s and not ϵ^s.

where $x_{i,j}$ represents the exposure of bank i to factor j.[4] We summarize a bank i's risk exposures in the $J \times 1$ vector

$$\mathrm{x}_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,J} \end{bmatrix},$$

The risk exposures $\{\mathrm{x}_i\}$ are not observed by the regulator. The banks have estimates of these exposures that they use in their (imperfect) internal models to predict losses. We can add bank-specific shocks to Equation (2) but these are not important for our analysis. The aggregate net worth of the banking system is

$$W(\mathrm{s}) \equiv \sum_{i=1}^{N} w_i(\mathrm{s}) = \overline{W} - \sum_{i=1}^{N} y_i(\mathrm{s}), \tag{3}$$

where $\overline{W} = \sum_{i=1}^{N} \overline{w}_i$.

1.2 Regulator's Preferences and Interventions

Following Acharya et al. (2016) we assume that the regulator has preferences $U(W)$ over the aggregate net worth of the banking system.[5] The regulator can affect W in two ways. She can impose capital requirements to set $\overline{W}$. She can also intervene to change banks' exposures to specific risks. These targeted interventions include capital and collateral requirements against specific types of loans or specific borrowers (e.g., LTV ratios in commercial real estate), assets sales and divestitures, as well as supervisory communications on matters requiring (immediate) attention (MRA and MRIA).

[4]We use the term "exposure" to denote the relevant elasticity that determines losses under a given realization of the macroeconomic state. Banks and regulators usually agree on the book value of positions. They can disagree about the value of illiquid positions, and in all cases, liquid or not, the impact of a scenario on the loss on that position needs to be estimated. What we call "exposure" combines the position (measured with near certainty in some cases) with its value under stress scenarios (estimated with error).

[5]The general case is $U([w_i]_{1..N})$, where the idiosyncratic failure of bank i matters regardless of the health of the banking sector as a whole. As in the systemic risk literature, we assume here that only $W = \sum_{i=1}^{N} w_i$ matters. As a result, a financial crisis only happens when the financial system as a whole is under-capitalized. See Philippon and Wang (2021) for a proof of how transfers of assets from under- to well-capitalized banks transform the value function $U([w_i]_{1..N})$ into $U(W)$.

The most granular description of interventions is at the bank×factor level. In some cases, however, a targeted intervention would affect exposures to several factors. We discuss in detail how we model these constraints in Section 3. For now we denote the action as a (large) vector $\mathbf{a}$ in some feasible set $\mathcal{A}$ with the understanding that higher actions reduces exposure more: the vector x_i becomes $(\mathbf{1}_{NJ\times1} - \mathbf{a}_i) \circ \mathrm{x}_i$ where $\mathbf{a}_i$ are the set of actions taken on bank i. Interventions are costly. We let $\mathcal{C}(\mathbf{a})$ denote the cost of action $\mathbf{a}$. Similarly we let $\mathcal{K}\left(\overline{W}\right)$ denote the cost of bank capital.

Let $\mathscr{S}$ denote the information set of the regulator at the time when she chooses her interventions. The regulator chooses $\left(\overline{W}, \mathbf{a}\right)$ to maximize her expected utility

$$\mathbb{E}\left[U\left(\overline{W} - \mathrm{s}\cdot\left(\sum_{i=1}^{N}(\mathbf{1}_{NJ\times1} - \mathbf{a}_i) \circ \mathrm{x}_i,\right)\right)\middle|\mathscr{S}\right] - \mathcal{C}(\mathbf{a}) - \mathcal{K}\left(\overline{W}\right).$$

The more general form $U\left([w_i]_{1..N}\right)$ would depend on the entire distribution of net worth across banks as opposed to just the sum W. The learning (Kalman filter) and design choice would be similar, but the mapping from actions to design would be more complicated.

1.3 Prior beliefs and stress tests

The regulator has prior beliefs over the distribution of exposures within banks and across banks. These prior beliefs come from historical experiences and the regulator's own risk models. We stack the banks' exposures vectors $\{\mathrm{x_i}\}$ in one large $NJ \times 1$ vector as follows

$$\mathbf{x} \equiv \begin{bmatrix} \mathrm{x}_1 \\ \vdots \\ \mathrm{x}_N \end{bmatrix}.$$

We assume that the regulator's prior over the vector of exposures $\mathbf{x}$ is given by

$$\mathbf{x} \sim N\left(\overline{\mathbf{x}}, \Sigma_{\boldsymbol{x}}\right),$$

where the $NJ \times 1$ vector of unconditional means and the $NJ \times NJ$ covariance matrix are, respectively,

$$\overline{\mathbf{x}} = \begin{pmatrix} \overline{\mathrm{x}}_1 \\ \vdots \\ \overline{\mathrm{x}}_N \end{pmatrix} \quad \text{and } \Sigma_{\boldsymbol{x}} = \begin{bmatrix} \Sigma_{\boldsymbol{x}}^1 & \Sigma_{\boldsymbol{x}}^{1,2} & \cdots & \Sigma_{\boldsymbol{x}}^{1,N} \\ \Sigma_{\boldsymbol{x}}^{1,2} & \Sigma_{\boldsymbol{x}}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Sigma_{\boldsymbol{x}}^{(N-1),N} \\ \Sigma_{\boldsymbol{x}}^{1,N} & \cdots & \Sigma_{\boldsymbol{x}}^{(N-1),N} & \Sigma_{\boldsymbol{x}}^N \end{bmatrix}$$

where $\Sigma_{\boldsymbol{x}}^{i} = \mathbb{V}\text{ar}\left(\mathrm{x}_i\right)$ is the $J \times J$ covariance of exposures of bank i, and $\Sigma_{\boldsymbol{x}}^{i,h} = \mathbb{C}\text{ov}\left(\mathrm{x}_i, \mathrm{x}_h\right)$ for all $i \neq h$ is the covariance of exposures across banks.[6] If $\Sigma_{\boldsymbol{x}}^{i}$ is diagonal the regulator expects the exposures of bank i to the different factors to be independent of each other. If $\Sigma_{\boldsymbol{x}}^{i,h} = 0$, the regulator's prior is that the risk exposures of banks i and h are independent. In almost all empirically relevant cases the covariance matrices are not diagonal.

The regulator uses stress tests to learn about the banks' risk exposures and improve the accuracy of her intervention. In a stress test, the regulator asks the banks to estimate and report their losses under a particular realization of the future macroeconomic state. We assume that banks must truthfully report their expected losses (see discussion below). The choice of macroeconomic state is a *scenario* $\hat{s}$.

Definition 1. (Scenario) A scenario $\hat{\mathrm{s}} = (\hat{s}_1, \ldots, \hat{s}_J)'$ is a realization of the vector of states s.

A scenario $\hat{s}$ is a row-vector of size J that represents an aggregate state of the economy. We entertain two interpretations of the size of the state space, J. The simplest way is to think of J as exogenously given. There might be a limited number of macroeconomic variables (GDP, unemployment, house prices) that everyone agrees need to be included in the test. The other way to think about J is as a large number capturing the set of all possible risk factors and in any given tests many have zero loadings. A non-zero weight is then a statement about whether that risk factor is included in the particular stress test. Our model can also then shed light on which risk factors should be used.

Given our normalization of the baseline state to $s = 0$, a scenario close to 0 is a scenario close to the baseline of the economy. A scenario $\hat{s}$ in which element $\hat{s}_j$ is large, represents a large deviation from the baseline along the dimension of factor j. The larger $|\hat{s}_j|$, the more extreme the scenario along dimension j.

Definition 2. (Stress test) A stress test is a collection of M scenarios $\{\hat{\mathrm{s}}^m\}_{m=1}^{M}$ presented by the regulator and a collection of estimated losses $\{\hat{y}_i^m\}_{i=1..N}^{m=1..M}$ reported by the banks.

When designing a stress test, the regulator specifies a *set* of scenarios for which the banks need to report their losses. For each scenario m, each bank i estimates and reports its net losses $\hat{y}_i^m$ given the input parameters in scenario $\hat{\mathrm{s}}^m$.

[6] We assume that the risk exposures of banks are normally distributed in order to apply the Kalman filter. Technically, therefore, it can happen that $x < 0$ but, as usual, we choose parameters to ensure that this is a negligible possibility.

1.4 Stress test results

Banks use imperfect models to predict their losses under the stress test scenarios. Bank i's estimated loss under scenario $\hat{\mathrm{s}}^m$ is

$$\hat{y}_i\left(\hat{\mathrm{s}}^m, M\right) = \hat{\mathrm{s}}^m \cdot \mathrm{x}_i + \hat{\epsilon}_{i,m}\left(\left\|\hat{\mathrm{s}}^m\right\|, M\right), \tag{4}$$

where the error term $\hat{\epsilon}_i\left(\|\hat{\mathrm{s}}\|, M\right)$ is a random variable that captures measurement error and model uncertainty. When a stress test contains multiple scenarios, the results of the stress test reported by one bank i are summarized in the $M \times 1$ vector

$$\hat{\mathrm{y}}_i\left(\hat{S}\right) = \hat{S}\,\mathrm{x}_i + \hat{\varepsilon}_i, \tag{5}$$

where $\hat{\mathrm{y}}_i\left(\hat{S}\right)$ represents the estimated reported losses for bank i under the $M \times J$ matrix $\hat{S}$ that gathers the M scenarios, and the errors in bank i's reported losses are gathered in the $M \times 1$ vector $\hat{\varepsilon}_i$, i.e.,

$$\hat{\mathrm{y}}_i\left(\hat{S}\right) = \begin{bmatrix} \hat{y}_i\left(\hat{\mathrm{s}}^1, M\right) \\ \vdots \\ \hat{y}_i\left(\hat{\mathrm{s}}^M, M\right) \end{bmatrix}, \quad \hat{S} \equiv \begin{bmatrix} \left(\hat{\mathrm{s}}^1\right)' \\ \vdots \\ \left(\hat{\mathrm{s}}^M\right)' \end{bmatrix}, \quad \text{and} \quad \hat{\varepsilon}_i = \begin{bmatrix} \hat{\epsilon}_{i,1}\left(\left\|\hat{\mathrm{s}}^1\right\|, M\right) \\ \vdots \\ \hat{\epsilon}_{i,M}\left(\left\|\hat{\mathrm{s}}^M\right\|, M\right) \end{bmatrix},$$

where $\hat{\varepsilon}_i$ is a normally distributed random vector with mean zero and variance-covariance $\boldsymbol{\Sigma}^i_{\hat{\varepsilon}} \equiv \mathbb{V}ar\left[\hat{\varepsilon}_i\right]$, and $\hat{\varepsilon}_i \perp \mathrm{x}_j$ for all i, j. We assume that the banks' models perform worse for scenarios that are further from the baseline and when they have more losses to report, i.e., $\mathbb{V}ar\left(\hat{\epsilon}_i\left(\|\hat{\mathrm{s}}\|, M\right)\right)$ is increasing in the norm of the scenarios $\|\hat{\mathrm{s}}\|$ and in the number of scenarios M. Moreover, since banks build their internal risk models using historical data, the mistakes they make are likely to be correlated *across scenarios*. This correlation is captured by the non-diagonal elements of $\boldsymbol{\Sigma}^i_{\hat{\varepsilon}}$.

Differences in $\boldsymbol{\Sigma}^i_{\hat{\epsilon}}$ across banks reflect differences in priors, in the amount or quality of data available to each bank, or in the bank's information processing capacity. We assume that x_i and $\hat{\varepsilon}_i$ are independent, but we allow banks to make correlated mistakes. To guarantee an interior solution, we assume that $\mathbb{V}ar\left(\hat{\epsilon}_i\left(\|\hat{\mathrm{s}}\|, M\right)\right)$ is continuous and that $\lim_{\hat{s}_j \to \infty} \frac{\left(\hat{s}_j\right)^2}{\mathbb{V}ar\left(\hat{\epsilon}_i\left(\|\hat{\mathrm{s}}\|, M\right)\right)} = 0$ for all m and for all j.

Remark 1. **(Truthful reporting)** We assume throughout the paper that banks truthfully report their expected losses to regulators. Several reasons explain our choice to abstract from incentives

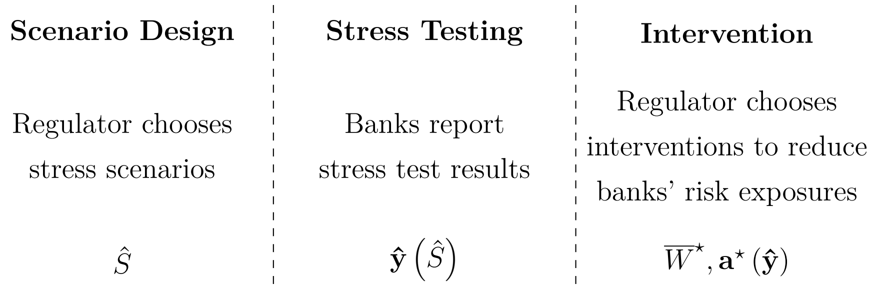


Figure 1: Timeline

issues in the reporting of stress tests results. A pragmatic reason is that the model without incentives is already rich and complex to understand. Another reason is that actively lying to regulators is unlikely to be a good or even feasible strategy. Running stress tests is a massive undertaking involving many employees and it would be difficult for the top management to convince all of them to deliberately mislead their regulator without getting caught. In addition the regulator can compare results across banks. As long as banks do not collude, the regulator can spot a bank that produces an excessively optimistic loss estimate.

Remark. The more relevant issues, then, are lying by omission and statistical bias. The fact that banks can lie by omission is actually supportive of our model. Banks do not need to answer questions that are not asked and this reinforces the importance of designing the right scenarios. Statistical bias is a different matter. Each risk unit within the bank would accept at face value results that look fine, but would have an incentive to investigate and modify a model that delivers a large predicted loss. This issue is similar to p-hacking in academia. While we agree that this is a real concern, it does not necessarily invalidate our approach. The average bias drops out in a rational expectation equilibrium since the regulator can always infer it (Holmström, 1999). What is left is noise, as assumed in the paper. This does not mean that inducing truthful reporting is not important, but it suggests that our approach is useful even in a world of imperfect reporting.

1.5 Timing

To summarize, there are three stages in our model: the scenario design stage, the stress testing stage, and the intervention stage. The regulator first chooses the stress scenarios. The regulator then extract information from the results of the test. The regulator finally chooses her targeted interventions and capital requirements. Figure (1) shows the timeline of the model. The regulator takes into account that her choices affect the informativeness of the test and the efficiency of her interventions.

2 Learning

The bank's model summarized in Equation (5) implies that reported losses contain information about the true vector of exposures $\mathbf{x}$. The Bayesian regulator updates her prior beliefs before deciding on her interventions. In our linear Gaussian setting the regulator's learning can be expressed as a filtering problem in which the regulator's optimal updating is given by the Kalman filter.[7]

2.1 A Kalman Filter

A key insight of our paper is that stress test results can be interpreted as signals about the banks' risk exposures by defining the error terms and the signals appropriately. Before diving into the regulator's updating/filtering problem, it is useful to briefly summarize all the sources of risks and information in our model:

1. For each bank i, $\hat{y}_i\left(\hat{S}\right)$ is an $M \times 1$ vector that summarizes the estimated losses in the various scenarios. This is the key source of information for the regulator.

2. For each bank i, the $M \times M$ matrix $\mathbf{\Sigma}^i_{\hat{\varepsilon}} \equiv \mathbb{V}ar\left[\hat{\varepsilon}_i\right]$ contains the size and correlations of estimation errors across the M scenarios. Element m of the $M \times 1$ vector of errors $\hat{\boldsymbol{\varepsilon}}_i$ is given by $\hat{\epsilon}_{i,m}\left(\|\hat{\mathrm{s}}^m\|, M\right)$. The estimation errors can be correlated within and across banks.

3. For all banks and risk factors, the $NJ \times NJ$ covariance matrix $\Sigma_{\boldsymbol{x}}$ contains the priors of the regulators regarding exposures within and across banks. The covariance matrix $\Sigma_{\boldsymbol{x}}$ is predetermined and unaffected by the scenarios.

[7]See Chapter 2 in Veldkamp (Veldkamp) for a textbook analysis of the Kalman filter.

To interpret the stress test results as signals about the banks' risk exposures it is useful to define the reported losses and error terms in the $NM \times 1$ vectors

$$\hat{\mathbf{y}} \equiv \begin{bmatrix} \left[\hat{y}_1\left(\hat{S}\right)\right] \\ \vdots \\ \left[\hat{y}_N\left(\hat{S}\right)\right] \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} [\hat{\varepsilon}_1] \\ \vdots \\ [\hat{\varepsilon}_N] \end{bmatrix}.$$

Then, using these definitions, the state-space representation of the reported losses is

$$\hat{\mathbf{y}} = \hat{\mathbf{S}}\mathbf{x} + \hat{\boldsymbol{\varepsilon}}, \tag{6}$$

where $\hat{\mathbf{S}} \equiv \left(\mathbf{I}_N \otimes \hat{S}\right)$ simply repeats $\hat{S}$ on its diagonal, and $\hat{\boldsymbol{\varepsilon}} \sim N\left(0, \boldsymbol{\Sigma}_{\hat{\varepsilon}}\right)$.

The regulator observes $\hat{\mathbf{y}}$ and wants to learn about $\mathbf{x}$. Expressing the stress test as in equation (6) allows us to apply the Kalman filter and to obtain a full characterization of the posterior beliefs of the Bayesian regulator.

Lemma 1. ***Kalman Filter.*** *After observing the results* $\hat{\mathbf{y}}$ *of the stress test, the posterior beliefs of the regulator regarding the banks' risk exposures are*

$$\mathbf{x}|\,\hat{\mathbf{y}} \sim N\left(\hat{\mathbf{x}}, \hat{\Sigma}_{\mathbf{x}}\right) \ ,$$

where the posterior mean $\hat{\mathbf{x}}$*, the Kalman gain* K*, and the residual covariance matrix* $\hat{\Sigma}_{\mathbf{x}}$ *are given by*

$$\hat{\mathbf{x}} = \left(\mathbf{I}_{NJ} - K\hat{\mathbf{S}}\right)\bar{\mathbf{x}} + K\hat{\mathbf{y}}\,, \tag{7}$$

$$K = \Sigma_{\mathbf{x}}\hat{\mathbf{S}}'\left(\hat{\mathbf{S}}\Sigma_{\mathbf{x}}\hat{\mathbf{S}}' + \boldsymbol{\Sigma}_{\hat{\varepsilon}}\right)^{-1}\ , \tag{8}$$

$$\hat{\Sigma}_{\mathbf{x}} = \Sigma_{\mathbf{x}} - K\hat{\mathbf{S}}\Sigma_{\mathbf{x}}\,. \tag{9}$$

Lemma 1 is a direct application of the Kalman filter. The Kalman gain K is an $NJ \times MN$ matrix. A few special cases can give some intuition. With one bank ($N = 1$), then $K_{j,m}$ is a measure of the amount of information about the exposure to risk factor j contained in the results from scenario m. With one scenario ($J = 1$) and uncorrelated exposures among banks, $K_{j,m}$ also measures the reduction in uncertainty about bank j's exposure to the risk factor.

The posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}}$ plays a critical role in our analysis. The true exposures are distributed around $\hat{\mathbf{x}}$ with covariance $\hat{\Sigma}_{\mathbf{x}}$. Thus, $\hat{\Sigma}_{\mathbf{x}}$ measures the *residual uncertainty* that persists after observing the results of the stress test. The goal of the stress test is to reduce this residual uncertainty as much as possible, along dimensions that depend on the objective function.

In the standard state-space representation in Equation (6), the stress test scenarios determine the structure of the signals observed by the regulator by controlling the weight of each exposure in the reported losses. The scenarios also determine the precision of the banks' reported losses in Equation (4). Increasing $|s_j|$ in a scenario makes the results more informative about exposures to factor j, but extreme scenarios reduce the precision of the banks' estimates and the noise might spill over to the measurement of other exposures. On the other hand, the regulator can improve her learning by taking into account the fact that true exposures are correlated across positions and across banks.

When designing the scenarios, the regulator must anticipate how she will interpret and use the results of the test. The extent to which learning takes place is captured *ex ante* by the *distribution of the posterior mean*, given by

$$\hat{\mathbf{x}} \sim N\left(\overline{\mathbf{x}}, \Sigma_{\hat{\mathbf{x}}}\right), \tag{10}$$

where the *expected variance of the posterior mean*, $\Sigma_{\hat{\mathbf{x}}}$, is given by

$$\Sigma_{\hat{\mathbf{x}}} \equiv \Sigma_{\mathbf{x}} - \hat{\Sigma}_{\mathbf{x}} = K\hat{\mathbf{S}}\Sigma_{\mathbf{x}}.$$

The matrix $\Sigma_{\hat{\mathbf{x}}}$ represents the expected amount of learning from the stress test. If the stress test is pure noise, $K = 0$, the regulator learns nothing, $\Sigma_{\hat{\mathbf{x}}} = 0$, and $\hat{\Sigma}_{\mathbf{x}} = \Sigma_{\mathbf{x}}$. If the test is fully informative, then $\hat{\Sigma}_{\mathbf{x}} = 0$ and the regulator learns exactly all the exposures, i.e., $\Sigma_{\hat{\mathbf{x}}} = \Sigma_{\mathbf{x}}$. The regulator seeks to maximize $\Sigma_{\hat{\mathbf{x}}} = K\hat{\mathbf{S}}\Sigma_{\mathbf{x}} = K\left(\mathbf{I}_N \otimes \hat{S}\right)\Sigma_{\mathbf{x}}$ – or equivalently to minimize the residual uncertainty $\hat{\Sigma}_{\mathbf{x}}$ – so as to be able to design an accurate policy intervention. When choosing what to learn, the regulator takes into account that the Kalman gain K is itself a function of the scenarios, given by equation (8).

Remark 2. **(Endogenous noise)** If $\sigma_{\hat{\epsilon}}$ is exogenous, then learning is trivially maximized by sending $\hat{s}_1 \to \infty$. In reality extreme scenarios are more difficult to estimate. This is why we assume that $\sigma_{\hat{\epsilon}}$ increases when the scenario deviates more from the baseline. We obtain an interior solution as long as $\sigma_{\hat{\epsilon}}$ is convex enough in $\|\hat{s}\|$.

2.2 Feasible Information Sets

Every set of stress scenarios $\hat{S}$ delivers a unique posterior covariance matrix. More precisely, the Kalman filter provides a mapping of $\frac{JN(JN-1)}{2}$ equations from the $J \times M$ elements in the scenario matrix $\hat{S}$ into the elements of $\Sigma_{\hat{\mathbf{x}}}$. Choosing a set of scenarios $\hat{S}$ is therefore equivalent to choosing a posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}}$ in the set $\mathbf{\Sigma}$ given by Equations (8) and (9). The shape of the feasibility set $\mathbf{\Sigma}$ is determined by the regulator's priors $\Sigma_{\boldsymbol{x}}$ and by the errors in banks' models $\mathbf{\Sigma}_{\hat{\boldsymbol{\varepsilon}}}$.

We can therefore think of the scenario design problem as choosing an information point in a feasible set determined by the Kalman filter. Figure (2) shows the feasible set of posterior variances $\left\{\hat{\Sigma}_{\mathbf{x},11}, \hat{\Sigma}_{\mathbf{x},22}\right\}$ in a model with one bank and two risk factors, for different values of prior correlations among risk exposures.[8]

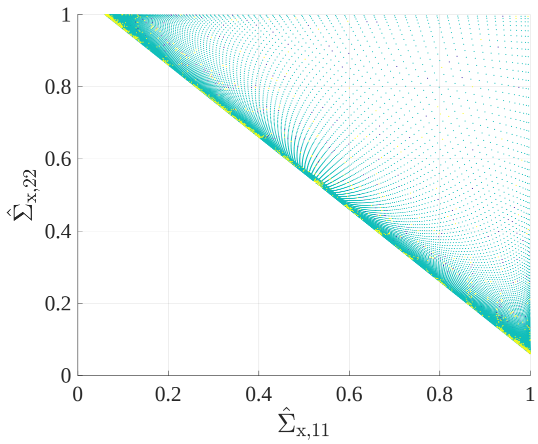
Perfect learning is not feasible, as can be seen in Figures 2 where the budget does not include a variance of zero. As discussed above, choosing a more extreme scenario has two effects on the amount of information that the regulator can acquire. On the one hand, a higher value of $\hat{s}_i$ increases the weight the bank's stress test results put on the bank's exposure to factor i. On the other hand, more extreme scenarios increase the noise $\mathbf{\Sigma}_{\hat{\boldsymbol{\varepsilon}}}$ in the stress tests result.

Prior correlation among risk exposures makes it easier to learn and reduces the posterior variances. The regulator cannot learn about the bank's exposure to factor 1 without learning about the bank's exposure to factor 2. Hence, as it can be seen from panels *a, b, c* and *d* in Figure 2, the boundary of set of feasible posterior precisions, $\mathbf{\Sigma}$, becomes more convex. When the prior correlation is high, the boundary of the feasible set slopes up in the tails. When s_1 is already large, it can become more efficient to learn about x_1 by increasing s_2 instead of s_1.

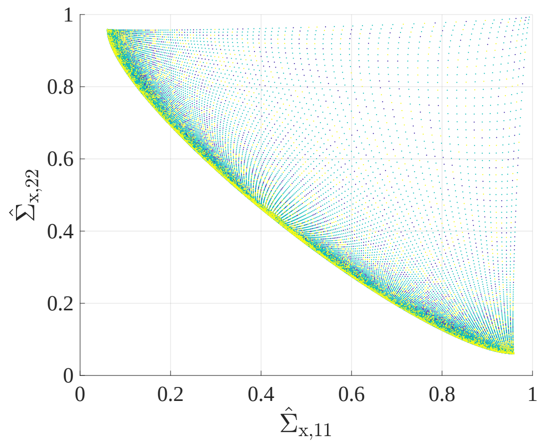
3 Taking Action

The regulator values information from the stress test because it allows her to intervene more accurately. In return, the design of optimal scenarios depends on the actions that the regulator expects to take. Regulators typically have two ways of intervening in the banking sector. They can mandate a broad increase in capital, or they can restrict specific activities, for instance by

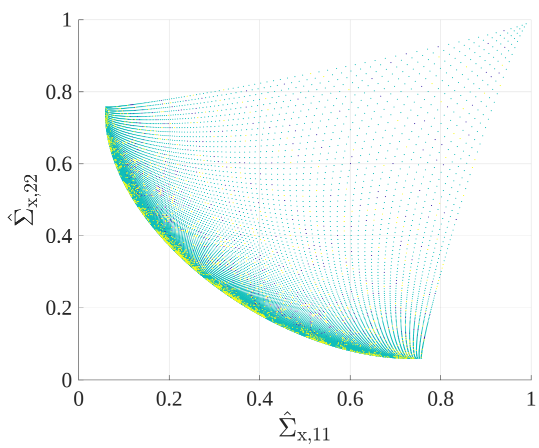
[8] When the number of banks is higher than the number of scenarios ($N > M$), the design problem boils down to choosing residual variances about the risk exposures of any M banks since it is equivalent for the regulator to choose the stress scenarios or to choose $J \times M$ elements of the residual covariance matrix.



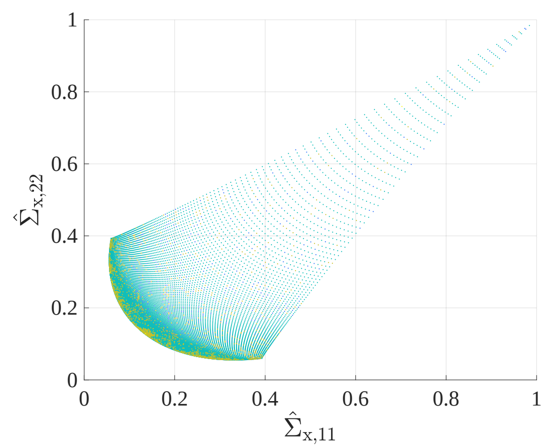
(a) Prior correlation in exposures $\Sigma_{x,12} = 0$



(b) Prior correlation in exposures $\Sigma_{x,12} = 0.2$



(c) Prior correlation in exposures $\Sigma_{x,12} = 0.5$



(d) Prior correlation in exposures $\Sigma_{x,12} = 0.8$

Figure 2: Feasible set of residual variances, $\left(\hat{\Sigma}_{\mathbf{x},11}, \hat{\Sigma}_{\mathbf{x},22}\right)$ when there are two factors and one representative bank for different values of the regulator's prior correlation among the bank's risk exposures to factors 1 and 2.

Note: Figures 1 illustrates the set of feasible posterior variances, $\mathbf{\Sigma}$ for different values of prior correlations among risk exposures when $\sigma_{\tilde{\epsilon}}^2 = \beta \left\|\hat{s}\right\|^2 + \beta_\tau e^{\tau \|\hat{s}\|^2}$. The parameters used are $M = 1$, $N = 1$, $J = 2$, $\theta = 0$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\beta = 5$, $\beta_\tau = 5$, $\tau = 2$, and $\mathbb{E}\left[\epsilon^2\right] = 1$.

imposing loan-to-value ratios or collateral requirements.

3.1 General Case

If the regulator takes action $\mathrm{a}_i = \{a_{i,j}\}_{j=1..J}$ on bank i, the exposure of bank i to factor j becomes $(1 - a_{i,j})\, x_{i,j}$. Aggregate banking wealth is then given by

$$W\left(s, x; \mathbf{a}, \overline{W}\right) = \overline{W} - \sum_{i=1}^{N} \sum_{j=1}^{J} (1 - a_{i,j})\, x_{i,j} s_j, \tag{11}$$

where $\overline{W}$ is the capital required by the regulator. The regulator takes action after observing the results of the test, therefore her expected utility in the intervention stage is a function of her information set

$$V(\mathscr{S}) = \max_{\overline{W}, \mathbf{a} \in \mathcal{A}} \mathbb{E}\left[U\left(W\left(s, x; \mathbf{a}, \overline{W}\right)\right) \mid \mathscr{S}\right] - \mathcal{C}(\mathbf{a}) - \mathcal{K}\left(\overline{W}\right),$$

where $\mathscr{S}$ denotes the information set after the stress test is conducted.

In an interior solution, the first order conditions equate the marginal cost of an intervention to its expected marginal benefit. For capital requirements we obtain

$$\mathcal{K}'\left(\overline{W}\right) = \mathbb{E}\left[U'(W) \mid \mathscr{S}\right]. \tag{12}$$

Since capital is useful in all states of the world the optimality condition simply states that the marginal cost of banking capital be equal to the expected marginal utility of banking net worth. Similarly, the optimal targeted intervention on the exposure of bank i to factor j is

$$\frac{\partial \mathcal{C}(\mathbf{a})}{\partial a_{i,j}} = \mathbb{E}\left[x_{i,j} s_j U'(W) \mid \mathscr{S}\right]. \tag{13}$$

The expected marginal benefit of reducing the risk exposure to factor j in bank i depends on the covariance between the marginal social utility $U'(W)$ and the contribution of factor j to bank i's losses, $x_{i,j} s_j$. Risk reduction is more valuable when the planner expects high losses in states of the world where U' is also large.

3.2 Pseudo Mean-Variance Preferences

The first order conditions (12) and (13) characterize the planner's solution for a *given* information set. The key point of our model, however, is that the information set is endogenous. The FOCs thus represent only an intermediate step in the planner's optimization problem. To keep the analysis tractable, we assume a specific form for the planner's utility function:

$$U(W) = \begin{cases} W & \text{if} \quad \mathscr{D} = 0, \\ W - \gamma\theta(\mathbf{W} - W) - \frac{\gamma}{2}(\mathbf{W} - W)^2 & \text{if} \quad \mathscr{D} = 1, \end{cases} \tag{14}$$

where $\mathscr{D}$ is a dummy variable for economy-wide financial distress and $\mathbf{W}$ is a satiated level of capital. Distress is simply defined as a partition over the states $\{s_j\}$. We group all the normal states under $\mathscr{D} = 0$, with probability $1 - p$. There are no externalities in these states and $U(W) = W$. In the states where $\mathscr{D} = 1$, on the other hand, a capital shortfall creates negative externalities for the economy. We assume that the satiated level of capital $\mathbf{W}$ is high enough that $W < \mathbf{W}$ when $\mathscr{D} = 1$, and with a slight abuse we say that these preferences are *linear quadratic*.[9] The utility function implies that the regulator's absolute risk aversion is higher in bad states of the world – as it would be under CRRA preferences for example – while retaining the tractability of mean-variance analysis. The parameters γ and θ capture the regulator's risk aversion: θ measures the marginal utility of bank capital when the economy is in distress, and γ pins down the elasticity of capital requirements to changes in fundamentals.

For ease of notation we define the probability of distress as

$$p \equiv \Pr[\mathscr{D} = 1].$$

and we define $\tilde{\mathbb{E}}$ as the expectation conditional on the economy being in distress, i.e., $\tilde{\mathbb{E}}[.] \equiv \mathbb{E}[. \mid \mathscr{D} = 1]$. We assume that $\tilde{\mathbb{E}}[s] > 0$ and we define $\tilde{\Sigma}_{\mathrm{s}} \equiv \mathbb{V}ar[\mathrm{s} | \mathscr{D} = 1]$. We can then write the expected utility of the regulator as

$$\mathbb{E}[U(W)] = \mathbb{E}[W] - p\gamma\theta\tilde{\mathbb{E}}[(\mathbf{W} - W)] - \frac{p\gamma}{2}\tilde{\mathbb{E}}\left[(\mathbf{W} - W)^2\right], \tag{15}$$

Since we have normalized $\mathbb{E}[\mathrm{s}] = 0$ we see from (11) that $\mathbb{E}[W] = \overline{W}$. Therefore, with linear quadratic preferences the planner's intervention problem is

$$\max_{\overline{W}, \mathbf{a} \in \mathcal{A}} \overline{W} - \mathcal{K}\left(\overline{W}\right) - p\gamma\theta\tilde{\mathbb{E}}[(\mathbf{W} - W)] - \frac{p\gamma}{2}\tilde{\mathbb{E}}\left[(\mathbf{W} - W)^2 \mid \mathscr{S}\right] - \mathcal{C}(\mathbf{a}). \tag{16}$$

[9] We must of course make sure that the realized values of W are consistent with our assumptions, i.e., that W is high in the normal states and lower than $\mathbf{W}$ in the distress states.

with $W = \overline{W} - \sum_{i=1}^{N}\sum_{j=1}^{J}(1-a_{i,j})\,x_{i,j}s_j$. The first order condition for the optimal capital requirement is

$$\mathcal{K}'\left(\overline{W}^{\star}\right) = 1 + p\gamma\theta + p\gamma\left(\mathbf{W} + \sum_{i=1}^{N}\sum_{j=1}^{J}(1-a_{i,j})\,\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right] - \overline{W}^{\star}\right) \tag{17}$$

and the first order conditions for optimal actions are

$$\frac{\partial\mathcal{C}(\mathbf{a})}{\partial a_{i,j}} = \hat{x}_{i,j}\mathbb{E}\left[s_j\right] + p\gamma\theta\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right] + p\gamma\tilde{\mathbb{E}}\left[x_{i,j}s_j\left(\mathbf{W} - W\right) \mid \mathscr{S}\right]. \tag{18}$$

For the remainder of the paper we assume the following functional forms.

Assumption L. *The cost of bank capital and the cost of targeted actions are linear, i.e.,*

$$\mathcal{K}\left(\overline{W}\right) = (1+\kappa)\,\overline{W}, \quad \text{and} \quad \mathcal{C}(\mathbf{a}) = \Phi'\mathbf{a}.$$

We assume that the costs of bank equity is linear for simplicity since we have included curvature in the utility function. To keep the analysis symmetric between targeted interventions and bank capital, we also assume a linear cost of targeted interventions.[10]

3.3 Standard Capital Requirement

As it can be seen from Equation (17)optimal capital requirements are increasing in risk aversion γ, in the probability of distress p, and in the estimated risk exposures $\hat{x}$. Capital requirements do not depend directly on the variance-covariance matrices of states or exposures but, in general, the requirements are mitigated by targeted interventions. Suppose, for example, that a stress test uncovers excessive exposures to commercial real estate lending. Without targeted interventions the regulator would have to increase overall requirements. With targeted interventions, on the other hand, the regulator might instead mandate lower LTV ratios for that specific class of loans.

To understand the connection between our optimal capital requirement and actual stress tests it is useful to consider first a model *without* targeted interventions since actual stress tests are conducted "all else equal", i.e., assuming no actions from the regulator. Let us then assume $\mathbf{a} = 0$. We obtain the following proposition.

[10] All results are valid qualitatively with $\mathcal{C}(\mathbf{a}) = \Phi_0'\mathbf{a} + \frac{1}{2}\mathbf{a}'\Phi_1\mathbf{a}$ or with a convex cost of capital. Also note that Assumption L only needs to hold over the relevant range.

Lemma 2. ***(Standard capital requirement)*** *Optimal capital requirements (assuming* $\mathbf{a} = 0$*) are a linear function of expected losses under distress:*

$$\overline{W}^{\star} + \frac{\kappa}{p\gamma} = \mathbf{W} + \theta + \sum_{i=1}^{N} \mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right], \tag{19}$$

where $\mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right] = \tilde{\mathbb{E}}\left[\mathrm{s}\right] \cdot \mathbb{E}\left[\mathrm{x_i} \mid \mathscr{S}\right]$.

The net marginal cost of bank equity $\mathcal{K}'\left(\overline{W}^{\star}\right) - 1$ is compared to the adjusted risk of distress $p\gamma$.[11] Net of this cost, capital requirements are set to cover losses in the adverse scenario. It is important to understand the similarities and the differences between our Lemma 2 and what regulators do in practice. Exactly as in standard stress tests, our model says that the requirements should be set to cover losses under an adverse scenario. The adverse scenario in our model is the expectation of the state conditional on potential distress, $\tilde{\mathbb{E}}\left[\mathrm{s}\right] = \mathbb{E}\left[\mathrm{s} \mid \mathscr{D} = 1\right]$.

The main difference is that, in our model, the regulator uses expected exposures from the Kalman filter $\mathbb{E}\left[\mathrm{x_i} \mid \mathscr{S}\right]$. In general, therefore $\mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right] \neq \hat{y}_i\left(\tilde{\mathbb{E}}\left[\mathrm{s}\right]\right) = \tilde{\mathbb{E}}\left[\mathrm{s}\right] \cdot \mathrm{x}_i + \hat{\epsilon}_i\left(\|\tilde{\mathrm{s}}\|\right)$: the optimal forecast of losses under the adverse scenario differ from the losses the bank would report under the adverse scenario. Without noise, of course, $\hat{y}_i\left(\tilde{\mathbb{E}}\left[\mathrm{s}\right]\right) = \tilde{\mathbb{E}}\left[\mathrm{s}\right] \cdot \mathrm{x}_i$ would be exact. Given unavoidable measurement errors and model mis-specifications, however, the regulator optimally uses reported losses from other scenarios ($\hat{\mathrm{s}} \neq \tilde{\mathrm{s}}$) and from other banks ($j \neq i$) to compute $\mathbb{E}\left[\mathrm{x_i} \mid \mathscr{S}\right]$.

Remark 3. **(unconditional capital requirements)** Under Assumption LQ we have a form of certainty equivalence. The expected level of capital requirements depends only on the regulator's priors:

$$\mathbb{E}_{\hat{x}}\left[\overline{W}^{\star}\right] = \mathbf{W} + \theta + \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[\mathrm{s}\right] \cdot \overline{\mathrm{x}}_{\mathrm{i}} - \frac{\kappa}{p\gamma}. \tag{20}$$

We can therefore implement the optimal capital requirement in two steps. The regulator sets unconditional requirements based on her priors as in (20) and then makes (mean-zero) adjustments based on the result of the test. The adjustments are valuable because they allow the regulator to tighten requirements when exposures are high and loosen them when they are low. This result

[11] Without any cost of raising bank equity, it would trivially be optimal to set requirements at the satiated at the level $\mathbf{W}$that brings the marginal utility of net worth back to 1 in all states of the world. As explained earlier, the formula above assumes an interior solution where $W < \mathbf{W}$ when $\mathscr{D} = 1$, which is the empirically realistic case.

is consistent with Orlov et al. (2017) that argues that the optimal sequential stress test consists of a precautionary recapitalization (our expected capital requirement) followed by contingent recapitalizations based on an informative stress test, as needed (our adjustments based on the stress test results).

3.4 Optimal Interventions and the Distress Uncertainty Matrix

Targeted interventions require more information than standard capital requirements. More specifically, targeted interventions depend on the posterior mean exposures and on a specific covariance matrix.

Using the optimal capital requirement from equation (17) the regulator's choice of targeted actions simplifies to[12]

$$\min_{\mathbf{a}\in\mathcal{A}} \kappa \overline{W}^{\star}(\mathbf{a}) + \frac{p\gamma}{2}\tilde{\mathbb{E}}\left[\left(\frac{\kappa}{p\gamma} + \sum_{i=1}^{N}\sum_{j=1}^{J}(1-a_{i,j})\left(x_{i,j}s_j - \hat{x}_{i,j}\tilde{\mathbb{E}}[s_j]\right)\right)^2\right] + \mathcal{C}(\mathbf{a}).$$

The optimal interventions are therefore given by

$$\mathbf{a}^{\star} = \left(p\gamma\tilde{\mathbb{V}}\right)^{-1}\left(\kappa\left(\mathbf{1}_{N\times N}\otimes\tilde{\mathbb{E}}[\mathrm{s}]\right)\circ\hat{\mathbf{x}} - \Phi + p\gamma\tilde{\mathbb{V}}\mathbf{1}_{NJ\times 1}\right), \tag{21}$$

where $\tilde{\mathbb{V}}$ is the **Distress Uncertainty Matrix** given by

$$\tilde{\mathbb{V}} \equiv \mathbb{COV}\left[(\mathbf{1}_N\otimes\mathrm{s})\circ\mathbf{x} \mid \mathscr{S}, \mathscr{D}=1\right]. \tag{22}$$

The matrix $\tilde{\mathbb{V}}$ is the covariance of the NJ vector $(\mathbf{1}_N\otimes\mathrm{s})\circ\mathbf{x} = (x_{i,j}s_j)_{i=1:N}^{j-1:J} = [s_1x_{1,1},..,s_Jx_{1,J},s_1x_{2,1}..,s_Jx_{N,J}]$. This covariance is conditional on the stress test results – ex-post mean $\hat{x}$ and residual variance $\hat{\Sigma}_{\mathbf{x}}$ – and on the distribution of risks under distress – $\tilde{\mathbb{E}}[\mathrm{s}]$ and $\tilde{\Sigma}_{\mathrm{s}}$. The residual uncertainty is known in advance since the evolution of the covariance matrix is deterministic, but the posterior mean depends on the random realization the test itself, since $\hat{\mathbf{x}} = \bar{\mathbf{x}} + K(\hat{\mathrm{y}} - \hat{\mathrm{s}}'\bar{\mathbf{x}})$. Since s and $\hat{\mathbf{x}}$ are independent we can write the covariance matrix as[13]

[12]Note that $p\gamma\theta\left(\frac{\kappa}{p\gamma} + \sum_{i=1}^{N}\sum_{j=1}^{J}(1-a_{i,j})\tilde{\mathbb{E}}[(x_{i,j}s_j - \hat{x}_{i,j}\tilde{s}_j)]\right) = 0$.

[13]We have assumed that the error term ε in stress results is independent of future realization of the risk factors s. While this is an obvious assumption to make at this point, we note that it is not without loss of generality if we consider endogenous financial crises. Suppose, for example, that banks are too optimistic about mortgage risk: ε is negative and their perceived exposures are lower than their true exposures. This might lead to excessive lending, real estate price appreciation, and this might increase the probability of a future decrease in real estate prices. This would violate the assumption of correlation between ε and $\tilde{\mathrm{s}}$.

$$\tilde{\mathbb{V}} = \left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}} \otimes \tilde{\Sigma}_{\mathrm{s}}\right) \circ \left(\hat{\mathbf{x}}\hat{\mathbf{x}}'\right) + \left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}} \otimes \left(\tilde{\Sigma}_{\mathrm{s}} + \tilde{\mathbb{E}}\left[\mathrm{s}\right]\tilde{\mathbb{E}}\left[\mathrm{s}\right]'\right)\right) \circ \hat{\Sigma}_{\mathbf{x}}.$$

$\tilde{\mathbb{V}}$ therefore combines uncertainty about the macro state under distress $\tilde{\Sigma}_{\mathrm{s}}$ with residual uncertainty about banks' exposures $\hat{\Sigma}_{\mathbf{x}}$.[14] For example the matrix is large when estimated exposures $\hat{\mathbf{x}}\hat{\mathbf{x}}'$ are high in states where conditional risk $\tilde{\Sigma}_{\mathrm{s}}$ is also high.

Equation (21) says that the regulator intervenes more against high and uncertain exposures to bad and uncertain states. Her interventions are limited by the cost Φ and the uncertainty itself. High residual uncertainty limits the responsiveness of the targeted interventions to the expected exposures $\hat{\mathbf{x}}$. More accurate interventions translate into lower variation in ex post net exposure since the regulator intervenes more when it is needed, and less when exposures are low.

4 Designing Optimal Scenarios

We can finally characterize the design of optimal scenarios. Taking into account optimal future actions the interim utility of the regulator $V(\mathscr{S})$ depends on the scenarios chosen and on the stress test results $\hat{\boldsymbol{y}}$. At the design stage the regulator chooses the stress scenarios $\hat{S}$ to maximize the expected value of her information. The scenario design problem is therefore

$$\hat{S}^{*} = \arg\max_{\hat{S}} \mathbb{E}\left[V(\mathscr{S}) \mid \hat{S}\right]. \tag{23}$$

We could incorporate a cost of creating additional scenarios for the regulator: choosing M scenarios for the stress test could have a cost $\mathscr{C}(M)$. In that case the objective function would simply be $\mathbb{E}_{\hat{\boldsymbol{y}}}\left[V(\mathscr{S}) \mid \hat{S}\right] - \mathscr{C}(M)$ and the regulator would also choose the number of scenarios to include in the stress test. The relevant cost function depends on institutional details (e.g., stress testing insurance portfolios) and we leave this for future applied work.

4.1 Scenario Design for Standard Capital Requirements

In the case of standard capital requirements we obtain a particularly simple result.

[14] The matrix notations are somewhat complicated but in the one dimensional case the formula is simply the variance of a product of independent variables: $\tilde{\mathbb{V}}(xs) = \tilde{\mathbb{V}}(x)\tilde{\mathbb{V}}(s) + \tilde{\mathbb{V}}(x)\left(\tilde{\mathbb{E}}[s]\right)^2 + \tilde{\mathbb{V}}(s)\left(\tilde{\mathbb{E}}[x]\right)^2$.

Lemma 3. ***Scenarios for Standard Capital Requirements.*** *Under LQ the planner designs capital stress scenarios ($\mathbf{a} = 0$) to minimize the Distress Uncertainty Matrix:*

$$\min_{\hat{\Sigma}_{\mathbf{x}} \in \mathbf{\Sigma}} \mathbf{1}_{1\times NJ} \mathbb{E}\left[\tilde{\mathbb{V}}\right] \mathbf{1}_{NJ\times 1}, \tag{24}$$

where $\tilde{\mathbb{V}}$ is given by (22) and $\mathbf{\Sigma}$ is the set of feasible residual uncertainty implied by the Kalman filter.

The key simplification comes from the fact that the FOC for capital, $\overline{W}^{\star} = \mathbf{W} - \frac{\kappa}{p\gamma} + \sum_{i=1}^{N} \mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right]$, is linear in y. The expected capital cost $\kappa\mathbb{E}\left[\overline{W}^{\star}\right]$ is therefore independent of $\hat{\Sigma}_{\mathbf{x}}$. As a result the regulator only cares about minimizing uncertainty. The program is equivalent to $\min_{\hat{\Sigma}_{\mathbf{x}} \in \mathbf{\Sigma}} \mathbf{1}_{1\times NJ} \left(\mathbf{1}_{N\times N} \otimes \left(\tilde{\Sigma}_s + \tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\right) \circ \hat{\Sigma}_{\mathbf{x}} \mathbf{1}_{NJ\times 1}$. The value of learning about factor j depends on the unit cost of exposure $\left(\tilde{\Sigma}_s + \tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)$ and on residual exposure uncertainty $\hat{\Sigma}_{\mathbf{x}}$.

4.2 General Scenario Design

Given that $\hat{\mathbf{y}}$ is normally distributed and that macro factors are independent from risk exposures, we can integrate the indirect value function $\mathbb{E}\left[V\left(\mathscr{S}\right) | \hat{S}\right]$ and express it as function of the covariance matrices $\hat{\Sigma}_{\mathbf{x}}$ and $\tilde{\Sigma}_s$. To see this, note that the regulator's objective depends on the result of the stress test $\hat{\mathbf{y}}$ only through $\hat{\mathbf{x}}$. Moreover, the optimal targeted interventions only depend on $\hat{\mathbf{x}}$ and on the posterior variance. The regulator then solves[15]

$$\min_{\hat{\Sigma}_{\mathbf{x}} \in \mathbf{\Sigma}} \mathbb{E}\left[\kappa\overline{W}^{\star} + \frac{1}{2}\left(\left(1_{NJ\times 1} - \mathbf{a}^{\star}\right)' p\gamma\tilde{\mathbb{V}}\left(1_{NJ\times 1} - \mathbf{a}^{\star}\right) + \Phi'\mathbf{a}^{\star}\right)\right]. \tag{25}$$

It is useful to compare (25) with (24). If we force $\mathbf{a}^{\star} = 0$ in (25) we obtain (24) since, when $\mathbf{a}^{\star} = 0$, $\mathbb{E}\left[\overline{W}^{\star}\right]$ is independent of $\hat{\Sigma}_{\mathbf{x}}$. Three changes occur when $\mathbf{a}^{\star}$ is optimally chosen. First $\mathbb{E}\left[\overline{W}^{\star}\right]$ now depends on $\hat{\Sigma}_{\mathbf{x}}$ via $\mathbf{a}^{\star}$. Second, $\mathbf{a}^{\star}$ mitigates the cost of uncertainty, as seen in the middle term by limiting the ex-post exposures to the macro factors. Finally, the cost of targeted action appears as $\Phi'\mathbf{a}^{\star}$. As usual in linear-quadratic problems, we can substitute the optimal controls (actions and capital requirements) to re-write the optimal scenario design problem.

Lemma 4. *The regulator's scenario design with capital and targeted interventions solves*

$$\min_{\hat{\Sigma}_{\mathbf{x}} \in \mathbf{\Sigma}} \mathbb{E}_{\hat{\mathbf{x}}}\left[\kappa\overline{W}^{\star} + \Phi\mathbf{a}^{\star}\right], \tag{26}$$

[15] Note that $-p\gamma\theta\left(\mathbf{W} - \overline{W}^{\star} + \left(\mathbf{1}_N \otimes \tilde{\mathrm{s}}\right) \circ \mathbf{x}\left(1_{NJ\times 1} - \mathbf{a}^{\star}\right)\right) = -p\gamma\theta\left(\frac{\kappa}{p\gamma} - \theta\right)$.

where $\mathbf{a}^\star$ is given by (21), $\overline{W}^\star$ by (17), and $\mathbf{\Sigma}$ is the set of feasible residual uncertainty implied by the Kalman filter.

The simplicity of Equation (26) comes from the linear cost of functions and the quadratic benefits of targeted actions. Consider for simplicity the one dimensional case, $NJ = 1$. Then we have $\overline{W}^\star = \mathbf{W} + \theta - \frac{\kappa}{p\gamma} + \hat{x}\tilde{s}\left(1 - a^\star\right)$ and the program is $\min \mathbb{E}\left[\kappa\overline{W}^\star + \phi a^\star + \frac{1}{2}p\gamma\tilde{\mathbb{V}}\left(1 - a^\star\right)^2\right]$. The optimal action $a^\star = \frac{p\gamma\tilde{\mathbb{V}} - \phi + \kappa\hat{x}\tilde{s}}{p\gamma\tilde{\mathbb{V}}}$ implies $p\gamma\tilde{\mathbb{V}}\left(1 - a^\star\right)^2 = \left(\phi\left(1 - a^\star\right) - \left(1 - a^\star\right)\kappa\hat{x}\tilde{s}\right)$ and therefore the program is equivalent to $\min \mathbb{E}\left[\kappa\hat{x}\tilde{s}\left(1 - a^\star\right) + \phi a^\star\right]$.

The regulator anticipates that she will intervene optimally after observing the results of the test and she chooses a posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}}$ to maximize the accuracy of her interventions weighted by the relevant costs. The set $\mathbf{\Sigma}$ restricts the feasible information set. The benefit of learning more about exposure j, i.e. of decreasing $\hat{\Sigma}^j_{\mathbf{x}}$, is given by

$$\mathbb{E}\left[\left(\mathbf{1}_{NJ\times 1} - \mathbf{a}^{\star\prime}\right)\left(\kappa\left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathrm{s}}\right) \circ \frac{\partial \hat{\mathbf{x}}}{\partial \hat{\Sigma}^j_{\mathbf{x}}}\right) + \left(\Phi - \kappa\left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathrm{s}}\right) \circ \hat{\mathbf{x}}\right)\frac{d\mathbf{a}^{\star\prime}}{d\hat{\Sigma}^j_{\mathbf{x}}}\right]. \tag{27}$$

The value of learning depends on the responsiveness of interventions to new information. The first term in Equation (27) represents the impact of information on the posterior expected risk exposure. The more precise this information, the more sensitive $\hat{\mathbf{x}}$ is to the new information in the stress test. Since $\frac{\partial \overline{W}^\star}{\partial \hat{\mathbf{x}}} = \left(\mathbf{1}_{NJ\times 1} - \mathbf{a}^{\star\prime}\right)\left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathrm{s}}\right)$, the first term in Equation (27) represents the reduction in the cost of the capital requirements when the test improves information along dimension j. The second term in Equation (27) captures the benefit of changing the targeted interventions when $\hat{\Sigma}^j_{\mathbf{x}}$ is lower. The effect of $\hat{\Sigma}^j_{\mathbf{x}}$ on $\mathbf{a}^\star$ deserves further attention. Since $\mathbf{a}^\star\left(\hat{\mathbf{x}}, \tilde{\mathbb{V}}\right)$ we have

$$\frac{d\mathbf{a}^\star}{d\hat{\Sigma}^j_{\mathbf{x}}} = \frac{\partial \mathbf{a}^\star}{\partial \tilde{\mathbb{V}}}\frac{d\tilde{\mathbb{V}}}{d\hat{\Sigma}^j_{\mathbf{x}}} + \frac{\partial \mathbf{a}^\star}{\partial \hat{\mathbf{x}}}\frac{d\hat{\mathbf{x}}}{d\hat{\Sigma}^j_{\mathbf{x}}} \tag{28}$$

The first term in Equation (28) captures changes in targeted interventions in response to changes in residual uncertainty. The second term measures the value of intervening more accurately. It depends on the sensitivity of targeted interventions to the ex-post expected exposures $\frac{\partial \mathbf{a}^\star}{\partial \hat{\mathbf{x}}}$ and on how new information changes this posterior mean, $\frac{d\hat{\mathbf{x}}}{d\hat{\Sigma}^j_{\mathbf{x}}}$. If the regulator's priors are very precise, only extreme realizations of $\hat{\mathbf{y}}$ can move her beliefs and the sensitivity of the intervention policy to new information, $\frac{d\hat{\mathbf{x}}}{d\hat{\Sigma}^j_{\mathbf{x}}}$, is low. In this case, increasing the precision of the test along dimension j does not improve the accuracy of interventions and the value of learning is low. Similarly, the value of information is low if interventions are too costly to be responsive to new information.

4.3 Comparative Statics

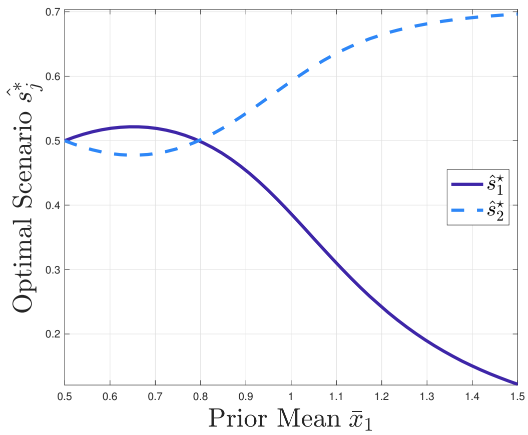
The weights of the different risk factors in the optimal scenarios depend on how targeted interventions respond to new information, which in turn depend on prior beliefs and intervention costs. In this section, we provide comparative statics to illustrate the determinants of the optimal stress scenarios. Most examples below use two factors and one bank and, unless explicitly stated otherwise, we assume that the regulator's priors about exposures are uncorrelated across factors, and we use the following functional form for the variance of the error term in the stress test: $\sigma_{\hat{\epsilon}} = \alpha + \beta \left\|\hat{s}\right\|^2 + \beta_\tau e^{\tau\|\hat{s}\|}$.[16]

Prior mean exposure

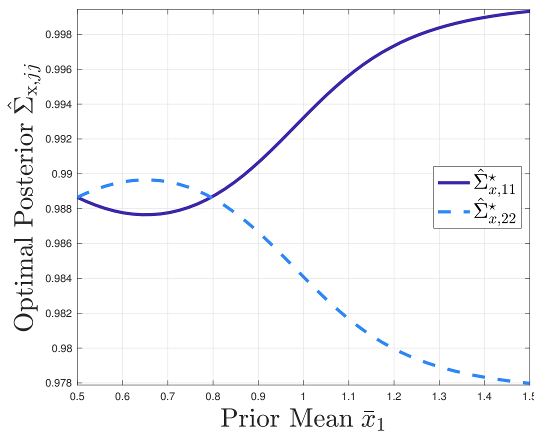
Prior mean of exposures have a non monotone effect on the optimal scenario and posterior precisions, as it can be seen from Panels (a) and (b) in Figure 3. The optimal scenario is non monotonic because there are two opposing forces. Targeted interventions increase with expected exposure which increases the value of learning about high exposures. On the other hand, a tight prior reduces the value of information. When the prior mean exposure to factor 1 is high, the regulator's posterior mean is anchored around this value and the posterior mean is likely to be large regardless of the information produced by the tests. New information is not very valuable and the weight of factor 1 decreases. When the prior is high enough, the regulator finds it optimal not to stress that factor at all.

Figures 3c and 3d respectively show optimal capital requirements and targeted interventions as a function of the prior mean exposure to factor 1. Targeted interventions increase with the prior mean. The effect on capital requirements is non monotonic, however, because there are two opposing forces. Targeted interventions increase with expected exposure which increases the value of learning about high exposures. On the other hand, when the prior mean exposure to factor 1 is high, the posterior mean is likely to be large regardless of the information produced by the tests. New information is not very valuable since the planner except to intervene in any case, and the weight of factor 1 decreases. When the prior is high enough (for a given variance), the regulator

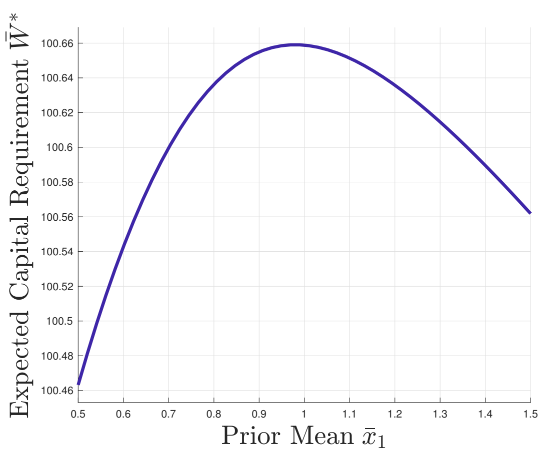
[16] If the regulator's prior expectation is that exposures are correlated, it may still be beneficial for her to stress factor j even if doing so does not improve the accuracy of her intervention along dimension j. In this case, stressing factor j would only be valuable to learn about the exposures to other factors and improve the accuracy of the targeted interventions along these other dimensions.



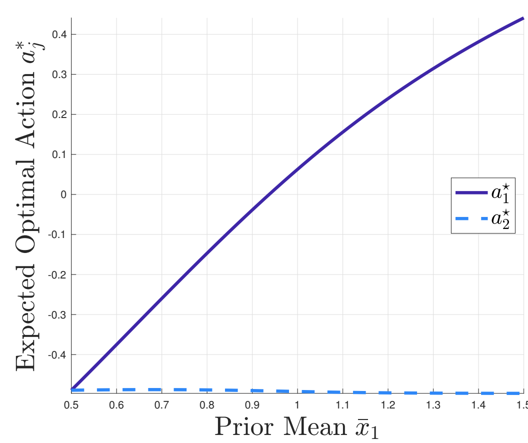
(a) Optimal scenario choice



(b) Optimal posterior variances



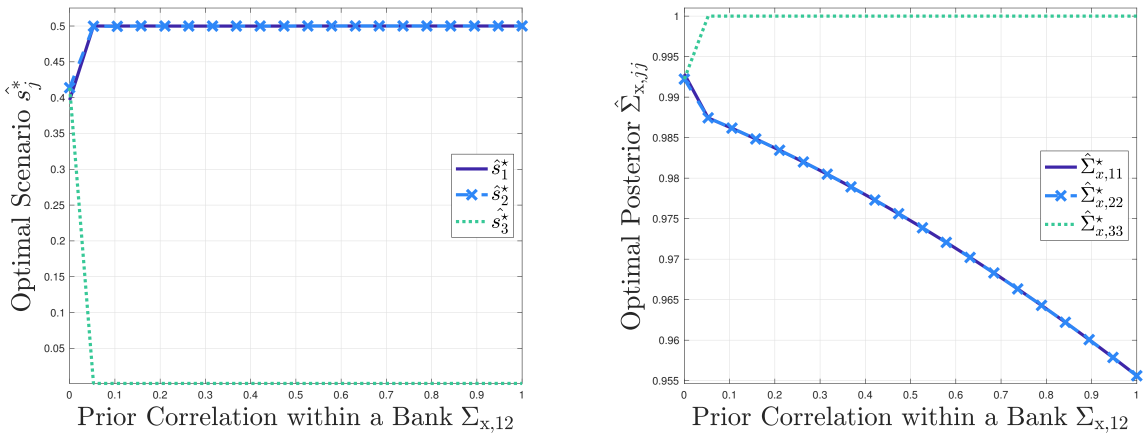
(c) Expected optimal capital requirements



(d) Expected optimal targeted action

Figure 3: Optimal scenario, information choice and expected interventions as a function of the prior mean $\bar{x}_1$.

Note: The parameters used are $M = 1$, $N = 1$, $J = 2$, $\theta = 0$, $p\gamma = 0.1$, $\Phi = [0.75, 0.75]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 0$, $\beta = 5$, $\beta_\tau = 7$, $\tau = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [1, 1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = 3 \times \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathcal{W} = 100$, and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.



(a) Optimal scenario choice

(b) Optimal posterior variance

Figure 4: One bank, Three Risk Factors. Optimal scenario and information as a function of $\Sigma_{\mathbf{x},12}$, the prior correlation among exposures 1 and 2.

Note: The parameters used are $M = 1$, $N = 1$, $J = 3$, $\theta = 0$, $p\gamma = 0.1$, $\Phi = [0.1, 0.1, 0.1]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 0$, $\beta = 5$, $\beta_\tau = 7$, $\tau = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [1, 1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\Sigma_s} = 3 \times \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathbf{W} = 100$, , and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

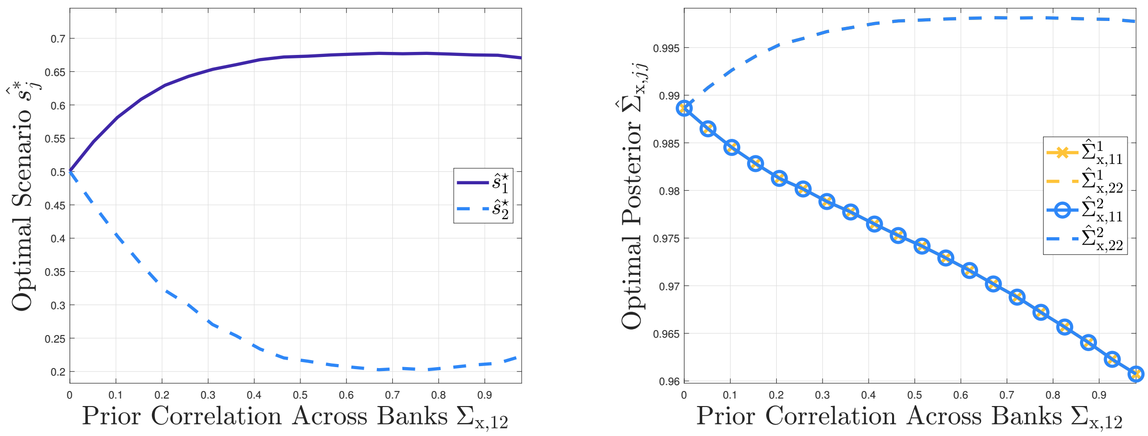
finds it optimal not to stress that factor at all.

Intervention costs imply relatively similar comparative to those of prior means, as it can be seen from Panels (a) and (b) in Figure **??**.

Correlated risks

Let us now consider the role of correlation among risk exposures within and across banks. We saw earlier in Figure (2) that correlations affect the shape of $\mathbf{\Sigma}$, the feasible set of posterior precisions. When correlations are low, stressing one risk factor conveys little information about other risk exposures. When correlation are high, signals about one exposure contain information about the others.

Panel (a) in Figure 4 plots the optimal stresses among 3 factors as a function of prior correlation among the first two exposures. Panel (b) shows that the amount of information that the regulator can learn increases with the prior correlation. Note that if the correlation among two factors is high enough, the regulator may choose not to learn about the other independent dimension.



(a) Optimal scenario choice

(b) Optimal posterior variance

Figure 5: Two factors, Two banks. Optimal scenario and information choice as a function of the prior correlation between the bank's risk exposures to factor 1, $\Sigma^{11}_{\mathbf{x},12}$.

Note: Figure 5 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of prior correlation in risk exposures. The parameters used are $N = 2$, $J = 2$, $\theta = 0$, $p\gamma = 0.1$, $\Phi = [0.05, 0.05]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 0$, $\beta = 5$, $\beta_\tau = 7$, $\tau = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [1, 1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\Sigma_s} = 3 \times \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathbf{W} = 100$, , and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

Risk exposures are also likely to be correlated among banks. Figure 5 shows the optimal stresses as a function of the prior correlation between exposures to factor 1 among two banks. When the exposures to factor 1 are correlated across banks, reported losses from one bank contain information about that bank's exposures but also about the other bank's exposure to the correlated factor. Learning about factor 1 becomes more valuable. When the correlation is strong the regulator barely learns about the other factor. The posterior variance of risk exposures to factor 2 tends towards the prior variance.

5 Calibrated Analysis with Multiple Scenarios

The qualitative patterns described above do not change when we add more scenarios because the number of scenarios only affects the set of feasible precisions $\mathbf{\Sigma}$ and not the value of information.

Considering multiple scenarios is important, however, because it allows us to connect our model to actual stress tests which often feature several scenarios. In this section, we consider several regulatory problems that allow us to assess the value of well designed scenarios. To improve the practical relevance of our analysis, we provide a calibration of our model in the context of U.S. banking stress tests.

5.1 Calibration

We calibrate our model for U.S. stress tests using quarterly bank-level and macroeconomic data. We focus only on credit losses, measured as total net charge offs. There are four groups of variables we need to calibrate: the distribution of the macroeconomic risk factors, the regulator's priors, the regulator's preferences, and the uncertainty of the stress test results.

We first need to choose a set of risk factors. We follow the Capital and Loss Assessment under Stress Scenarios (CLASS) model and data developed in Hirtle et al. (2014). We regress banks' total net charge off (NCO) rate on standard macroeconomic variables: GDP growth, short-term and long-term interest rates, unemployment rate, housing prices, equity prices, and credit rates.[17] We implement this regression at the aggregate banking system level and at the bank level for the panel of banks participating in Dodd-Frank Act Stress Test (DFAST) exercises. We find that GDP growth rate and real estate prices explain more than 80% of the variation in NCO rates in our sample.[18] We therefore calibrate our model using two macro risk factors: GDP growth rate and a real estate price index that puts equal weights on residential and commercial properties. We get the distribution of macroeconomic states and the probability of distress from historical data. To make our results easy to interpret we standardize the risk factors to have mean zero and unit variance. Stress scenario magnitudes therefore represents units of standard deviations. We set the mean of the risk factors under distress to $\tilde{\mathbb{E}}[s] = 1$ and with a conditional volatility of 1.5 times the historical one to take into account that uncertainty is higher in distress.

We then need to set the priors of the regulator. As shown earlier these beliefs play a crucial role in determining the optimal stress scenario. Fortunately, the CLASS model gives us some insight into how the Federal Reserve Bank thinks about the link between macroeconomic scenarios and

[17] We use macro historical data from 1991-2013 for this exercise. In line with the CLASS model, we include a one-quarter lag of the dependent variable and cluster standard errors over time.

[18] We include the cumulative GDP growth rate of the previous four quarters, the 1-quarter lag of the level of the real estate price index and the four-quarter lag of the dependent variable in this regression.

bank outcomes. Based on the regressions in the CLASS model for NCO rates, we use the point estimates and asymptotic variances of the coefficients in a regression of NCO rates on our two factors to back out the regulator's priors.

For our calibration exercise we assume that the cost of capital requirements is piece-wise linear. We take the average dead weight loss of capital requirements from Kisin and Manela (2016) who estimate a shadow cost of 0.3pp for every point increase in capital requirements, and we choose the marginal cost so that a 10% decrease in the marginal cost increases capital requirements by half a percentage point. We set the satiated level of capital $\mathbf{W}$ to 7 percent of assets (not risk weighted) in line with Basel requirements, we set the effective distress aversion of the regulator to $p\gamma = 1$, and we choose the marginal value of capital in distress, θ, to target an optimal (non risk-weighted) capital requirements of 5.6% in a model.

Finally, we require a parameterization of the error in the stress test responses. To do so we compute the forecasting errors of the CLASS model in the cross section of banks and regress their standard deviation on the squared norm of the macro risk factors. Table 1 shows the descriptions and values of the calibrated parameters.

5.2 Results

We consider four "problems" for the regulator. The first three problems focus on capital requirements without targeted interventions. The fourth problem studies targeted interventions.

- Problem 1: standard capital requirements and two scenarios chosen optimally. This is the scenario design problem studied earlier where the regulator chooses both scenarios to optimize her learning.

- Problem 2: standard capital requirements with one scenario fixed at $\tilde{\mathbb{E}}[\mathsf{s}]$ and one scenario chosen optimally. This problem captures an important feature of actual stress tests where one scenario is often used to set capital requirements under a *plausible* adverse scenario. We therefore constrain one scenario to be the expected adverse state $\hat{\mathrm{s}}^1 = \tilde{\mathbb{E}}[\mathsf{s}]$, and we let the regulator optimize over the second scenario. This allows us to quantify the efficiency loss of using one scenario in this mechanical way.

- Problem 3: standard capital requirements with one scenario fixed at $\tilde{\mathbb{E}}[\mathsf{s}]$ and two scenarios chosen optimally. The first scenario is the expected adverse state. Scenarios 2 and 3 are

Parameter	Description	Value
N	number of banks	1
J	number of risk factors	2
$[s_1, s_2]$	cumulative GDP growth over 4 quarters and real estate price index	normalized, mean zero and variance ~ 1
Σ_s	covariance matrix of normalized risk factors	$\begin{bmatrix} 0.990 & 0.664 \\ 0.664 & 1.046 \end{bmatrix}$
$\left[\tilde{\mathbb{E}}\left[s_1\right], \tilde{\mathbb{E}}\left[s_2\right]\right]$	mean under distress	$[1, 1]$
$\tilde{\Sigma}_s$	covariance matrix of normalized risk factors under distress	$3 \times \begin{bmatrix} 0.990 & 0.664 \\ 0.664 & 1.046 \end{bmatrix}$
Regulator priors	computed from CLASS model	
$\overline{\mathbf{x}}$	prior mean exposures to risk factors	$\begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$
$\Sigma_{\mathbf{x}}$	prior covariance matrix of exposures to risk factors	$\begin{bmatrix} .0036 & 0 \\ 0 & .0036 \end{bmatrix}$
Regulator preferences		
$p\gamma$	effective distress aversion	1
θ	marginal value of capital in distress	3
$\mathbf{W}$	satiated level of capital	7%
κ^a	average cost of bank capital	0.3
κ^m	marginal cost of bank capital	5
Stress test results		
$\sigma\left(\hat{\epsilon}_i\left(\lVert\hat{\mathrm{s}}\rVert, M\right)\right)$	standard deviation of model error	$\alpha + \beta \lVert\hat{\mathrm{s}}\rVert^2$
$[\alpha, \beta]$	model error parameter	$[0.55, 0.11]$

Table 1: Calibration

chosen optimally. This allows us to quantity the value of adding one exploratory scenario to the standard stress test of problem 2.

- Problem 4: capital requirements with targeted interventions and two chosen optimally scenarios. This allows us to quantify the value of targeted interventions.

As discussed earlier, targeted interventions can take the form of notices of Matters Requiring (Immediate) Attention (MR(I)A) used by bank examiners to communicate concerns about a bank's management.[19] They represent an important part of the supervision process but are sometimes weakly enforced. We can use problem 4 to gauge the potential value of MRAs, especially if they are enforceable and not just mere suggestions brought up for consideration by the bank's management or board of directors.

Figure 6 shows that we obtain sensible magnitudes for our optimal scenarios in Problems 1 (circles), 2 (triangles) and 3 (crosses). In Problems 2 and 3 the first scenario is fixed $\tilde{\mathbb{E}}[\mathsf{s}] = 1$. In Problem 1 both scenarios are chosen optimally. The optimal range of scenario is between 1 and 2 additional historical sigmas of negative shocks.

Figures 7a and 7b show the welfare gains from various types of stress tests, relative to the no learning case, as a function of uncertainty in the distress states. The horizontal axis is the multiplicative factor λ on the prior standard deviation of exposures, i.e., the prior variance is $\lambda^2 \Sigma_x$. Welfare gains are small when uncertainty is small and the historical calibration can give a false sense of confidence in estimated exposures, especially during distress. The vertical axis show the welfare gains from the stress tests normalized by the welfare gain from a hypothetical decrease in the cost of capital.

To provide an intuitive interpretation of our results we use as a benchmark a well defined experiment: the welfare gains from lowering capital costs κ by some fraction. The baseline capital requirement is calibrated at 5.6%. In figure 7 the benchmark is a 10% decrease in κ^m which corresponds to an increase in the optimal capital ratio from 5.6% to 6.1%. Figures 7a and 7b allow us to compare the welfare gains from the stress test to this simple benchmark. In Figure 7b, we calibrate the cost of targeted interventions such that the optimal targeted intervention is zero without learning, i.e., targeted interventions have no value by themselves.

Figure 7a shows that the ex-ante welfare gains from learning are small when the regulator can *only* adjust overall capital requirements. The option to increase the requirements when risks build

[19] https://www.federalreserve.gov/supervisionreg/srletters/sr1313a1.pdf

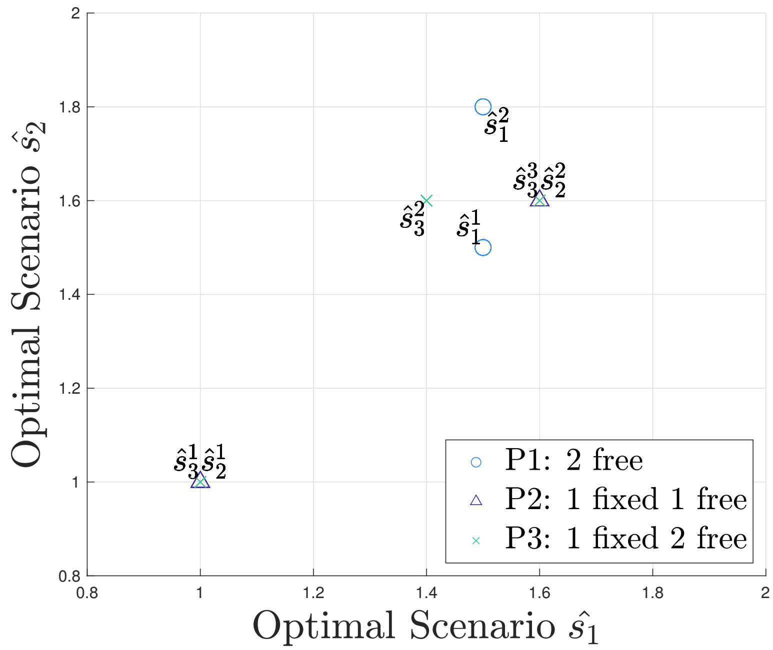
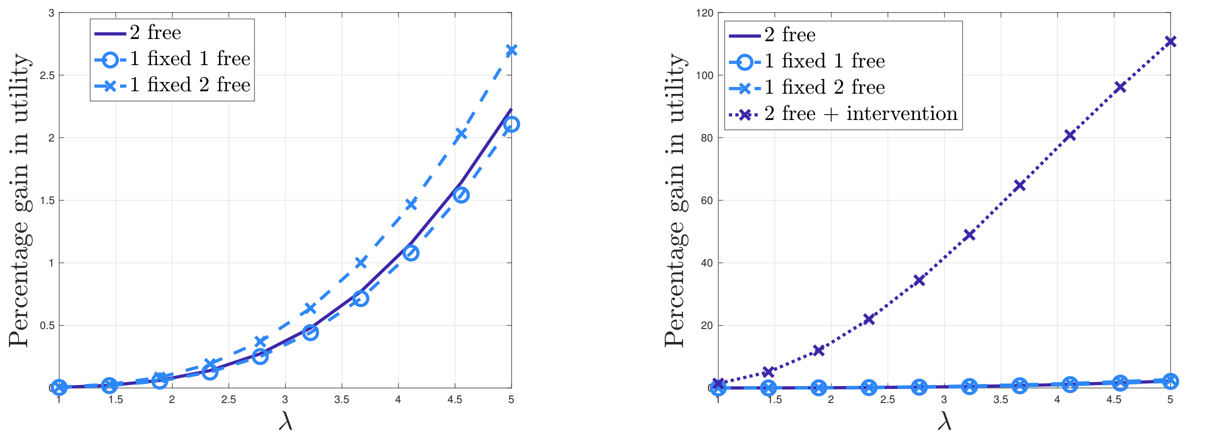


Figure 6: Optimal scenarios

up is worth less than 1% of the gain from a 10% lower capital cost. The reason is that requirements are correct on average and that large shocks are unlikely. An envelope theorem therefore applies to the value of information revealed by the stress tests. Of course this does not invalidate the usefulness of stress tests as a way to set the benchmark requirement ex-ante as in Lemma 2. The regulator can always *define* the capital requirement as the level of equity that guarantees that a bank remains well-capitalized in an adverse scenario. Our results here only refer to the value of updating the regulator's prior based on *new* information revealed by the stress tests.

Figure 7b, on the other hand, shows that information is valuable when the regulator can make targeted interventions. The welfare gains from stress testing are of the same order of magnitude as a 10% decreased in the cost of equity capital. The reason is that the regulator can now tailor its interventions to the specific information revealed by the tests and change exposures to individual risks as necessary.



(a) Percentage utility gain with standard capital requirements

(b) Percentage utility gain with targeted interventions

Figure 7: Percentage utility gain for the regulator relative to $\kappa' = 0.9 \times \kappa$

6 Discussion and Conclusion

Despite the growing importance of stress testing for financial regulation and risk management, economists still lack a theory of the design of stress scenarios. We model stress testing as a learning mechanism and show how to map the scenario choice problem into an information acquisition problem. In this framework, we derive optimal scenarios and characterize how their design depends on the cost of interventions, the prior beliefs of the regulator, the precision of regulatory information, the uncertainty about the risk factors, and the volatility of systemic risk factors.

Our comparative static exercises above shed light on the optimal stress scenario design in the presence of systemic factors, in times of distress, over time, and its relation to capital requirements. We conclude our paper by highlighting some important findings.

Correlation and Uncertainty Can Lead to Specialization Our analysis on correlated exposures among banks suggests that the optimal stress scenario would put relatively more weight on risk factors that lead to correlated losses among banks. Moreover, if the correlation of the banks' exposures to some factor is high enough, the optimal stress scenario may put weight *only* on this factor.

Similarly we find that it is optimal to focus the stress test on factors that are more uncertain. Moreover, an across the board increase in uncertainty or in the regulator's risk aversion can lead the optimal design to put more weight on fewer factors. This implies a positive correlation between the uncertainty in the economy and the focus and severity of the stress scenarios.

Testing to Learn or to Set Capital Requirements? Our model sheds light on the debate about the fundamental value of stress tests. We find that learning is of limited value when the regulator can only adjust capital requirements. In this case, one can think of stress tests as a sensible way to set ex-ante capital requirements using plausible adverse scenarios, in line with current practice.

On the other hand, our quantitative analysis highlights the complementarities between learning and specific interventions. Designing scenarios explicitly to learn about exposures is valuable when the regulator can make targeted interventions. For instance, if the response to excessive duration exposure is to require specific hedging or modifications to loan maturities, then it is valuable to design a specific scenario to learn about these exposures. This scenario is likely to be quite different from the plausible adverse scenario used to set overall capital requirements.

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Appendix

The Appendix contains some auxiliary calculation for formulas in the text. It needs to be completed.

A Proofs

A.1 Learning from stress tests

Mapping scenarios to posterior precisions

The Kalman filter in Equations (8) and (9) imposes restrictions on the set of posterior variances that can be attained. More specifically, the Kalman filter maps the $J \times M$ elements in the stress scenarios $\hat{S}$ to the the $\frac{NJ(NJ+1)}{2}$ elements of the posterior precision $\hat{\Sigma}_{\mathbf{x}}$ from the set $\overline{\mathbf{\Sigma}}$. Moreover, the regulator's choice will always be on the frontier of the feasibility set, given by $\overline{\mathbf{\Sigma}} \equiv \left\{ K\left(\hat{S}_\omega\right)\hat{S}_\omega' \Sigma_{\boldsymbol{x}} \quad \text{for all} \quad \omega \in [0,1]^{N\times J} \quad \text{with} \quad \sum_{h=1}^{N\times J}\omega_h \right\}$, where

$$\hat{S}_\omega \equiv \arg\max_{\hat{S}} \omega' K\left(\hat{S}\right)\hat{S}'\Sigma_{\boldsymbol{x}}\omega.$$

Given our assumptions on $\hat{\epsilon}_i\left(\hat{\mathrm{s}}, M\right)$, $\hat{S}_\omega$ is unique. Hence, since the objective function of the regulator depends on $\hat{S}$ only through the posterior variance, the regulator's scenario choice problem can be cast in term of choosing $\hat{\Sigma}_{\mathbf{x}}$.

When $M < N$,as long as all risk dimensions are spanned, choosing the $J \times M$ elements in $\hat{S}$ is equivalent to choosing $J \times M$ elements of the posterior precision $\hat{\Sigma}_{\boldsymbol{x}}$. Without loss of generality, one can focus on the posterior variances of the risk exposures of M banks from the set

$$\overline{\mathbf{\Sigma}}_M \equiv \left\{ K\left(\hat{S}_\omega\right)\hat{S}_\omega'\Sigma_{\boldsymbol{x}} \quad \text{for all} \quad \omega \in [0,1]^{N\times J} \quad \text{with} \quad \sum_{m=1}^{M}\sum_{j=1}^{J}\omega_{(j-1)J+m} \right\}.$$

A.2 Taking action

Under linear quadratic preferences, the first order conditions that characterize the optimal capital requirement is

$$\mathcal{K}'\left(\overline{W}^\star\right) = 1 + p\gamma\theta + p\gamma\tilde{\mathbb{E}}\left[\left(W - \mathbf{W}\right) \mid \mathscr{S}\right] \tag{A.1}$$

and the first order condition that characterizes the regulator's optimal targeted intervention policy is

$$\frac{\partial \mathcal{C}\left(\mathbf{a}^\star\right)}{\partial a_{i,j}} = \hat{x}_{i,j}\mathbb{E}\left[s_j\right] + p\gamma\theta\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right] - p\gamma\tilde{\mathbb{E}}\left[x_{i,j}s_j\left(W - \mathbf{W}\right) \mid \mathscr{S}\right], \tag{A.2}$$

where $W = \overline{W}^\star - \left(\left(\mathbf{1}_{N\times 1} \otimes \mathrm{s}\right) \circ \mathbf{x}\right)'\left(\mathbf{1}_{NJ\times 1} - \mathbf{a}^\star\right)$. Using Equation (A.1) in Equation (13) we get

$$\frac{\partial \mathcal{C}\left(\mathbf{a}^\star\right)}{\partial a_{i,j}} = \kappa\gamma\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right] + p\gamma\sum_{h=1}^{N}\sum_{l=1}^{J}\left(1 - a_{h,l}\right)\tilde{\mathbb{C}ov}\left(x_{i,j}s_j, x_{h,l}s_l\right) \quad \forall i, j. \tag{A.3}$$

Using the functional form for $\mathcal{C}$ and rewriting the system in Equations (A.3) in vector form for $\mathbf{a}^\star$, gives

$$\mathbf{a}^\star = \left(p\gamma\tilde{\mathbb{V}}\right)^{-1}\left(\kappa\left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathrm{s}}\right)\circ\hat{\mathbf{x}} - \Phi + p\gamma\tilde{\mathbb{V}}\mathbf{1}_{NJ\times 1}\right),$$

where

$$\tilde{\mathbb{V}} = \tilde{\mathbb{C}ov}\left[\left(\mathbf{1}_{N\times 1}\otimes\mathrm{s}\right)\circ\mathbf{x}\mid\mathscr{S}\right] = \left(\left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}}\otimes\tilde{\Sigma}_{\mathrm{s}}\right)\circ\hat{\Sigma}_{\mathbf{x}}\right) + \left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}}\otimes\tilde{\Sigma}_{\mathrm{s}}\right)\circ\left(\hat{\mathbf{x}}\hat{\mathbf{x}}'\right) + \left(\mathbf{1}_{N\times N}\otimes\tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\circ\hat{\Sigma}_{\mathbf{x}}.$$

A.3 Optimal scenario choice

The objective function of the regulator is given by

$$\begin{aligned}\mathbb{E}_{\hat{\mathbf{x}}}[O] &= \mathbb{E}_{\hat{\mathbf{x}}}\left[\mathbb{E}_{\mathrm{s},\eta,\mathrm{x}}\left[U\left(W\left(\mathbf{a}^\star,\overline{W}^\star\right)\right)|\mathscr{S}\right] - \mathcal{C}\left(\mathbf{a}^\star\right) - \mathcal{K}\left(\overline{W}^\star\right)\right]\\ &= \mathbb{E}_{\hat{\mathbf{x}}}\left[\mathbb{E}_{\mathrm{s},\eta,\mathrm{x}}\left[W\left(\mathbf{a}^\star,\overline{W}^\star\right)|\mathscr{S}\right] - \frac{p\gamma}{2}\tilde{\mathbb{E}}_{\mathrm{s},\eta,\mathrm{x}}\left[\left(W\left(\mathbf{a}^\star,\overline{W}^\star\right)-\mathcal{W}\right)^2|\mathscr{S}\right] - \mathcal{C}\left(\mathbf{a}^\star\right) - (1+\kappa)\overline{W}^\star\right].\end{aligned}$$

Proof 3

The optimal capital requirement in the absence of targeted actions is

$$\overline{W}^\star = \mathcal{W} + \theta - \frac{\kappa}{p\gamma} + \left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathbb{E}}\left[\mathrm{s}\right]\right)'\hat{\mathbf{x}}.$$

The total wealth of the regulator is then

$$W = \mathcal{W} + \theta - \frac{\kappa}{p\gamma} - \left(\left(\mathbf{1}_{N\times 1}\otimes\mathrm{s}\right)'\mathbf{x} - \left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathbb{E}}\left[\mathrm{s}\right]\right)'\hat{\mathbf{x}}\right),$$

which implies

$$\begin{aligned}\mathbb{E}\left[W\right] &= \overline{W}^\star\\ \tilde{\mathbb{E}}\left[W\,|\mathscr{S}\right] &= \mathcal{W} + \theta - \frac{\kappa}{p\gamma}\end{aligned}$$

and

$$\tilde{\mathbb{E}}\left[\left(W-\mathcal{W}\right)^2|\mathscr{S}\right] = \left(\theta - \frac{\kappa}{p\gamma}\right)^2 + \mathbf{1}_{1\times NJ}\tilde{\mathbb{V}}\mathbf{1}_{NJ\times 1},$$

where $\tilde{\mathbb{V}} \equiv \tilde{\mathbb{C}ov}\left[\left(\mathbf{1}_{N\times 1}\otimes\mathrm{s}\right)'\mathbf{x}\,|\mathscr{S}\right]$. Then, the objective function of the regulator becomes

$$\begin{aligned}\mathbb{E}[O] &= \mathbb{E}\left[\mathbb{E}\left[W\,|\mathscr{S}\right] + p\gamma\theta\tilde{\mathbb{E}}\left[W-\mathcal{W}\,|\mathscr{S}\right] - \frac{p\gamma}{2}\tilde{\mathbb{E}}\left[\left(W-\mathcal{W}\right)^2|\mathscr{S}\right] - (1+\kappa)\overline{W}^\star\right]\\ &= -\kappa\left(\mathcal{W} - \frac{\kappa}{p\gamma} + \left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathbb{E}}\left[\mathrm{s}\right]\right)'\bar{\mathbf{x}}\right) + p\gamma\theta\left(\theta - \frac{\kappa}{p\gamma}\right) - \frac{p\gamma}{2}\left(\theta - \frac{\kappa}{p\gamma}\right)^2 - \frac{p\gamma}{2}\mathbf{1}_{1\times NJ}\mathbb{E}\left[\tilde{\mathbb{V}}\right]\mathbf{1}_{NJ\times 1}.\end{aligned}$$

Note that the only term that the regulator can affect by choosing a stress scenario, or alternatively a posterior covariance matrix, is $\tilde{\mathbb{V}}$. Therefore, the regulator's objective in the design problem is to minimize her residual uncertainty $\mathbb{E}\left[\tilde{\mathbb{V}}\right]$, which is given by

$$\mathbb{E}\left[\tilde{\mathbb{V}}\right] \equiv \mathbb{E}\left[\tilde{\mathbb{C}ov}\left[(\mathbf{1}_{N\times1}\otimes \mathrm{s})'\,\mathbf{x}\,|\mathscr{S}\right]\right] = \left(\mathbf{1}_{N\times N}\otimes\tilde{\Sigma}_s\right)\circ\mathbb{E}\left[\mathbb{E}\left[\mathbf{x}\mathbf{x}'\mid\mathscr{S}\right]\right] + \left(\mathbf{1}_{N\times N}\otimes\left(\tilde{\Sigma}_s+\tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\right)\circ\hat{\Sigma}_{\mathbf{x}},$$
$$= \left(\mathbf{1}_{N\times N}\otimes\tilde{\Sigma}_s\right)\circ\left(\Sigma_{\mathbf{x}}+\bar{\mathbf{x}}\bar{\mathbf{x}}'\right) + \left(\mathbf{1}_{N\times N}\otimes\tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\circ\hat{\Sigma}_{\mathbf{x}}$$

which is separable in $\bar{\mathbf{x}}\bar{\mathbf{x}}'$ and $\hat{\Sigma}_{\mathbf{x}}$ because

$$\mathbb{E}\left[\mathbb{E}\left[\mathbf{x}\mathbf{x}'\mid\mathscr{S}\right]\right] = \mathbb{E}\left[\mathbf{x}\mathbf{x}'\right] = \Sigma_{\mathbf{x}}+\bar{\mathbf{x}}\bar{\mathbf{x}}'.$$

Proof of Proposition 4

When utility is linear quadratic, and targeted intervention costs and capital requirement costs are linear, the optimal capital requirement is given by

$$\overline{W}^{\star} = \mathbf{W}+\theta-\frac{\kappa}{p\gamma}+\left(\mathbf{1}_{N\times1}\otimes\tilde{\mathbb{E}}\left[s\right]\right)'\left(\hat{\mathbf{x}}\circ\left(1_{NJ\times1}-\mathbf{a}^{\star}\right)\right),$$

and the optimal targeted interventions are given by

$$\mathbf{a}^{\star} = \left(p\gamma\tilde{\mathbb{V}}\right)^{-1}\left(\kappa\left(\mathbf{1}_{N\times1}\otimes\tilde{\mathrm{s}}\right)\circ\hat{\mathbf{x}}-\Phi+p\gamma\tilde{\mathbb{V}}\mathbf{1}_{NJ\times1}\right).$$

Moreover, the regulator's expected utility is given by

$$\mathbb{E}_{\hat{\mathbf{x}}}\left[-\kappa\overline{W}^{\star}+p\gamma\theta\tilde{\mathbb{E}}\left(W-\mathbf{W}\,|\mathscr{S}\right)-\frac{p\gamma}{2}\tilde{\mathbb{E}}\left(\left(W-\mathbf{W}\right)^2\,|\mathscr{S}\right)-\Phi'\mathbf{a}^{\star}\right].$$

Note that given the expression for $\overline{W}^{\star}$above, we have

$$\tilde{\mathbb{E}}\left(W-\mathbf{W}\,|\mathscr{S}\right) = \theta-\frac{\kappa}{p\gamma}$$

and

$$\tilde{\mathbb{E}}\left(\left(W-\mathbf{W}\right)^2\,|\mathscr{S}\right) = \left(\theta-\frac{\kappa}{p\gamma}\right)^2+\left(1_{NJ\times1}-\mathbf{a}^{\star}\right)'\tilde{\mathbb{V}}\left(1_{NJ\times1}-\mathbf{a}^{\star}\right),$$

where we used that

$$\tilde{\mathbb{C}ov}\left[\left(\mathbf{1}_{N\times1}\otimes\mathrm{s}\right)'\left(\mathbf{x}\circ\left(1_{NJ\times1}-\mathbf{a}^{\star}\right)\right)\,|\mathscr{S}\right] = \left(1_{NJ\times1}-\mathbf{a}^{\star}\right)'\tilde{\mathbb{V}}\left(1_{NJ\times1}-\mathbf{a}^{\star}\right).$$

Moreover, note that the expression for $\mathbf{a}^{\star}$ above implies

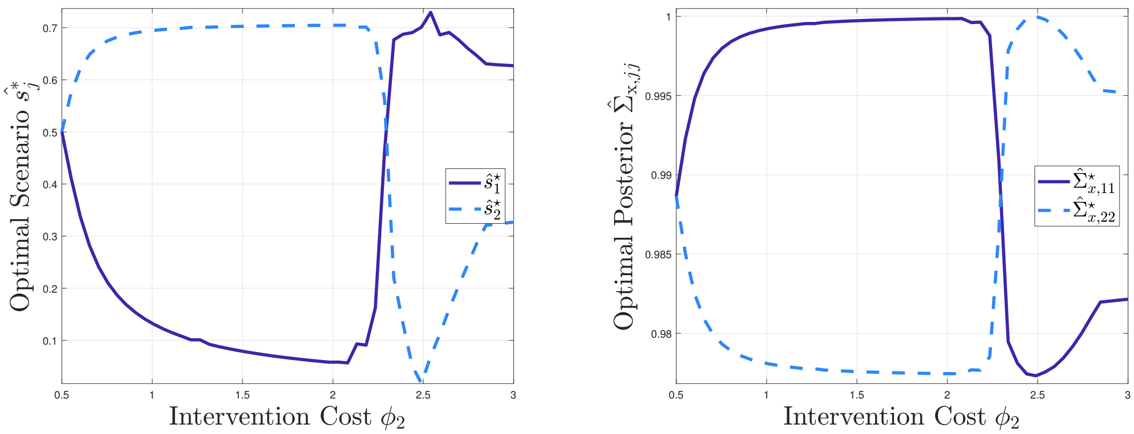
$$\left(1_{NJ\times1}-\mathbf{a}^{\star}\right)'p\gamma\tilde{\mathbb{V}}\left(1_{NJ\times1}-\mathbf{a}^{\star}\right) = \left(-\kappa\left(\mathbf{1}_{N\times1}\otimes\tilde{\mathrm{s}}\right)\circ\hat{\mathbf{x}}+\Phi\right)\left(1_{NJ\times1}-\mathbf{a}^{\star}\right).$$

Therefore, the regulators expected utility can be written as

$$\mathbb{E}_{\hat{\mathbf{x}}}\left[-\frac{1}{2}\kappa\overline{W}^{\star}-\frac{1}{2}\left(\mathbf{W}+\theta-\frac{\kappa}{p\gamma}\right)+p\gamma\theta\left(\theta-\frac{\kappa}{p\gamma}\right)-\frac{p\gamma}{2}\left(\theta-\frac{\kappa}{p\gamma}\right)^2-\frac{1}{2}\Phi'1_{NJ\times1}-\frac{1}{2}\Phi'\mathbf{a}^{\star}\right]$$

and therefore, the regulator's problem is equivalent to minimizing

$$\mathbb{E}_{\hat{\mathbf{x}}}\left[\kappa\overline{W}^{\star}+\Phi'\mathbf{a}^{\star}\right].$$



(a) Optimal scenario choice (b) Optimal posterior variances

Figure B.1: Optimal scenario and information choice as a function of the intervention cost Φ_2

Note: Figure B.1 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of the intervention cost Φ_2. The parameters used are $N = 2$, $J = 2$, $\theta = 0$, $p\gamma = 0.3$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 0$, $\beta = 5$, $\beta_\tau = 7$, $\tau = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [1, 1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\Sigma_s} = 3 \times \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathbf{W} = 100$, , and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$, $\mathcal{A} = [-0.7, 1.25]$.

B Additional comparative statics

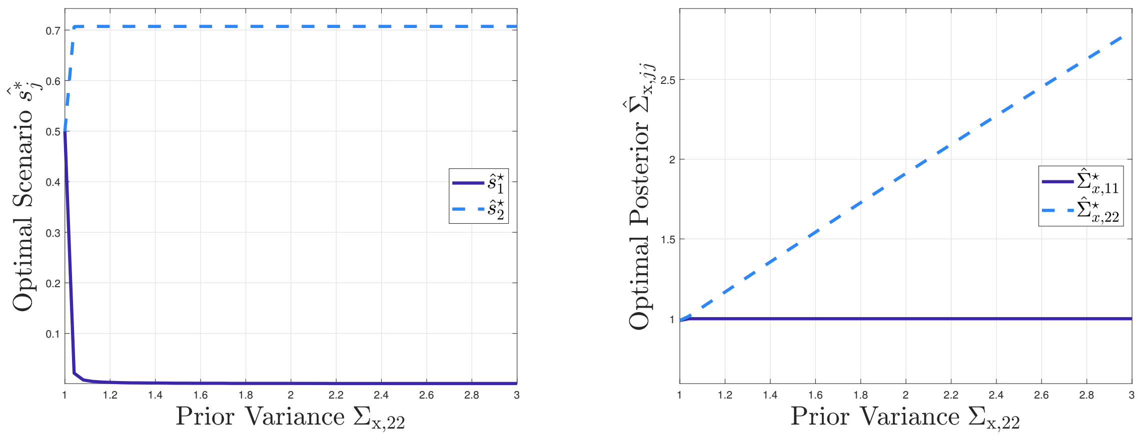
B.1 Intervention costs

The first important point is that intervention costs have a non monotone impact on scenario design. When intervention costs are low, the regulator can intervene preemptively to reduce exposures. Inaccurate interventions are not too costly and the regulator cares little about learning about that factor. When the intervention costs are intermediate, interventions are sensitive to the information produced by the stress tests and the regulator values learning to avoid wasteful interventions. Finally, when the intervention costs are high, the ex-post interventions hit the boundary and learning is less valuable for the regulator.

B.2 Uncertainty

Two dimensions of uncertainty shape the regulator's choice of stress scenarios: uncertainty about risk exposures and uncertainty about risk factors. The regulator intervenes more along dimensions about which she is more uncertain. When the regulator is more uncertain about exposures to risk factor j, her targeted intervention along dimension j is more responsive to the information contained in the stress test results and information is more valuable.

Figure B.2 shows the effect of prior uncertainty regarding exposures to factor 2, $\Sigma_{\mathbf{x},22}$, on the optimal stress on factors 1 and 2. When $\Sigma_{\mathbf{x},22}$ is high the regulator stresses factor 2 to improve the efficiency



(a) Optimal scenario choice

(b) Optimal posterior variances

Figure B.2: Optimal scenario and information choice as a function of the regulator's prior uncertainty of the exposure to factor 2, $\Sigma_{\mathbf{x},22}$.

Note: Figure B.2 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of the regulator's prior variance of the exposure to factor 2, $\Sigma_{\mathbf{x},22}$. The parameters used are $N = 2$, $J = 2$, $\theta = 0$, $p\gamma = 0.3$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 0$, $\beta = 5$, $\beta_\tau = 7$, $\tau = 2$, $\mathbb{E}[s] = [0, 0]'$, $\Phi = [0.1, 0.1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\Sigma_s} = 3 \times \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathbf{W} = 100$, , and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$, $\mathcal{A} = [-0.7, 1.25]$.

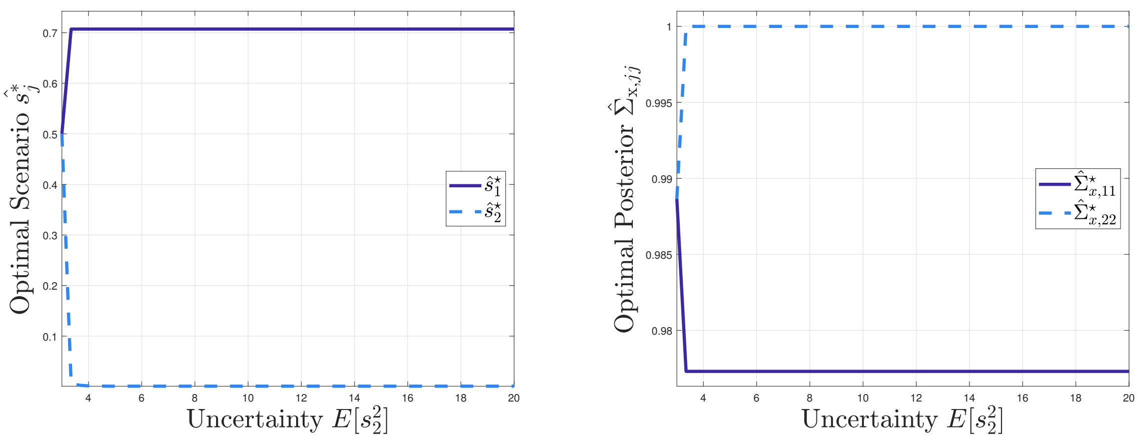
of her expected interventions. The consequences of uncertainty about the risk factors themselves are similar, as shown in Figure B.3 in the Appendix.

B.3 Uncertainty about risk factors

If one risk factor has a very low variance and will stay close to the the baseline, then it is less valuable to learn about the exposures to it and to intervene to reduce them. In this case, the factor's weight on the expected losses will be small and uncertainty about the exposure to it is less costly. However, if the variance of a risk factor is large it has the potential to be an important driver of bank losses depending on the risk exposures to it. In this case, the regulator has more incentives to learn and intervene along the dimension of this factor to curve its potential impact on losses. Therefore, the regulator will stress a risk factor more in the optimal scenario the highest the uncertainty about it. Figures B.3 show the optimal scenario choice as a function of the uncertainty about risk factor 2, $\mathbb{E}[s_2^2]$.

Online Appendix

The Appendix contains some auxiliary calculation for formulas in the text. It needs to be completed.



(a) Optimal scenario choice

(b) Optimal posterior

Figure B.3: Optimal scenario and information choice as a function of the uncertainty about risk factor 2, $\mathbb{E}\left[s_2^2\right]$.

Note: Figure B.3 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of the uncertainty about risk factor 2, $\mathbb{E}\left[s_2^2\right]$. The parameters used are $N=1$, $J=2$, $\theta=0$, $p\gamma=0.1$, $\Phi=[0.75,0.75]'$, $\bar{x}=[1,1]'$, $\Sigma_{\boldsymbol{x}}=\boldsymbol{I}_J$, $\alpha=0$, $\beta=5$, $\beta_\tau=7$, $\tau=2$, $\mathbb{E}[s]=[0,0]'$, $\tilde{\mathbb{E}}[s]=[1,1]'$, $\Sigma_s=\boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s]=3\times\boldsymbol{I}_J$, $\kappa=0.1$, $\mathcal{W}=100$, and $\mathbb{E}[\epsilon\epsilon']=\boldsymbol{I}_N$.

A A Kalman Filter: Example

Consider the case of one bank ($N=1$), one scenario ($M=1$), and two risk factors ($J=2$). To simplify the notation, we omit the argument M and the bank-specific subscript i, and we denote $\sigma_1^2\equiv\Sigma_{x,11}$, $\rho\sigma_1\sigma_2\equiv\Sigma_{x,12}$ and $\sigma_\epsilon^2(\hat{\mathrm{s}})\equiv\mathbb{V}\mathrm{ar}\left[\hat{\epsilon}\left(\|\hat{\mathrm{s}}\|,1\right)\right]$. The stress test result under scenario $\hat{s}$ is

$$\hat{y}=\hat{s}_1x_1+\hat{s}_2x_2+\hat{\epsilon}.$$

For this illustration, we assume $\hat{\epsilon}(\hat{\mathrm{s}})=\alpha\epsilon_0+\beta\left(\|\hat{\mathrm{s}}\|^{\frac{1}{2}}+\|\hat{\mathrm{s}}\|^{1+\theta}\right)\epsilon_1$.

The Kalman gain in this case is a 2×1 vector $K=\begin{bmatrix}k_1\\k_2\end{bmatrix}$ where

$$k_1=\frac{\sigma_1^2\hat{s}_1+\rho\sigma_1\sigma_2\hat{s}_2}{\sigma_1^2\hat{s}_1^2+2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2+\sigma_2^2\hat{s}_2^2+\sigma_{\hat{\epsilon}}^2}\quad\text{and}\quad k_2=\frac{\sigma_2^2\hat{s}_2+\rho\sigma_1\sigma_2\hat{s}_1}{\sigma_1^2\hat{s}_1^2+2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2+\sigma_2^2\hat{s}_2^2+\sigma_{\hat{\epsilon}}^2}.$$

The posterior mean is then

$$\hat{\mathrm{x}}=\begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix}=\overline{\mathrm{x}}+K\left(\hat{y}-\hat{s}'\overline{\mathrm{x}}\right)$$

and the learning matrix is

$$\Sigma_{\hat{\mathbf{x}}} = \begin{bmatrix} k_1 \hat{s}_1 & k_1 \hat{s}_2 \\ k_2 \hat{s}_1 & k_2 \hat{s}_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

The chosen scenario $\hat{s}$ affects the Kalman gain in two ways: directly via the correlation between the results of the stress test and the unknown risk exposures; and indirectly, through the amount of noise in the stress test result, $\sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})$. Using the above formula we see that the top left term of the learning matrix is

$$\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} = \frac{(\sigma_1 \hat{s}_1 + \rho\sigma_2 \hat{s}_2)^2}{(\sigma_1 \hat{s}_1 + \rho\sigma_2 \hat{s}_2)^2 + (1-\rho^2)(\sigma_2 \hat{s}_2)^2 + \sigma_{\hat{\epsilon}}^2}. \tag{A.1}$$

The upper bound $\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} = 1$ corresponds to learning everything about exposure x_1. When x_1 and x_2 are correlated we can learn about x_1 by increasing $\hat{s}_2$, but the potential to learn is bounded by this correlation. Holding constant $\sigma_{\hat{\epsilon}}$, if we let $\hat{s}_2 \to \infty$ the limit is $\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} \to \rho^2$.

Equation (A.1) shows the relevance of the *endogenous* noise term. If $\sigma_{\hat{\epsilon}}$ is exogenous, then learning is trivially maximized by sending $\hat{s}_1 \to \infty$. In reality extreme scenarios are more difficult to estimate and $\sigma_{\hat{\epsilon}}$ is increasing in the size of the deviation of the stress scenario from the baseline. We obtain an interior solution for the regulator's scenario choice problem as long as $\sigma_{\hat{\epsilon}}$ is convex enough in $\|\hat{s}\|$. To make this idea more precise consider the case where σ_2 is small, and, therefore, the planner only wishes to learn about x_1. Normalizing $\sigma_1 = 1$ and defining $\sigma_{\hat{\epsilon}}^2 = z(s^2)$ we see that the planner solves max $\frac{s^2}{s^2 + z(s^2)}$ and the FOC is $s^2 z' = z$. The class of noise models $z = \alpha s_1^2 + \beta e^{\theta s_1^2}$ is then particularly useful because the uni-dimensional learning solution is simply $\theta \hat{s}_1^2 = 1$, which does not depend on σ_1, α, β. This motivates our functional form

$$\sigma_{\hat{\epsilon}}^2 = \alpha \|\hat{s}\|^2 + \beta e^{\theta \|\hat{s}\|^2}$$

in our application. In this case, when $\rho = 0$, the first learning coefficient is

$$\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} = \frac{\sigma_1^2 \hat{s}_1^2}{(\alpha + \sigma_1^2)\hat{s}_1^2 + (\alpha + \sigma_2^2)\hat{s}_2^2 + \beta e^{\theta\left(\hat{s}_1^2 + \hat{s}_2^2\right)}}. \tag{A.2}$$

Lemma 5. *When $\rho = 0$, $\Sigma_{\hat{\mathbf{x}}}(1,1)$ is increasing in $\hat{s}_1^2$ when $\beta e^{\theta\left(\hat{s}_1^2 + \hat{s}_2^2\right)}(\theta \hat{s}_1^2 - 1) < (\alpha + \sigma_2^2)\hat{s}_2^2$ and decreasing afterwards.*

The overall effect of an increase in $\hat{s}_1^2$ on the amount of learning about x_1 depends on the trade off between the salience of the stress on factor 1 and the increase in noise associated with more extreme scenarios. When the departure from the baseline is small the direct effect dominates and stressing factor 1 leads to more learning about factor 1. When the scenario is more extreme the second effect dominates. Note that the presence of the second factor expands the range where $\Sigma_{\hat{\mathbf{x}}}(1,1)$ is increasing in $\hat{s}_1^2$. When $\hat{s}_2^2 = 0$ the range is simply $\left[0, \theta^{-1/2}\right]$. When $\hat{s}_2^2 > 0$ it expands beyond $\theta^{-1/2}$ because of the baseline noise from factor 2 in the denominator of Equation (A.2).

The effect of the scenario choice on learning is more complex when the risk exposures are correlated because the planner can learn about x_1 by increasing $\hat{s}_2$ instead of $\hat{s}_1$. In addition, in some risk settings the planner might care a lot about the covariance term

$$\frac{\Sigma_{\hat{\mathbf{x}}}(1,2)}{\sigma_1\sigma_2} = \frac{\rho\left((\sigma_1 \hat{s}_1 + \sigma_2 \hat{s}_2)^2\right) + (1-\rho)^2 \sigma_1\sigma_2 \hat{s}_1 \hat{s}_2}{(\sigma_1 \hat{s}_1 + \sigma_2 \hat{s}_2)^2 - 2(1-\rho)\sigma_1\sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_{\hat{\epsilon}}^2}.$$

When $\rho = 0$ we get $\frac{\Sigma_{\hat{\mathbf{x}}}(1,2)}{\sigma_1\sigma_2} = \frac{\sigma_1\sigma_2\hat{s}_1\hat{s}_2}{\sigma_1^2\hat{s}_{11}^2+\sigma_2^2\hat{s}_2^2+\sigma_{\hat{\epsilon}}^2}$, when $\rho = 1$ we get $\frac{\Sigma_{\hat{\mathbf{x}}}(1,2)}{\sigma_1\sigma_2} = \frac{(\sigma_1\hat{s}_1+\sigma_2\hat{s}_2)^2}{(\sigma_1\hat{s}_1+\sigma_2\hat{s}_2)^2+\sigma_{\hat{\epsilon}}^2}$, and we can show that $\frac{\partial\Sigma_{\hat{\mathbf{x}}}(1,2)}{\partial\rho} > 0$ so learning about the posterior mean is easier when the exposures are correlated.

Proof of Lemma 5

When $N = 1$, $M = 1$, and $J = 2$, the Kalman gain is given by

$$K = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} \left(\begin{bmatrix} \hat{s}_1 & \hat{s}_2 \end{bmatrix} \left(\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right) \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} + \sigma_{\hat{\varepsilon}}^2(\hat{\mathrm{s}}) \right)^{-1},$$

which implies

$$k_1 = \frac{\sigma_1^2\hat{s}_1 + \rho\sigma_1\sigma_2\hat{s}_2}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}, \quad \text{and} \tag{A.3}$$

$$k_2 = \frac{\sigma_2^2\hat{s}_2 + \rho\sigma_1\sigma_2\hat{s}_1}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}. \tag{A.4}$$

Moreover,

$$\Sigma_{\hat{\mathbf{x}}}(j,j) = k_j\hat{s}_j\left(\sigma_j^2 + \frac{\hat{s}_h}{\hat{s}_j}\rho\sigma_1\sigma_2\right),$$

where

$$k_j\hat{s}_j = \frac{\sigma_j^2\hat{s}_j^2 + \rho\sigma_1\sigma_2\hat{s}_2\hat{s}_1}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = 1 - \frac{\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_h^2\hat{s}_h^2 + \sigma_{\hat{\epsilon}}^2}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}.$$

Note that

$$\frac{\partial k_j}{\partial\sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = -\frac{k_j}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(v)} \leq 0$$

and

$$\frac{\partial\Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial\sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = -\frac{\Sigma_{\hat{x}}(j,j)}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} \leq 0$$

Since

$$\hat{\epsilon}(\hat{\mathrm{s}}) = \alpha\epsilon_0 + \beta\left(\|\hat{\mathrm{s}}\|^{\frac{1}{2}} + \|\hat{\mathrm{s}}\|^{1+\theta}\right)\epsilon_1,$$

we have

$$\sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}}) = \alpha^2 + \beta^2\left(\left(\hat{s}_1^2 + \hat{s}_2^2\right)^{\frac{1}{2}} + \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}\right),$$

which is increasing in $|s_j|$ for $j = 1, 2$. Therefore, more extreme scenarios decrease the amount of learning. The effect of an increase in noise on the amount of learning is negligible close to the baseline, i.e.,

$$\lim_{|\hat{s}_j|\to 0}\frac{\partial\Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial\sigma_{\hat{\epsilon}}^2(\hat{s})}\frac{\partial\sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}{\partial|\hat{s}_j|} = -\frac{\Sigma_{\hat{\mathbf{x}}}(j,j)\,\beta^2\left(\left(\hat{s}_1^2 + \hat{s}_2^2\right)^{-\frac{1}{2}} + (1+\theta)\left(\hat{s}_1^2 + \hat{s}_2^2\right)^{\theta}2\right)|\hat{s}_j|}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = 0 \quad \forall j = 1, 2. \tag{A.5}$$

Moreover, the direct effect of a more extreme scenario on the amount of learning is given by

$$\frac{\partial \Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial |\hat{s}_j|} = \frac{\partial (k_j \hat{s}_j)}{\partial |\hat{s}_j|}\sigma_j^2 + \frac{\partial k_j}{\partial |\hat{s}_j|}\hat{s}_h \rho \sigma_1 \sigma_2 \quad \forall j,h=1,2,\ j \neq h.$$

When $\rho = 0$,we have

$$\frac{\partial \Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial |\hat{s}_j|} = \frac{\partial (k_j \hat{s}_j)}{\partial |\hat{s}_j|}\sigma_j^2 = \frac{\sigma_h^2 \hat{s}_h^2 + \sigma_{\hat{\epsilon}}^2}{\left(\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})\right)^2}\sigma_j^4 2 |\hat{s}_j| \geq 0.$$

Then, using Equation A.5 we have that

$$\Sigma_{\hat{\mathbf{x}}}(j,j) = \frac{\sigma_j^2 \hat{s}_j^2}{\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2 + \beta^2\left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}}$$

$$\begin{aligned}
\frac{d\Sigma_{\hat{\mathbf{x}}}(j,j)}{d\hat{s}_j^2} &= \frac{\sigma_j^2\left(\sigma_h^2 \hat{s}_h^2 + \alpha^2 + \beta^2\left(\hat{s}_1^2+\hat{s}_2^2\right)^{1+\theta}\right) - \sigma_j^2 \hat{s}_j^2 \beta^2 (1+\theta)\left(\hat{s}_1^2+\hat{s}_2^2\right)^{\theta}}{\left(\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2 + \beta^2\left(\hat{s}_1^2+\hat{s}_2^2\right)^{1+\theta}\right)^2} \\
&= \frac{\sigma_j^2\left(\frac{\sigma_h^2 \hat{s}_h^2 + \alpha^2}{\beta^2\left(\hat{s}_1^2+\hat{s}_2^2\right)^{1+\theta}} + 1\right) - \frac{\sigma_j^2 \hat{s}_j^2 (1+\theta)}{\left(\hat{s}_1^2+\hat{s}_2^2\right)}}{\left(\frac{\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2}{\beta^2\left(\hat{s}_1^2+\hat{s}_2^2\right)^{1+\theta}} + 1\right)^2}
\end{aligned}$$

$$\lim_{\hat{s}_j \to 0} \frac{d\Sigma_{\hat{\mathbf{x}}}(j,j)}{d\hat{s}_j^2} > 0 \quad \text{and} \quad \lim_{\hat{s}_j \to \infty} \frac{d\Sigma_{\hat{\mathbf{x}}}(j,j)}{d\hat{s}_j^2} = -\theta \sigma_j^2 < 0$$

When $\rho > 0$, using Equation (A.3) and the definition of $\sigma_{\hat{\epsilon}}^2(\hat{s})$, we have that

$$k_1 \hat{s}_1 = \frac{\sigma_1^2 \hat{s}_1^2 + \rho \sigma_1 \sigma_2 \hat{s}_2 \hat{s}_1}{\sigma_1^2 \hat{s}_1^2 + 2\rho\sigma_1\sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2 + \beta^2\left(\hat{s}_1^2+\hat{s}_2^2\right)^{1+\theta}}$$

and

$$\Sigma_{\hat{\mathbf{x}}}(j,j) = \frac{\sigma_j^2 \hat{s}_j^2 + \rho\sigma_1\sigma_2 \hat{s}_2 \hat{s}_1 + \rho \sigma_j^2 \hat{s}_j \hat{s}_h + (\rho\sigma_1\sigma_2)^2 \hat{s}_h^2}{\sigma_1^2 \hat{s}_1^2 + 2\rho\sigma_1\sigma_2 \hat{s}_1 \hat{s}_2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2 + \beta^2\left(\hat{s}_1^2+\hat{s}_2^2\right)^{1+\theta}}.$$