Selling to Advised Buyers

Andrey Malenko*
MIT Sloan School of Management
Anton Tsoy
EIEF

October 28, 2015

Abstract

In many cases, agents that make purchase decisions are uninformed and rely on the advice of biased experts. For example, when contemplating an acquisition, the board of the bidder relies on the advice of the managerial team when deciding what offer to make for the target. In this paper, we study how to sell assets to such "advised buyers" (i) if the goal is to maximize revenues; (ii) if the goal is to maximize allocative efficiency. In static mechanisms, such as first- and second-price auctions, advisors communicate a coarsening of information and a version of the revenue equivalence theorem holds. In contrast, dynamic mechanisms, such as multiple-round auctions, result in more informative communication between buyers and their advisors, which leads to more efficient allocations and may lead to higher revenues. When advisors are biased toward overpaying, an ascending-price auction dominates any static mechanism in terms of both efficiency and revenues. When advisors are biased towards underpaying, a descending-price auction dominates static mechanisms in terms of efficiency but often results in lower revenues.

*Malenko: amalenko@mit.edu. Tsoy: tsoianton.ru@gmail.com. We thank Alessandro Bonatti, Peter DeMarzo, Doug Diamond, Glenn Ellison, Bob Gibbons, Navin Kartik, Nadya Malenko, Jonathan Parker, Jean Tirole, and participants at the MIT Theory Lunch, ASSA 2015, 2015 Stony Brook International Conference on Game Theory for helpful comments.

1 Introduction

In many applications, agents that make purchase decisions have limited information about their valuations of the assets for sale. As a consequence of limited information, they often rely on the advice of informed experts, who however often have misaligned preferences. Consider the following examples:

1. *A firm competing for a target in a takeover contest.* While the board of directors typically has formal authority over submitting offers for the target, the firm's managers are more informed about the valuation of the target. The managers, however, could be prone to overbidding because of career concerns and empire building preferences.

2. *Bidding in spectrum auctions.* Large telecommunication companies bidding in spectrum auctions have research teams in charge of preparing for the auction, producing valuation estimates of auctioned frequencies, and advising the top management and board of directors on bidding. These research teams may have different incentives as winning the auction could give a positive signal to the market and help the research team attract future business.

3. *Suppliers competing in procurement.* When a construction company competes on cost for a project in a procurement auction, managers who will work on a project are privately informed about the actual cost of the project for the firm, while the top management of the company makes offers for the project. If the cost at which the construction company determines the budget that managers will operate with, they have a bias for overstating the cost.

4. *Realtors in real estate transactions.* A potential buyer of a house gets advice from a realtor about what offer to make. The realtor has information about the value of the house that the buyer does not have. The realtor may be biased toward overpaying, since she is compensated as a percentage of the transaction price and conditional on the transaction happening.

These examples have a common element: There is a separation of authority over bidding decisions and information about the buyer's valuation. We call such players "advised buyers" and ask the following question in the paper: Do agency problems between buyers and their advisors affect how the seller should design the sale process? We analyze this question both from the position of maximizing expected revenues, which is likely the goal if the designer is the seller, and from the position of allocative efficiency, which could be a more important concern than revenues if the designer is the government.

We study a canonical setting where the seller has an asset for sale to auction among a number of potential buyers who have independent private values. But we depart from the canonical setting in one crucial aspect: We assume that each potential buyer is a bidder-advisor pair, where the bidder is the party with formal authority over bidding decisions (e.g., the board of the firm) but without information about her valuation, while the advisor (e.g., the firm's manager) is the party that knows the valuation of the bidder but has a conflict of interest. Our initial focus is on the case where advisors have a bias toward overbidding, that is, given the same information, the advisor's maximum willingness to pay exceeds that of the bidder she advises by $b > 0$. Later we consider the case where the advisor's bias is toward underbidding, which could be more relevant in procurement.

This overbidding bias captures empire-building motives or career concerns in the examples above. Before the bidder submits an offer, the advisor communicates with the bidder via a game of cheap talk. Specifically, if the sale process consists of a single round, there is only one round of communication. In contrast, if the sale process consists of multiple rounds, the advisor can communicate with the bidder at each stage of the auction. In this environment, there is an interesting interaction between communication and the design of the sale process. On one hand, the amount of information transmitted from each advisor to her bidder affects bids submitted in the auction and through them the efficiency and revenue of each auction format. On the other hand, the auction format affects the incentives of advisors to reveal their information to their bidders.

We analyze equilibria of the model under the NITS (no incentive to separate) condition adapted from Chen, Kartik, and Sobel (2008).[1] In fact, even if the NITS condition does not select the unique equilibrium in the communication game, our analysis about the comparison of auctions does not require any further refinement. We first study static (i.e., single-round) auctions. As one could expect from the standard game of "cheap talk" (Crawford and Sobel, 1982), communication in this case takes a partition form: all types of the advisor are partitioned into intervals and types in each interval induce the same bid. Even though our game is not a special case of Crawford and Sobel (1982), since the payoffs of the buyer and her advisor are endogenous, their logic directly applies here. Despite the endogenous cheap-talk game, we show that efficient equilibria of the static auction still have a partition structure with an upper bound on the number of partition intervals, and that the most informative equilibrium satisfies the NITS condition. We prove a version of the revenue equivalence theorem for static auctions: modulo the existence of efficient equilibria in a particular static auction, all efficient equilibria of any static auction bring the same expected revenue and generate the same communication. In other words, in the class of static sale procedures, it is not possible to manipulate the rules of the auction to extract extra revenues or induce better communication. Thus, among static auctions, advising relationships do not matter for the auction design, at least in the canonical environment of independent private values.

Surprisingly, this conclusion drastically changes if the asset is sold via dynamic mechanisms, such as multi-round auctions. To see this, consider an ascending-bid (English) auction, where the price continuously increases until only one bidder remains in the auction who then pays the price at which the previous bidder dropped out. From the position of a potential buyer, the ascending-bid auction is a stopping time problem: At what price level to drop out. Information transmission is perfect at the bottom of the type distribution and there is pooling at the top in a sense that types at the top induce the same bid. Because of the superior information transmission, the English auction generally outperforms any static auction both in terms of efficiency and expected revenue. In particular, the revenue equivalence does not hold between static and dynamic auctions.

[1] Intuitively, the NITS condition says that the weakest type (i.e., the lowest valuation if the advisor has an overbidding bias) has the option to credibly reveal herself at any stage of the auction, which puts a lower bound on her expected payoff at any stage.

The key distinction of the English auction is that the advisor can reveal information over time. Under the simplest communication protocol, the advisor reveals her information to the bidder right before the advisor's optimal quitting time. Under such a communication protocol, perfect information transmission is possible for types at the bottom of the distribution for the following reason. If the bidder observes her values, then it is optimal for bidders to quit the auction when the running price equals her value. Because of the overbidding bias, the advisor prefers to quit the auction later than the bidder. If the advisor perfectly reveals the value at her optimal quitting time, then it is optimal for the bidder to immediately quit the auction. Indeed, at this point the bidder is already past her break-even price and any further delay will result into a higher chance of winning at a price that brings negative profit. Because in the English auction the bidder is restricted to submit only bids higher than the current auction price, types at the bottom of the distribution are able to communicate their private information perfectly and induce the bidder to quit at their optimal price.

However, even in the English auction, information cannot be transmitted perfectly for all types when the support of the distribution of values is finite. As the price of the auction approaches the highest possible valuation of the asset, the uncertainty of each bidder about his value decreases. At some point, he can accurately predict his value as well as the fact that he will overpay for the asset if he wins, because the advisor waits until the advisor's optimal price to quit. Therefore, the bidder will always quit when her uncertainty is sufficiently reduced before types of the advisor at the top reveal themselves.

The information transmission affects the efficiency of auction formats. Because of imperfect information transmission, static auctions are necessarily inefficient, as ties occur with positive probability. At the same time, in the English auction the information transmission for types at the bottom of the distribution is perfect, hence allocation is more efficient for these types compared to static auctions. It turns out that even taking into account the pooling at the top, the English auction is always more efficient than any static auction, as no static auction makes these types at the top separate even partially.

The information transmission also affects the revenue of auction formats. Under the assumption that the distribution of valuations satisfies the monotone hazard ratio property, the expected revenue of the auction is higher in the English auction than in any static auction format. Hence, when advisors are biased toward overbidding, there is no trade-off between revenue and efficiency: the English auction dominates any static format on both dimensions. This fact has an important practical implication. While in many contexts the expected revenue is the primary objective of the seller, in other contexts, such as FCC auctions and privatizations, efficiency can be an equally important or even the primary goal. Moreover, the bias for overbidding is relevant in many applications, because of the empire-building and career concerns described above. Our results suggest that the English auction is the preferred method of selling assets in this environment no matter whether the seller is concerned about efficiency, revenue or both.

The intuition for the higher revenue comes from the fact that the seller would prefer to sell directly to advisors, as they have a higher willingness to pay for the asset. However, because bidders have formal authority over bidding and advisors can only affect them through the information they provide, the equilibrium bids reflect a mix of interests of bidders and advisors, and so, are lower. The English auction is an auction format that allows the seller to essentially eliminate bidders and sell directly to advisors, as bids are optimal for advisors.

Do dynamic auction also dominate static auctions if advisors are biased towards underbidding? Suprisingly, the answer is no: the comparison of dynamic and static auctions is ambiguous in this case. If the bidder knew his value, he would submit a bid that wins with higher probability than an optimal bid of the advisor. Hence, with bias toward underbidding, the Dutch auction that restricts bidders to submit bids not higher than the current price of the auction allows for a better information transmission. We construct an equilibrium of the Dutch auction that exhibits pooling at the bottom and perfect information transmission at the top of the distribution. This equilibrium is more efficient than any efficient equilibrium of any static auction, but it can bring lower expected revenue to the seller. The reason for this is that when the advisor is biased toward underbidding, selling directly to advisors no longer guarantees the highest expected revenue, as advisors have lower willingness to pay. Because of that, it is possible that the seller benefits from imperfect communication between the advisor and the bidder, as it results into an upward bias of bids relative to the bids submitted directly by advisors.

This paper is related to two strands of the literature: The literature on the comparison of auction formats and the literature on communication of non-verifiable information. First, it is related to the literature on communication of non-verifiable information (cheap talk), pioneered by Crawford and Sobel (1982). This literature usually focuses on exogenous payoffs of players from the decision made by the decision-maker (buyer in our context) and exogenous timing of the game (typically, one round of communication). In contrast, the payoffs and the game itself are endogenous. In particular, we show how by using communication over time, the seller can make communication between bidders and their advisors more efficient, which sometimes (but not always) leads to higher revenues. The result that decisions over the timing lead to very different equilibria than static decisions due to irreversibility of time is due to Grenadier, Malenko, and Malenko (2015). The new features of our model is that there are multiple sender-receiver (bidder-advisor) pairs, and the design of the game is strategic by the seller. A number of papers study cheap talk models with other, less related to ours, dynamic aspects of communication.[2]

Second, the paper is related to the mechanism design literature on the comparison of auction formats. The central result in the literature is the celebrated revenue equivalence theorem in the independent private values setting (Myerson 1981; Samuelson, 1981). As we show, it continues to hold when bidders are "advised buyers" and sale mechanisms are static, but breaks down when they are multi-round. The revenue equivalence theorem can fail for other reasons, such as affiliation of values (Milgrom and Weber, 1982), bidder asymmetries (Maskin and Riley, 2000), or budget constraints (Che and Gale, 1998). To our knowledge, we are the first to study the problem of the design of sale procedures when potential buyers are advised by informed experts. A few of papers study related aspects. Compte and Jehiel (2007) shows that the ascending-bid auction brings higher revenues than static auctions when bidders can acquire information about their values. The key conditions for this result are the asymmetry of bidders in information endowments and knowledge of the number of remaining bidders. Our paper provides a different rationale for multiple-round auctions, which does not rely on bidder asymmetries and knowledge of the number of remaining bidders. Burkett

[2]See Sobel (1985), Morris (2001), Aumann and Hart (2003), Krishna and Morgan (2004), Ottaviani and Sorensen (2006a, 2006b), Golosov et al. (2014).

(2015) studies a principal-agent relationship in the auction context when the principal decides on the budget of the biased agent who submits the bid. He shows that the ranking of first- and second- price auctions usually agrees with those from models without budget constraints. Our result on revenue equivalence for static auctions is related to his. Finally, Ye (2007), Kos (2012), Inderst and Ottaviani (2013), and Quint and Hendricks (2013) study other interactions between mechanism design and communication. In particular, Ye (2007) and Quint and Hendricks (2013) study two-stage auctions, where the actual bidding is preceded by the indicative stage, which is a form of cheap talk between bidders and the seller. Kim and Kircher (2015) study how auctioneers with private reservation values compete for potential bidders by announcing cheap-talk messages.

The structure of the paper is the following. Section 2 introduces the model and illustrates our main findings with a simple example. Section 3 characterizes equilibria of static auctions and establishes a version of the revenue equivalence for static auctions. Section 4 characterizes equilibria of the English auction under the NITS condition when bidders have overbidding bias, and shows that the English auction outperforms any static auction. Section 5 analyzes the case of advisors' preferences for underbidding. Section 6 shows that the effect of the advisors' bias on the efficiency and revenues is quantitatively significant. Section 7 concludes. Key proofs are provided in the text and the rest of the proofs are relegated to Appendix.

2 Model

Consider the standard setting with independent private values. There is a single indivisible asset for sale. The value of the asset to the seller is normalized to zero. There are N ex-ante identical potential buyers (bidders). The valuation of bidder i, v_i, is an i.i.d. draw from distribution with c.d.f. F and p.d.f. f. The distribution F has full support on $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} < \overline{v} \leq \infty$. In the analysis, we will frequently refer to the distribution of valuation of the strongest opponent of a bidder. We denote by $\hat{v}$ the maximum of $N-1$ i.i.d. random variables distributed according to F and its c.d.f. by G: $G(\hat{v}) = F(\hat{v})^{N-1}$.

The novelty of our setup is that each bidder i does not know his valuation v_i, but can consult an advisor who does. Let advisor i denote the advisor to bidder i. Advisor i knows v_i, but has no information about valuations of other bidders except for their distribution F, which is common knowledge. While advisor i knows v_i, she is biased relative to the bidder. Specifically, the payoffs from acquiring the asset by bidder i are

$$\text{Bidder } i \quad : \quad v_i - p, \tag{1}$$
$$\text{Advisor } i \quad : \quad v_i + b - p, \tag{2}$$

where b is the bias of the advisor. The value that all players get from not acquiring the asset is zero. Bias b is commonly known.[3] Our primary focus is on the preference of advisors for overbidding, $b > 0$, as it is most prominent in applications. In Section5, we also consider the case of $b < 0$, which shares several similarities with the case of $b > 0$, but also differs from it in a number of important aspects.

[3]For many of our results it is sufficient to assume that b is commonly known by bidders and advisors, while the seller knows only the sign of the bias.

Our formulation $(1)-(2)$ captures the empire building motives described in the introduction. For example, consider a publicly traded firm bidding for a target. The board of the firm has formal authority over the bidding process, maximizes firm value, but does not know valuation v_i. Suppose that the CEO of the firm knows v_i, but is biased. Specifically, if the CEO owns fraction α of the stock of the company and gets a private benefit of B from acquiring the target and managing a larger company, her payoff is $\alpha(v_i - p) + B$. Normalizing this payoff by α and denoting $b = \frac{B}{\alpha}$, we obtain $(1)-(2)$.

In this paper, we compare how different selling mechanisms affect the seller's expected revenue and the allocative efficiency. Several formats are commonly used in practice and studied in the academic literature:

1. **Second-price auction (SPA)**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pays the second-highest bid.

2. **First-price auction (FPA)**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pay her bid.

3. **Ascending-price (English) auction**. The seller continuously increases the price p, which we refer to as the *running price,* starting from zero. At each moment, each bidder decides whether to continue participating or to *quit* the auction. Once a bidder quits, she cannot re-enter the auction. Once only one bidder remains, she is declared the winner and pays the price at which the last of her opponents quit the auction.

4. **Descending-price (Dutch) auction**. The seller continuously decreases the price p, which we refer to as the *running price,* starting from a high enough level. At each moment, each bidder decides whether to *stop* the auction. The first bidder who stops the auction is declared the winner and pays the price at which she stopped the auction.

In all of these auction formats, if a tie occurs, the winner is drawn randomly from the set of tied bidders. We study a rich class of static auctions formally described in Section 3, but restrict attention to the ascending-price and descending-price auctions among dynamic mechanisms.

Communication between bidders and their advisors is modeled as game of cheap talk. If the auction format is static (i.e., it consists of a single round of bidding), the timing of the game is as follows:

1. Advisor i sends a private message $\tilde{m}_i \in M$ to bidder i where M is some infinite set of messages.

2. Having observed message $\tilde{m}_i$, bidder i chooses her action, i.e., what bid b_i to submit.

3. Given all submitted bids b_1, ..., b_N, the asset is allocated and payments are made according to the rule specified by the auction.

If the auction format is dynamic (i.e., it consists of multiple rounds of bidding), the advisor sends a message to the bidder before each round of bidding. In ascending-price and descending-price auctions, we index stages by corresponding running prices p.

A *(private) history* of bidder i at the beginning of stage p consists of all bidders' actions and all messages sent by advisor i in the previous rounds. In static auctions, there is a single round of bidding, so the history is empty. A history at stage p in the ascending-bid auction consists of the current running price p, the set of bidders remaining in the game, stages at which some of the bidders dropped out in the past, and messages sent by advisor i up to stage p. A history at stage p in the descending-bid auction consists of simply the current running price p and messages sent by advisor i up to stage p. Let $\mathcal{H}$ denote the set of all possible histories.

A strategy of advisor i is a measurable mapping $m_i : [\underline{v}, \overline{v}] \times \mathcal{H} \to M$ from the set of possible valuations and histories into a message sent to bidder i. A strategy of bidder i is a mapping $a_i : \mathcal{H} \times M \to A$ from the history and current message into the action chosen by the bidder. We focus on pure strategies in dynamic auctions, but allow for mixing by bidders in static auctions. In static auctions, A consists of all possible mixtures over bids. In the English/Dutch auction, $A = \{0, 1\}$ consists of a decision to quit the auction or continue.

The equilibrium concept is the Perfect Bayesian Equilibrium in Markov strategies (PBEM) where the state consists of the auction stage p and a bidder's posterior belief about her valuation v_i.[4] Of course, when checking that strategies constitute a PBEM, we allow for deviations that condition on all the history. Since all bidders are symmetric, we focus on symmetric equilibria in which strategies m_i and a_i do not depend on i. Thus, we suppress index i in the notation.

In dynamic auctions, the timing of actions becomes relevant. A convenient class of equilibria is the one in which the advisor gives a *real-time* recommendation of the auction to the bidder, such that the bidder follows her advisor's recommendation on equilibrium path.

Definition 1. *A PBEM in the dynamic auction is* in online strategies *if* $m : [\underline{v}, \overline{v}] \times \mathcal{H} \to A$ *and* $a(h, \tilde{a}) = \tilde{a}$ *for all* $h \in \mathcal{H}$ *and all* $\tilde{a}$ *in the image of* $m(\cdot, h)$.

Let us stress that action recommendations happen in real time. For example, the online strategy in an ascending-bid auction is such that the advisor recommends whether to quit or continue bidding at each current price level p. In particular, the strategy in which the advisor makes only one recommendation at the beginning of the auction is not an online strategy. The following lemma states that restriction to online strategies is without loss of generality. The proof is the manifestation of the sure-thing principle (Savage, 1954) stating that if an action is optimal for a decision maker in every state, then it is optimal unconditionally.

Lemma 1. *For any PBEM there is another outcome-equivalent PBEM in online strategies.*

There is in general a multiplicity of equilibria both in cheap-talk and auction games. To make meaningful comparison across auction formats, we rely on the following refinements. First, we assume that bidders play weakly dominant strategies if such strategies exist. In particular, it guarantees that in the second-price auction bidders bid their expected values.

[4] The state that we choose does not capture the whole payoff-relevant history which in addition includes the number of remaining bidders in the auction. The reason for this restriction is a technicality that arises in continuous-time games. When the running price in the English auction changes continuously, the outcome of the auction may be indeterminate, which is a common problem of formulating games in continuous time (Simon and Stinchcombe, 1989).

Second, we impose the "no incentive to separate" (NITS) condition, adapted from Chen, Kartik, and Sobel (2008), to select among equilibria in the communication game between bidders and advisors. Call type $v_w \equiv \underline{v}$ the *weakest type* of advisor. According to the NITS condition, the weakest type has an option to credibly reveal herself if she wants. Thus, an equilibrium violates the NITS condition if the payoff of the weakest type is less than what she would get from revealing herself to the seller. Intuitively, when an advisor is biased for overbidding, it is natural to assume that the recommendation to bid little would be perceived as credible by the bidder. Chen, Kartik, and Sobel (2008) show that NITS can be justified by perturbations of the cheap-talk game with non-strategic players and costs of lying.

For dynamic selling mechanisms, such as ascending- and descending-bid auctions, we require that the NITS condition holds at every stage of the game. Specifically, let $v_w^p \equiv \inf\{v | v \in supp(\mu^p)\}$ be the weakest remaining type of the advisor at stage p, where μ^p for a posterior belief of the bidder about her value at stage p. Similarly to Chen, Kartik, and Sobel (2008), an equilibrium violates the NITS condition if at stage p, the advisor of type v_w^p is better off claiming that she is the weakest remaining type than playing her equilibrium strategy:

Definition 2. *An equilibrium satisfies* the NITS condition *if for any p, type v_w^p of the advisor weakly prefers her equilibrium strategy to the action optimally chosen by the bidder at stage p who knows that her value is v_w^p.*

We refer to an equilibrium as *babbling* if regardless the realization of the valuation, each bidder always plays the same strategy. We refer to an equilibrium of the static auction as *the most informative* if it induces the largest number of actions. As we show later, the most informative equilibrium in the static auctions always satisfies NITS. However, NITS need not select the unique equilibrium, and for the comparison of auction formats, we do not need a selection beyond NITS.

2.1 Example: Two Bidders and Exponential Distribution

Before proceeding with the analysis, we first illustrate the main effects with a simple example with two bidders ($N = 2$) and exponential distribution of valuations with parameter λ. We keep the assumption that $b > 0$, i.e., advisors have a bias for overbidding. In particular, we show how equilibria in the second-price and ascending-bid auctions look like, how the NITS condition refines them, and why the ascending-bid auction dominates the second-price auction in both efficiency and revenues.

We start with the characterization of the most informative equilibrium of the second-price auction. After a bidder receives a message from her advisor, it is a weakly dominant strategy to submit the bid equal to her expected valuation. The bias of an advisor to overbid imposes a limit on how much information she can transmit to the advisor. The following proposition characterizes the most informative equilibrium in this game:[5]

Proposition 1. *The following strategies constitute the most informative equilibrium of the second-price auction. There exists a sequence $(\omega_k)_{k=0}^K$ with $\omega_0 = 0$ and $K < \infty$ such that*

[5]In fact, by varying K in the statement of Proposition 1, we can get any equilibrium of the second-price auction.

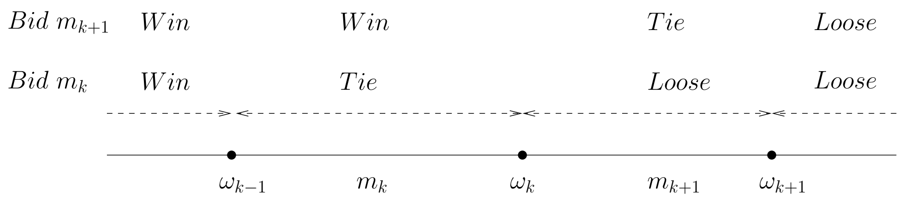


Figure 1: Thresholds in the partition equilibrium of the second-price auctions. Type ω_k of advisor is indifferent between pooling with types in $[\omega_{k-1}, \omega_k)$ by sending message m_k and types in $[\omega_k, \omega_{k+1})$ by sending m_{k+1}. The difference between messages m_k and m_{k+1} is that m_{k+1} wins for sure against types in $[\omega_{k-1}, \omega_k)$ and ties against types in $[\omega_{k-1}, \omega_k)$, while m_k ties with types in $[\omega_{k-1}, \omega_k)$ and looses against types in $[\omega_{k-1}, \omega_k)$.

- *for all $k = 1, \ldots, K < \infty$, the advisor with type $v \in [\omega_{k-1}, \omega_k)$ sends message $m_k \equiv \mathbb{E}[v | v \in [\omega_{k-1}, \omega_k)]$;*
- *the bidder submits a bid equal to the message received.*

For $b < \frac{1}{\lambda}$, $(\omega_k)_{k=0}^{K}$ are given by the following recursion

$$\frac{\omega_{k-1} e^{-\lambda \omega_{k-1}} - \omega_{k+1} e^{-\lambda \omega_{k+1}}}{e^{-\lambda \omega_{k-1}} - e^{-\lambda \omega_{k+1}}} = \omega_k + b - \frac{1}{\lambda} \tag{3}$$

with the terminal condition $\omega_{K+1} = \infty$ where K is the maximal length of recursion possible so that $\omega_1 > 0$. For $b \geq \frac{1}{\lambda}$, there is only a babbling equilibrium, i.e. $\omega_1 = \infty$.

The intuition behind Proposition 1 is similar to that in the standard cheap talk game (Crawford and Sobel, 1982) with one important difference. In the standard cheap talk game, the payoffs of the sender and the receiver are exogenous. In contrast, in the auction, they are endogenous, since the payoff of a bidder and her advisor from every bid depend on the communication and bidding strategies of other bidders and their advisors. Other that this change, the intuition is similar. Because of the bidder cannot commit to not "de-bias" the advisor's recommendation, full information revelation is not possible. If the conflict of interest between buyers and their advisors is high enough ($b \geq \frac{1}{\lambda}$), advisors cannot communicate any information and all bidders submit the same bid $\mathbb{E}[v] = \frac{1}{\lambda}$. However, if the conflict of interest is not too high ($b < \frac{1}{\lambda}$), some information transmission is possible because there is enough alignment of interests of buyers and their advisors. In this case, equilibria have a partition structure, where advisors with the same information in a partition communicate to the bidder that the valuation is in the partition, and the bidder reacts by bidding the expected valuation, conditional on learning that it is in the partition. Partition points $(\omega_k)_{k=0}^{K}$ are determined by the indifference condition that the advisor with valuation ω_k, $k \in \{1, ..., K-1\}$ is indifferent between communicating that $v \in [\omega_{k-1}, \omega_k]$ and communicating that $v \in [\omega_k, \omega_{k+1}]$. When doing the latter instead of the former, the advisor with valuation ω_k is better off because she wins with probability 100% (instead of 50%) against the rival bidder with valuation in $[\omega_{k-1}, \omega_k]$, but worse off because the bidder wins with probability 50% (instead of 0%) and pays more than the advisor's maximum willingness to pay when the rival bidder's valuation is in $[\omega_k, \omega_{k+1}]$.

As the next proposition shows, the equilibrium in the ascending-bid auction takes a very different form:

Proposition 2. *Suppose $b \neq \frac{1}{\lambda}$. The unique PBEM of the English auction satisfying the NITS condition is:*

- *Fully informative if $b < \frac{1}{\lambda}$. The advisor with valuation v sends message "continue," if and only if the current price $p < v + b$, and "quit," otherwise. The bidder follows the recommendation of her advisor.*

- *Babbling when $b > \frac{1}{\lambda}$. The bidder ignores messages from the advisor and quits when the running price p reaches $\frac{1}{\lambda}$.*[6]

To see that these strategies are indeed an equilibrium, notice that if the bidder follows the recommendation, then the strategy to quit when $p = v + b$ is optimal for the advisor. Indeed, it is a weakly dominant strategy in the English auction where the advisor decides when to quit. When the bidder gets message "quit", $p > v$. Since p is increasing over time, it is optimal for the bidder to quit immediately, as he is already past his break-even point. To complete the verification of the equilibrium, we show that the bidder does not want to quit earlier. Let $v_p \equiv p - b$ be the lowest type of advisor remaining in the auction at stage p. The expected utility of the bidder i at time t from following the recommendation of the advisor is

$$\begin{aligned} V(v_p) &= \mathbb{E}\left[(v - \hat{v} - b)1\{v > \hat{v}\}|v, \hat{v} > v_p\right] \\ &= \tfrac{1}{2}\left(\mathbb{E}\left[\max\{v, \hat{v}\}|v, \hat{v} > v_p\right] - \mathbb{E}\left[\min\{v, \hat{v}\}|v, \hat{v} > v_p\right] - b\right) \\ &= \tfrac{1}{2}\left(\mathbb{E}\left[\max\{v, \hat{v}\}\right] - \mathbb{E}\left[\min\{v, \hat{v}\}\right] - b\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{\lambda} - b\right), \end{aligned}$$

where the first equality is by the symmetry of the auction, the second equality is by the memoryless property of the exponential distribution, and the last equality is by $\mathbb{E}\left[\max\{v_i, v_j\}\right] = \frac{3}{2\lambda}$ and $\mathbb{E}\left[\min\{v_i, v_j\}\right] = \frac{1}{2\lambda}$. Hence, when $b \leq \frac{1}{\lambda}$, the bidder prefers to wait for a recommendation from the advisor rather than quit earlier, while for $b > \frac{1}{\lambda}$, the bidder prefers to quit in the beginning of the auction.

The dynamic communication strategy of the advisor attains perfect information transmission in the English auction. When the advisor recommends the bidder to quit at some price p, the bidder learns that her valuation is $p - b$. Of course, the bidder would want to go back in the auction and quit at price $p - b$. However, it is not possible, since bids can only go up over time. As a consequence, the best response of the bidder is to drop from the English auction immediately after he receives the recommendation from the advisor. Intuitively, the fact that the price only goes up in the ascending-price auction creates a commitment device for the bidder not to overrule her advisor's recommendation. Consequently, the advisor can credibly communicate her information to the bidder in the English auction, which is not possible in the second-price auction.

[6] When $b = \frac{1}{\lambda}$, there is a continuum of equilibria indexed by $v^* \in [0, \infty]$. In a v^*-equilibrium, the advisor with valuation v sends message "continue," if and only if the current price $p < v + b$, and the bidder follows the recommendation of the advisor until the running price reaches $v^* + b$, at which point the bidder quits.

One can immediately see that the equilibrium in Proposition 2 satisfies NITS. At any stage p, the lowest type of the advisor sends the recommendation to quit. This is also an optimal action of the bidder who has the lowest value at stage p. Importantly, NITS rules out other equilibria. This is quite surprising given that without the advisor's bias, the second-price and English auction are strategically equivalent. Without NITS, the equilibrium of the second-price auction also constitutes an equilibrium of the English auction. We can specify that all communication happens at the initial bidding stage where bidders learn their values m_k. However, this equilibrium does not satisfy the NITS condition in the English auction. Indeed, consider the running price p just below m_{k+1}: $p = m_{k+1} - \varepsilon$ for infinitesimal $\varepsilon > 0$. Once p is past m_k, the bidder infers that $v \geq \omega_k$, and there is no updating until p reaches m_{k+1}. However, the lowest remaining type ω_k is determined by the indifference condition (13) , stating that the benefit of the advisor with type ω_k from winning the auction at price m_k is equal to the loss from potentially winning the auction at price m_{k+1} and overpaying in this case. In particular, it implies that the advisor with the lowest remaining type ω_k strictly prefers not to win at price m_{k+1}. Therefore, she prefers to reveal that her type is the lowest remaining at price $p = m_{k+1} - \varepsilon$, and according to the NITS, the bidder will believe this message and drop from the auction at price $p = m_{k+1} - \varepsilon$. As we will show later, this is a general phenomenon. All equilibria of static auctions have a partition structure, while all equilibria of the English auction have perfect information transmission up to a (potentially infinite) cut-off.

We next show that the English auction dominates the second-price auction in terms of efficiency and revenue. First, the English auction is efficient for $b \in \left(0, \frac{1}{\lambda}\right]$. In the second-price auction, since the communication is imperfect, ties arise in the auction with positive probability and lead to inefficient allocation. To compare the revenue, we use Myerson (1981), to write the revenue from different auction formats as follows

$$2\left(\mathbb{E}[\varphi(v)(1\{m(v) > m(\hat{v})\} + \tfrac{1}{2}1\{m(v) = m(\hat{v})\})] - U_A(0)\right) \tag{4}$$

where $\varphi(v) = v + b - \frac{1}{\lambda}$ is the virtual valuation of advisor with type v, $U_A(0)$ is the expected utility of the advisor with type 0.[7] We claim that (4) is higher in the English auctions than in the second-price auction. Since φ is increasing and the English auction is more efficient, the first term in (4) is higher for the English auction (and strictly higher for $b < \frac{1}{\lambda}$). Hence, we are left to show that $U_A(0)$ is lower in the English auction. In the English auction, $U_A(0) = 0$ as the type 0 is the first to quit the auction and so, she wins with probability 0. In general, it is not true that $U_A(0)$ is non-negative in the second-price auction. As a simple example, consider the babbling equilibrium. The type 0 wins the auction with probability $\frac{1}{2}$, as both bidders ignore messages from their advisors and bid their expected value $\frac{1}{\lambda}$. Hence, in the babbling equilibrium, $U_A(0) = \frac{1}{2}\left(b - \frac{1}{\lambda}\right) < 0$ for $b < \frac{1}{\lambda}$. The key observation is that the babbling equilibrium fails the NITS condition for $b < \frac{1}{\lambda}$. At the same time, the most informative equilibrium of the second-price auction satisfies the NITS condition which implies that $U_A(0) \geq 0$ in this equilibrium and so, the revenue is higher in the English auction.

To explore these effects quantitatively let $\lambda = \frac{1}{5}$. The left panel of Figure 2 depicts the efficiency of the auction formats captured by the expected value of the winner. Because

[7] We suppress the history in notation of the strategy m.

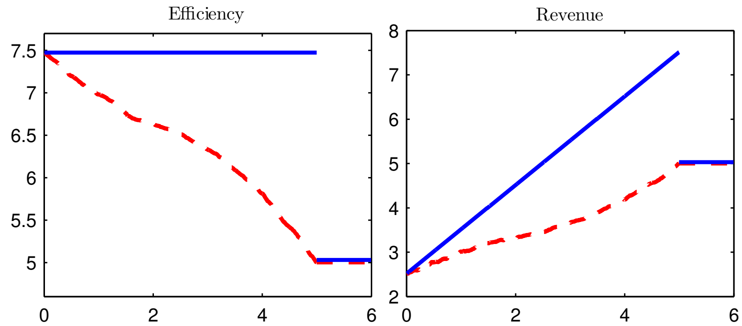


Figure 2: Efficiency and revenue comparison for $b > 0$: the English auction (solid line) and the second-price auction (dashed line). Bias b is plotted on the horizontal axis.

of the perfect information transmission, the English auction is efficient if $b \leq \frac{1}{\lambda}$ and so its efficiency does not vary with b on this interval. The gap in efficiency between the English auction and the second-price auction increases as b increases up to $\frac{1}{\lambda}$. This happens because communication in the second-price auction becomes less and less informative as the bias increases. In the right panel of Figure 2, we depict the revenue of the seller for different auction formats for $\lambda = \frac{1}{5}$. The gap in efficiency and revenue between two auctions increases with the size of the bias for $b < \frac{1}{\lambda}$. Notice the discontinuity with respect to b. If b is greater than $\frac{1}{\lambda}$, then all equilibria are babbling and the seller gets revenue $\frac{1}{\lambda}$. That is, the seller benefits from having the bias only when this bias is not too large.

3 Static Auctions

In this section, we show that the revenue equivalence principle extends to the setting when the interests of bidders and advisors are not aligned ($b \neq 0$), if selling mechanisms are static in the sense that they admit only a single round of communication between bidders and their advisors. We characterize equilibrium information transmission and show that there is necessarily an efficiency loss due to imperfect communication. In the next sections, we show that dynamic mechanisms lead to very different implications both in terms of information transmission and generated revenues.

Consider any static auction. After a bidder gets a message from her advisor, she updates her best estimate of her valuation. Denote by *type* $\theta_i \equiv \mathbb{E}[v_i|\tilde{m}_i] \in [\underline{v}, \overline{v}]$ *of bidder* i her expected value of the asset conditional on receiving message $\tilde{m}_i$ from her advisor. Denote by F_θ the distribution of types of each bidder generated through communication in equilibrium. If communication is imperfect, the support of F_θ could be a subset of $[\underline{v}, \overline{v}]$. In particular, it could be finite. We extend strategy a to types that are assigned probability zero under F_θ by simply specifying that they best respond to the the strategies of opponents.[8] Given equilibrium bidding strategy a, let $q_i(\theta_1, \ldots, \theta_N)$ be the probability that bidder i wins the auction (gets the asset) given types of all bidders. We define allocation to be *conditionally efficient* if the asset is allocated to the bidder with the highest type θ:

[8]Since these types have probability zero under F_θ, the extended strategy still constitutes an equilibrium.

Definition 3. *Allocation* $q_i(\theta_1, \ldots, \theta_N)$ *is* conditionally efficient *if*

$$q_i(\theta_1, \ldots, \theta_N) = \begin{cases} \frac{1}{n}, & \text{if } \theta_i \in \max\{\theta_1, \ldots, \theta_N\} \text{ and } n \equiv |\{j : \theta_j = \max\{\theta_1, \ldots, \theta_N\}| , \\ 0, & \text{otherwise.} \end{cases}$$

for all $i = 1, \ldots, N$ *and all* $(\theta_1, \ldots, \theta_N) \in [\underline{v}, \overline{v}]^N$*.*

The next lemma states a version of the revenue-equivalence for static auctions with communication.

Lemma 2 (**Revenue Equivalence**)**.** *Consider a conditionally efficient equilibrium of some static auction. There exists an equilibrium of the second price auction which generates the same expected revenue and the same distribution of bidders' types* F_θ*.*

In this paper, we are interested in whether the auction format affects information transmission and through it the revenue and efficiency. Lemma 2 tells us that it does not if one restricts attention to static auctions with the same allocation rule and rent to the lowest type. For example, one cannot expect to generate a better information transmission or higher revenue by switching from the second-price auction to the first-price auction or all-pay auction.

Lemma 2 provides a useful analytic tool. The second-price auctions are easier to analyze as they allow for a simple bidding equilibrium in weakly dominant strategies. At the same time, the equilibrium of the first-price auction with discrete types of bidders requires mixing by bidders. As we will see next, discrete types naturally arise in the communication between the bidder and the advisor.

Lemma 2 states that to characterize equilibria of a rich class of static auctions and compare their efficiency and revenue to dynamic auctions, one can simply analyze equilibria of the second-price auction. The next theorem uses this approach to characterize the information transmission in all efficient direct mechanisms

Theorem 1. *Suppose* $\overline{v} < \infty$*. The communication strategy in any conditionally efficient equilibrium of any static auction is characterized as follows. There exists a positive integer* $\overline{K}$ *such that for all* $1 \leq K \leq \overline{K}$*, there exists an equilibrium in which types of advisor* $v \in [\omega_{k-1}, \omega_k)$ *induce the same action of the bidder and signal to the bidder that bidder's value is equal to* $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$*. Thresholds* $(\omega_k)_{k=1}^K$ *are determined as follows:*[9]

$$G(\omega_{k-1}, \omega_k)(1 - \Lambda_k)(\omega_k + b - m_k) + G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}) = 0. \tag{5}$$

where

$$\Lambda_k = \frac{1}{G(\omega_{k-1}, \omega_k)} \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{1}{n+1}.$$

[9]Here and further, when a random variable v is distributed according to F, we use a short-hand notation $F(a, b)$ for $\mathbb{P}(v \in [a, b]) = F(b) - F(a)$.

Theorem 1 shows that in static auctions, the misalignment of interests on the bidder's side results into a coarsening of the information transmitted from the advisor to the bidder. In particular, this implies that with positive probability the object is allocated inefficiently when $b \neq 0$. Theorem 1 is a counter-part of Theorem 1 in Crawford and Sobel (1982). However, our result does not follow from their result. In our game, the cheap-talk game is endogenous. Each bidder and advisor play a cheap-talk game in which actions are bids. The attractiveness of each bid for the bidder and advisor is endogenous and depends on how opponents bid in the auction. The bidding behavior of opponents depends on the information communicated between opponent bidders and their advisors. Theorem 1 shows that main insights from the cheap-talk literature are still true even when the cheap-talk game is endogenously determined in equilibrium by the communication between opponents and their advisors.

Equation (5) reflects the incentive of threshold types ω_k in the second-price auction. Notice that Λ_k is the expected probability of winning a tie when the bidder submits bid m_k. Type ω_k is indifferent between sending message m_k and m_{k+1}. In the second-price auction, the bidder pays the second highest bid. Therefore, strategies bring different payoffs only when the bidder faces a highest opponent of type in the interval $[\omega_{k-1}, \omega_k)$ or in the interval $[\omega_k, \omega_{k+1})$. The first term in equation (5) represents the benefit from submitting a higher bid. A higher bid m_{k+1} increases the probability of winning a tie from Λ_k to 1. The second term in equation (5) is the cost associated with a higher bid. Sending message m_{k+1}, the advisor risks winning the auction at price m_{k+1}. Since the costs and the benefits, should be equalized for threshold types, the advisor with type ω_k prefers not to buy at a higher price m_{k+1}.

It will be useful to derive the necessary condition for informative communication in static auctions.

Corollary 1. *A necessary condition for a non-babbling equilibrium is*

$$b \leq \mathbb{E}v - \underline{v}.$$

It is also sufficient when $N = 2$.

Chen, Kartik, and Sobel (2008) shows that in the standard cheap-talk model, NITS always exists and selects equilibria that are sufficiently informative (induce a high number of actions). In particular, under some conditions, NITS selects the most informative equilibrium of the cheap-talk. We next verify that this result is also true in our model.

Theorem 2. *The most informative equilibrium of the second-price auction satisfies the NITS condition.*

The proof of Lemma 2 adapts the argument in Chen et al. (2008) showing that if there is an equilibrium in the cheap-talk game with K actions induced in equilibrium that fails to satisfy NITS, then there is also an equilibrium with $K+1$ induced actions. Again we cannot apply their result directly, as the cheap-talk game between the bidder and the advisor is endogenous. Their result implies that in our model for a fixed equilibrium, we can construct a different cheap-talk equilibrium for one bidder and her advisor that is more informative. However, this need not be an equilibrium of the model, as once we change the cheap-talk equilibrium of bidders, this changes the cheap-talk game played and hence, this will not be an equilibrium of the model.

4 Ascending-Bid Auction

In this section, we characterize PBEMs of the ascending bid (English) auction satisfying the NITS condition when advisors are biased toward overbidding. Equilibirum communication in the English auction takes the following form: types below some v^* completely separate over time, while types above v^* pool and induce the same bid. As a result, the English auction induces better information transmission than any static auction. We show that the English auction is preferred to any static auction both in terms of efficiency and revenue.

4.1 Characterization

This subsection shows that all PBEMs of the English auction satisfying NITS are in delegation strategies defined as follows.

Definition 4. *Strategies of players are* delegation strategies *if for some v^*:*

- *the advisor sends message "quit" when the running price equals $v + b$;*
- *the bidder quits if either the running price is above $v^* + b$ or she receives message "quit".*

If the advisor is in control of bids, then it is a weakly dominant strategy for her to quit when $p = v + b$. Hence, in delegation strategies, the bidder essentially delegates bidding to the advisor with the restriction that the advisor quits before the running price exceeds $v^* + b$.

An equilibrium in delegation strategies always exists. The advisor induces the bidder to quit either at her optimal price $v+b$ if $v \leq v^*$ or at price v^*+b if $v > v^*$, which is still better than quitting at any price below $v^* + b$. Hence, the communication strategy is optimal. On the other hand, message "quit" at price p implies that the bidder's value is $p-b < p$ and the bidder prefers to quit immediately and get utility zero, rather than wait longer and face the risk of winning the auction at a price that exceeds her value. Finally, the cutoff v^* can be chosen so that at stage $p^* = v^* + b$ the option value to the bidder of staying in the auction and waiting for the advisor's recommendation hits zero for the first time.

When players use delegation strategies, the full revelation below v^* is possible because of the dynamic nature of the English auction. The advisor reveals the value to the bidder only when the running price equals her optimal quitting price. Because of the overbidding bias, the bidder gets negative utility if she wins at the current or any future running price. Therefore, the bidder prefers to quit immediately after getting the recommendation to quit. This simple mechanism ensures perfect communication for types below v^*.

However, there are also other PBEMs in the English auction. In particular, for any equilibrium of the second price auction, there exists an outcome-equivalent PBEM equilibrium of the English auction. To construct such equilibrium, we simply specify that types in $[\omega_{k-1}, \omega_k)$ that send message m_k in the second-price auction, in the English auction, send the same message in the beginning of the auction from which the bidder infers her expected value m_k and quits when the running price reaches m_k. There are also more complicated equilibria in which the advisor reveals crude information about its type over time. In this case, intervals of types that pool with each other will depend on the number of remaining bidders.[10] The next theorem is the main characterization result and it shows that the NITS condition rules out these equilibria.

Theorem 3. *Suppose $b > 0$ and $\overline{v} \leq \infty$. Any PBEM in the English auction that satisfies the NITS condition is in delegation strategies with cutoff v^* characterized as follows:*

1. *v^* satisfies*
$$v^* + b = \mathbb{E}[v|v \geq v^*] \tag{6}$$
when $\underline{v} < v^* < \infty$, and $v^* + b \geq \mathbb{E}[v|v \geq v^*]$ when $v^* = \underline{v}$.

2. *Let $v_0^* = \underline{v}$, $v_{K+1}^* = \overline{v}$ and $v_1^* < \ldots v_k^* < \cdots < v_K^*$ be all solutions to equation (6). Then v^*equals to some $v_k^* \in \{v_0^*, \ldots, v_{K+1}^*\}$ such that for all v_j^*, $j < k$, and all $n = 1, \ldots, N-1$:*
$$\int_{v_j^*}^{v_k^*} (1 - F(s))(\mathbb{E}[v|v > s] - s - b)dG_n(s) \geq 0, \tag{7}$$
where G_n is the distribution of the maximum of n random variables independently, identically distributed according to F.

Theorem 3 shows that equilibria of the English auction are quite different from equilibria in static auctions: types at the bottom perfectly reveals themselves over time, while types at the top pool with each other. The NITS condition effective rules out partition equilibria corresponding to equilibria of the second price auction. To see this, consider an equilibrium described in Theorem 1. In the beginning of the game, type ω_k is indifferent between m_k and m_{k+1}. By the recursion (5), $\omega_k + b - m_{k+1} < 0$ and so, when price exceeds m_k, type ω_k strictly prefers to separate from other types contradicting NITS. The intuition is that in the beginning of the game type ω_k is willing to submit price m_{k+1} as it increases her probability of winning against types in $[\omega_{k-1}, \omega_k)$, despite the risk of winning at a higher price m_{k+1}. However, after the running price exceeds m_k, the benefits of submitting higher bid disappear, and only costs remain. Hence, at this stage, type ω_k would prefer to reveal herself and this way induce the bidder to quit immediately and avoid winning the auction which contradicts the NITS condition.

Conditions (6) and (7) on v^* reflect the option value to the bidder of following the advisor's recommendation. The bidder waits for the recommendation as long as this option value is positive. This option value can be calculated as follows. Suppose that the current running price is p, the lowest remaining type is v_p and there are n other bidders remaining in the auction. The bidder wins the auction if for some running price $s + b > p$, it holds $v > s = \hat{v}$. Then her expected payoff is $\mathbb{E}[v|v > s] - s - b$. Integrating over all s, we get that the option value to the bidder is equal to

$$\frac{1}{(1 - F(v_p))^n F(v_p)^{N-1-n}} \int_{v_p}^{v^*} (1 - F(s))(\mathbb{E}[v|v > s] - s - b)dG_n(s) \geq 0. \tag{8}$$

[10]The construction of such equilibria is very elaborate and we do not get into details as this turns out to be unnecessary under the NITS condition.

Condition (6) ensures that the bidder does not want to stop listening to the advisor slightly earlier or later. If $v^* + b < \mathbb{E}[v|v \geq v^*]$, then the bidder would prefer to quit slightly later, while she would prefer to quit slightly earlier if $v^* + b < \mathbb{E}[v|v \geq v^*]$. Condition (7) ensures that the option value stays positive up until $p = v^* + b$.

For many commonly-used parametric families of distributions this conditions on v^* can be further simplified. Introduce the *mean residual lifetime* function

$$MRL(s) = \mathbb{E}[v|v \geq s] - s,$$

which is well studied in statistics (see Bagnoli and Bergstrom (2005)). Many commonly-used distributions have monotone MRL. Function MRL is decreasing for such distribution as normal, logistic, extreme value, Weibull, gamma, power distribution with power greater than one, as well as their truncations from above or below.[11] For Pareto and log-normal distribution truncated from below at 1, MRL increasing.[12] For exponential distribution, MRL is constant.

We next characterize v^* for distributions with monotone MRL that cover most of the commonly-used distributions. After that we will return to the general characterization in Theorem 3 to discuss how equilibria look like for general distributions. We have already considered the exponential distribution in Section 2 which is a knife-edge case between increasing and decreasing MRL. The next corollary covers distributions with decreasing MRL.

Corollary 2. *Suppose that $\overline{v} \leq \infty$ and MRL is decreasing. Then for all $b > 0$ except $b = \mathbb{E}v - \underline{v}$, the unique PBEM of the English auction satisfying the NITS condition is in delegation strategies. The equilibrium is informative if and only if $\mathbb{E}v \geq \underline{v} + b$.*

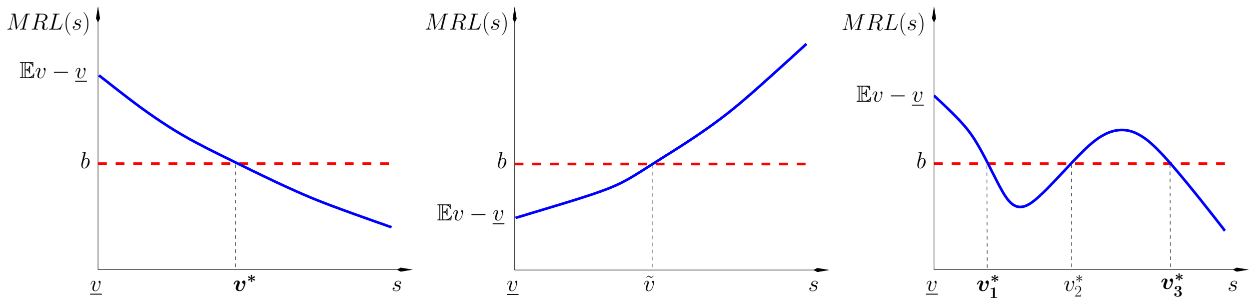
Figure 3*a* illustrates Corollary 2. When MRL is decreasing, there is v^* solving (6) if and only if $\mathbb{E}v - \underline{v}$, and it is unique whenever it exist. Since MRL crosses b from above, $\mathbb{E}[v|v \geq s] - s - b$ is positive for all $s < v^*$ and so, the option value (8) is positive for all $v_p < v^*$.

A new feature that was not present in the exponential example is that there is always pooling at the top, i.e. $v^* < \overline{v}$.[13] The bidder does not wait until all types of the advisor reveal themselves and at some point quits the auction before learning perfectly her value. Over the course of the auction, the bidder learns information about her value even if the advisor does not send any messages. The fact that there was no message so far indicates that her value cannot be lower than the running price minus bias b. When $\overline{v}$ is finite, after a certain time, the bidder knows that the value is close to $\overline{v}$. If she wins the auction, she will pay a price close to $\overline{v} + b$ and hence, it is very likely that she will overpay for the good. As a result, the bidder prefers to quit earlier and there is an interval of values at the top that she never learns.

[11] Bagnoli and Bergstrom (2005) shows that log-concavity of density f or log-concavity of reliability function $1 - F$, which are preserved by truncations, are sufficient for a weakly decreasing MRL.

[12] An (1998) shows that log-convexity of the density is sufficient for a weakly increasing MRL.

[13] Hence, if we consider a model with an exponential distribution truncated at the top at some $\overline{v}$ and let $\overline{v}$ go to infinity, then in equilibria of the English auction, there will be pooling at the top for each finite $\overline{v}$, but v^* will go to infinity as $\overline{v} \to \infty$ and in the limit, pooling will be degenerate and happen with probability zero.



(a) *decreasing* MRL (b) *increasing* MRL (c) *non-monotone* MRL

Figure 3: Graph of function MRL.

Condition in Corollary 2 for informative equilibria is the same as the necessary condition for informative equilibria for static auctions (see Corollary 1). In particular, when $N = 2$, there is an informative equilibrium in the English auction if and only if there is an informative equilibrium in the static auction. Moreover, as b increases, the set of types that pool in equilibrium increases and equilibria become less informative. This comparative statics is different for distributions with increasing MRL as the next corollary shows.

Corollary 3. *Suppose $b > 0$, MRL is increasing on $[\underline{v}, \infty)$, and let $\overline{b}$ be the largest b for which*

$$\int_{\underline{v}}^{\infty} (1 - F(s))(\mathbb{E}[v|v > s] - s - b)dG_n(s) \geq 0, \tag{9}$$

for all $n = 1, \ldots, N-1$. Then all PBEM of the English auction satisfying the NITS condition are in delegation strategies with v^* characterized as follows:

- *$v^* = \infty$ when $b \in [0, \mathbb{E}v - \underline{v})$;*
- *$v^* = \underline{v}$ or $v^* = \infty$ when $b \in [\mathbb{E}v - \underline{v}, \overline{b}]$;*
- *$v^* = \underline{v}$ when $b \in (\overline{b}, \infty)$.*

Figure 3*b* illustrates Corollary 2. When MRL is increasing, there is at most one solution $\tilde{v}$ to (6), however, it cannot be an equilibrium cutoff v^*. The reason is that since $\mathbb{E}[v|v \geq s] - s$ crosses b from below at $\tilde{v}$, if $v^* = \tilde{v}$ the option value to the bidder of waiting for advisor's recommendation is negative for $v < \tilde{v}$. Therefore, the equilibrium is either fully separating ($v^* = \infty$) or babbling ($v^* = 0$).[14] Intuitively, in the beginning of the auction, winning is a bad news, as the bidder gets negative utility if she wins. As the auction continues, eventually, the bidder gets positive utility from winning, as $\mathbb{E}[v|v \geq v_p]$ increases faster than the price $v_p + b$ that the bidder pays in case she wins. The bidder is willing to follow the advice of the bidder if the benefits of winning later in the auction outweigh the risk of winning early in the auction. Hence, the condition on b: when b is sufficiently low, there is a fully informative equilibrium. As b increases, at some point, the babbling equilibrium is possible, and a for

[14] Observe that it is necessary for MRL to be increasing that $\overline{v} = \infty$. Indeed, if $\overline{v} < \infty$, then $\lim_{s \to \overline{v}} \mathbb{E}[v|v > s] - s = 0 < \mathbb{E}v - \underline{v}$.

sufficiently high b, babbling equilibrium is the only equilibrium. Notice that the babbling equilibrium is an equilibrium only for sufficiently large b, for which the lowest type $\underline{v}$ gets positive utility $\frac{1}{N}(\underline{v} + b - \mathbb{E}v)$ from winning and so, the NITS condition is satisfied.

Corollary 3 allows the existence of non-babbling equilibria even when $b > \mathbb{E}v - \underline{v}$ and all equilibia of the static auctions are babbling as long as (9) holds. To give a concrete example, suppose $N = 2$ and $F(v) = 1 - \left(\frac{1}{v}\right)^2$ is a Pareto distribution on $[1, \infty)$. We have $\mathbb{E}[v] - \underline{v} = 1$ and so all equilibria in static auctions are babbling whenever $b > 1$. We can compute (9) as follows[15]

$$\int_1^\infty \frac{2\,(\hat{v} - b)}{\hat{v}^5} d\hat{v} = 2\int_1^\infty d\left(\frac{1}{\hat{v}^4} - \frac{b}{\hat{v}^5}\right) = 2\int_1^\infty d\left(\frac{b}{4\hat{v}^4} - \frac{1}{3\hat{v}^3}\right) = \frac{4-3b}{6},$$

which is positive whenever $b < \frac{4}{3}$. Hence, for $b \in \left(1, \frac{4}{3}\right)$ there exists an informative equilibrium, even though all equilibria of static auctions are babbling. The reason is that the term $\mathbb{E}[v|v > v^*] = 2v^*$ grows faster than $v^* + b$ so that for $b < \frac{4}{3}$, (9) holds.

Let us now return to the equilibria of the English auction for general distributions characterized in Theorem 3. For decreasing MRL, there is a unique candidate for v^* corresponding to the unique solution to (6), while for increasing MRL, there are two candidates $v^* = \underline{v}$ and $v^* = \overline{v}$. For general distributions, there can be multiple solutions to equation (6) which are, together with $\underline{v}$ and $\overline{v}$ (when $\overline{v} = \infty$), are candidates for v^*.

Condition (7) ensures that the option value to the bidder of following advisor's recommendation given by (8) is positive for all $v_p < v^*$. The integral (8) can be split into several integrals with limits of integration given by $(v_k^*)_{k=0}^{K+1}$. Since (8) should hold for every v_p up to v^*, only solutions to (6), in which MRL crosses b from above are possible candidates for the equilibrium cutoff. Moreover, the option value (8) is the smallest at the solutions to (6) where MRL crosses b from below.

As an illustration, consider general function MRL depicted in Figure 3c. There are three solutions v_1^*, v_2^*, and v_3^* to equation (6). By Theorem 3, there can be at most four equilibria satisfying NITS in this situation. First, v_0^* is not an equilibrium cutoff, as it fails the NITS ($\mathbb{E}v > \underline{v} + b$), and $v_4^* = \overline{v}$ is not an equilibrium cutoff, as MRL is below b at $\overline{v}$. Neither is v_2^*, as MRL crosses b from below at v_2^*. Hence, only candidates for the equilibrium cutoff are v_1^* and v_3^*. There is an equilibrium with cutoff v_1^*, as for any $v_p \le v_1^*$, the integrand in (8) is positive. There is an equilibrium with cutoff v_3^* if and only if the integral (8) for $v_p = v_2^*$ is positive.

The characterization in Theorem 3 can be extended in several directions. First, it is more realistic to assume that the seller instead of knowing bias b, has some prior beliefs F_b about b supported on $\mathbb{R}_+$. That is the seller knows that there is a conflict of interest on the bidder's side, but does not know the scope of this conflict. Our characterization of the English auction still obtain in this environment, as long as bidders and advisors know b. Importantly, the efficiency and revenue comparison that we carry out in the next subsections also hold in this case. Hence, the seller benefits from switching to the English auction from a static auction, and this does not require knowledge of fine details about the conflict of interest and only the direction of the bias.

[15] Observe that for Pareto distribution $f(v) = \frac{2}{v^3}$ and $\mathbb{E}[v|v > \hat{v}] = \hat{v}^2 \int_{\hat{v}}^\infty \frac{2}{v^2} dv = 2\hat{v}$.

Comparative Statics The characterization of equilibria in the English auction allows us to derive the following comparative statics.

Proposition 3. *Suppose MRL is strictly decreasing. Then the following hold.*

1. *The profit in the English auction is increasing in b in the neighborhood of $b = 0$ and decreasing in b in the neighborhood of $b = \overline{b}$ where $\overline{b} \equiv \mathbb{E}[v] - \underline{v}$.*

2. *For any bias $b > 0$, there exists $N(b)$ and $\varepsilon(b) > 0$ such that for all $N > N(b)$, the seller prefers bias $b - \varepsilon(b)$ to bias b.*

3. *Equilibrium strategies do not depend on N.*

The first statement of Proposition 3 shows that the seller prefers moderate values of the advisor's bias. In particular, for low values of the bias, the seller prefers to increase the bias, while for sufficiently high values of the bias the seller prefers to lower the bias. As an illustration, consider the case of two bidders and values distributed uniformly on $[0, 1]$. The profit of the seller given by

$$\mathbb{E}[\min\{\hat{v}, v^*\}] + b = v^*(1 - \tfrac{v^*}{2}) + b.$$

There are two contrary forces that affect the profit as b increases. On the one hand, as the bias increases, the distribution of advisors' values increases in the sense of first-order stochastic dominance, and hence, the profit of the seller is increased. On the other hand, the bias affects communication as it lowers v^*. Under the strictly decreasing MRL there is a decreasing relationship between b and v^*given by (6). The more biased the advisor, the more pooling at the top happens in equilibrium. In the uniform case, $v^* = 1 - 2b$ and when $b \geq \frac{1}{2}$ there is no information transmission in equilibrium. Using the relationship between b and v^* we can rewrite profit as $\frac{1}{2}(1 + v^* - v^{*2})$ which has an inverse-U-shaped and attains maximum when $v^* = \frac{1}{2}$ or $b = \frac{1}{4}$. Intuitively, initially the seller benefits from the increase in the bias, as it shift the distribution of advisors' values to the right. However, as the bias increases, the cost of reduced communication eventually outweighs the benefit from a first-order stochastic shift of advisors' values.

By the second statement of Proposition 3, as the environment becomes more competitive, the seller prefers a lower bias of advisors. This is quite intuitive and can be again seen clearly in the uniform example. When there are N bidders, then $\mathbb{E}[\min\{\hat{v}, v^*\}] = v^*(1 - \frac{v^{*N-1}}{N})$ and so, the seller's expected profit equals $\frac{1+v^*}{2} - \frac{v^{*N}}{N}$. As N increases, it becomes more likely that the second-order statistics of $N - 1$ draws from F is above v^*. However, pooling above v^* does not allow the seller to fully benefit from such a shift in the distribution of $\hat{v}$. Therefore, with the increase in N, the seller prefers lower biases as their allow for a finer discrimination of higher types.

It is interesting to observe how the number of bidders affects communication and efficiency of the English. First, when MRL is decreasing the number of bidders does not affect the communication in any way. Perhaps surprisingly, in a general case, increased competition reduces the scope of information transmission. Equation (6) for v^* does not depend on N, but condition (7) becomes more stringent as N increases, as it needs to hold for a larger set of ns. The reduction in the communication happens because the value of the advice for the bidder can depend on how competitive the auction is. Moreover, this dependence can be non-monotone. When n is larger, the value of the highest opponent bidder is higher in the sense of first-order stochastic dominance of distribution G_n. However, the integrand in (7) need not be a monotone function, and so, the option can both increase and decrease as n increases.

4.2 Efficiency Comparison

We next compare the efficiency of auction formats. In the exponential example, the efficiency comparison was clear, as the English auction always allocated the asset to the bidder with the highest value. Theorem 3 shows that in general, there can be pooling at the top which distorts the efficiency. This leads to a loss of efficiency and depending on the size of this pooling region, it is possible that the equilibrium of the static auction is more efficient if it generates a more efficient outcomes for types above v^*. The next theorem shows that this is not the case. More strongly, the superior efficiency stems from a superior information transmission. We say, that an equilibrium of an auction is *more informative* than an equilibrium of potentially different auction, if the partition of advisor types generated by the former is finer than the partition generated by the latter.

Theorem 4. *Suppose $b > 0$. Then any PBEM of the English auction satisfying NITS is more informative and more efficient than any conditionally efficient equilibrium of any static auction.*

Theorem 4 shows that there is no partition generated by a static auction that is finer than the partition generated by the English auction. That is, there is no $\omega_k > v^*$ where ω_k and v^* are as in Theorems 1 and 3. This implies that the English auction generates a finer information partition than any static auction and hence, is more efficient.

The argument for Theorem 4 can be sketched as follows. Suppose that there exists an equilibrium of the static auction such that $\omega_{k-1} = v^* < \omega_k < \omega_{k+1} = \overline{v}$. For simplicity, also assume that $N = 2$. Then equation (5) implies that

$$\frac{1}{2}F(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) = -\frac{1}{2}F(\omega_k, \omega_{k+1})(\omega_k + b - m_{k+1}),$$

or

$$\omega_k + b = \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})}m_k + \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}m_{k+1} = \mathbb{E}[v|v \geq v^*].$$

However, this contradicts the fact that $v^* < \omega_k$ solves (6). Intuitively, if there were a variation in bids among types above v^*, any type above v^* would strictly prefer to submit a higher bid and increase her chances of winning against types below. This happens because v^* is already sufficiently close to $\overline{v}$ and price m_{k+1} does not vary much from price m_k.

Theorem 4 also sheds light on the communication in static auctions. In static auctions, the dependence of the communication on N is more convoluted. The number of bidders N enters recursion (14) in a complicated way and from it, it is not clear how N affects the communication partition. However, from Theorem 4, the communication partition in the English auction is finer than the partition generated by any static auction. This implies that the communication in static auctions does not become perfect as we increase the competitiveness of the auction, which is a priori not obvious from recursion (14).

Finally, for distributions with increasing MRL we can go beyond the comparison with static auctions.

Corollary 4. *Suppose* $0 < b < \mathbb{E}v - \underline{v}$ *and* MRL *is increasing on* $[\underline{v}, \infty)$*. Then the unique PBEM of the English auction satisfying NITS is fully efficient.*

Corollary 4 solves the problem of efficient mechanism design for distributions with increasing MRL and moderate bias b. It shows that despite the conflict of interest, an efficient outcome is implementable as a unique outcome via the English auction. We conjecture that this holds more generally for any distribution. In order to show this one needs to show that no other auction format either static or dynamic can attain a finer communication for types above v^*. Theorem 4 guarantees that this is the case when the comparison is with efficient equilbiria of static auctions.

4.3 Revenue Comparison

We next compare the revenue from different auction formats. Denote by $\varphi(v) \equiv v + b - \frac{1-F(v)}{f(v)}$ the virtual valuation of advisor.

Theorem 5. *Suppose* $b > 0$ *and* φ *is increasing. Then any PBEM of the English auction satisfying NITS brings higher revenue than any conditionally efficient equilibrium of any static auction satisfying NITS.*

Proof. We can view the problem that the seller faces as an optimal mechanism design problem from informed advisors. The fact that bids are submitted not directly by advisors, but by bidders implies that there is a restriction on the set of mechanisms that the seller can implement. However, we can still use Lemma 3 in Myerson (1981) to write the expected revenue of the seller as follows:

$$N\left(\mathbb{E}[\varphi(v)p(v)] - U_A(0)\right), \tag{10}$$

where $p(v)$ is the probability that type v wins the auction and $U_A(0)$ is the expected utility of type 0 from the auction. In equation (10), only $p(\cdot)$ and $U_A(0)$ depend on the format of the auction. By Lemma 2, it is sufficient to compare the English auction with the second-price auction. By NITS, $U_A(0) \geq 0$ for the second-price auction, while $U_A(0) = 0$ for the English auction. To prove the comparison, we need to show that the first term in (10) is larger for the English auction. This is indeed the case, as φ is increasing and the English auction is more efficient by Theorem 2. □

The key insight of Theorem 5 is that we can view the problem of the seller of extracting maximal revenue as the a problem of designing a mechanism that extracts rents from informed advisors. In this case, the fact that there is a communication puts a restriction on the set of mechanisms that are the seller can implement. However, one can still use the envelope formula in Myerson (1981) to write the revenue in the form (10). The higher efficiency of the English auction implies that the first term in (10) is higher than in any static auction, while the NITS guarantees that the rent of the lowest type is positive in static auctions, while it is zero in the English auction.

While superior efficiency of the English auction because of the better information transmission is intuitive, it is a priori not clear if the English auction should also bring higher expected revenue. If types in some interval pool and induce the same bid in the second-price auction, then it can potentially increase the revenue of the seller. As we have already seen, at least some types above ω_k get negative utility from winning at price m_{k+1}, but with positive probability they end up winning the asset at this price. We show that despite this occasional overpaying, it does not occur often enough to reduce significantly the information rents of advisors. The key in ensuring this is the NITS condition. To see this, let us return to our exponential example and consider a babbling equilibrium in which all types pool and bidders submit bids $\frac{1}{\lambda}$. The revenue from such equilibrium of the second-price auction is $\mathbb{E}[v] = \frac{1}{\lambda}$. However, the equilibrium of the English auction brings revenue $\mathbb{E}[\min\{v_1, v_2\}] = \frac{1}{2\lambda} + b$ and so for $b < \frac{1}{2\lambda}$, the babbling equilibrium of the second-price auction brings higher revenue. In this case, a significant amount of low types make a bid that exceeds their value, as they cannot credibly transmit their value to the bidder. This way the seller extracts an extra revenue. However, for $b < \frac{1}{2\lambda}$, babbling equilibrium fails to satisfy the NITS conditions.

A natural next question is whether the revenue can be further improved. We know from Myerson (1981) that introducing the reservation price increases the revenue whenever the virtual valuation φ is negative for some types. Then introducing a reserve price $r = v_r + b$, where v_r is given by the solution to $\varphi(v_r) = 0$, increases further the revenue. By setting the reservation price at $v_r + b$, the seller does not allocate to types below v_r which contribute negatively into the expected profit (10). In our exponential example, $v_r = \frac{1}{\lambda} - b$ for $b < \frac{1}{\lambda}$. The knowledge of b is important to set the reservation price optimally as φ depends on b. Interestingly, in the family of distributions with increasing MRL we can go even further and find an optimal mechanism.

Corollary 5. *Suppose $0 < b < \mathbb{E}v - \underline{v}$ and MRL is increasing on $[\underline{v}, \infty)$. Then there exists a reservation price r such that the unique PBEM satisfying NITS of the English auction with a reservation price r is optimal.*

Corollary 5 follows from our characterization in Corollary 3. Indeed, Myerson (1981) shows that generally a second price auction with a reservation price is an optimal mechanism. In particular, it is an optimal mechanism from extracting rents from informed advisors and the question is whether the constraints imposed by the communication between bidders and advisors prevent us from implementing this outcome. Corollary 3 can be easily modified to allow for a reservation price by simply assuming that the seller start increasing price starting from r. Hence, for distributions with increasing MRL, the fact that bids are submitted by bidders does not prevent us from implementing an optimal outcome. We conjecture that this is true more generally which is related to the general efficiency of the English auction. If one shows that no mechanism induces finer partition of types above v^*, then this would imply that the English auction with the reservation price is an optimal mechanism for general distributions.

While the analysis of the English auction does not change with the introduction of the reserve price and one can easily compute an optimal reserve price, this is not the case in static auctions. Indeed, if the seller restricts bids in the second price auction to be above some r, then this affects the equilibrium communication. Essentially, after the introduction of the reserve price, the distribution of values is $F(\cdot|v \in [r, \overline{v}])$ and generally the partition of types generated in equilibrium changes, which in turn changes which types tie with each other in equilibrium. Hence, determining the revenue price is less straightforward in the static auction and requires more knowledge of the strategic environment from the seller, while only the knowledge of the distribution and b is necessary in the English auction.

When there are several competing buyers, the dynamic auction formats have advantages over static formats in the presence of the advisors' bias. This is not the case in negotiations where there is only one buyer. First, when there is only one buyer, it is efficient to allocate to this buyer by posting a price $\underline{v}$, as $\underline{v} \geq 0$. Second, under the increasing virtual valuation φ and $\mathbb{E}[v|v > \tilde{v}] - \tilde{v} \geq 0$, in the negotiation, it is optimal to post a price $\tilde{v}$ solving $\tilde{v} + b = \frac{1-F(\tilde{v})}{f(\tilde{v})}$. Indeed, such a price is optimal when the seller sells directly to the advisor. Moreover, if the advisor simply tells the bidder whether to buy or not at price $\tilde{v}$, under $\mathbb{E}[v|v > \tilde{v}] - \tilde{v} \geq 0$, it is optimal for the bidder to follow the advisor's recommendation. Intuitively, when there is only one buyer, a coarse information is sufficient to implement both the efficient and optimal allocations. Thus, there is no advantage of using dynamic auction formats. When there are several buyers, the seller needs to extract finer information about values to implement both efficient and optimal outcomes. Therefore, there is a benefit in using the dynamic auction formats, as they enable a better information transmission.

5 Bias toward Underbidding

Motivated by the empire-building and career concerns, we focused so far on the bias toward overbidding. This section considers the case $b < 0$ when advisors are biased toward underbidding that could be more relevant in procurement auctions.[16] As in the case of overbidding bias, dynamic auction formats can attain better information transmission and higher efficiency than static auctions, but unlike the case of overbidding bias, can lead to a lower revenue.

The analysis of static auctions is similar to the case of bias toward overbidding. In particular, the characterization of equilibria in Theorem 1 and the version of the revenue equivalence in Theorems 2 still hold. Thus, the communication strategy has a partition structure and one can focus on the second-price auction in the analysis of static conditionally efficient auctions. As an illustration, the next proposition describes a PBEM of the second-price auction for the exponential distribution.

Proposition 4. *The following strategies constitute an equilibrium of the second-price auction when F is exponential, $N = 2$, and $b < 0$. Let $x \in \left[-b, \frac{1}{\lambda} - b\right]$ be the solution to equation $x\frac{e^{2\lambda x}+1}{e^{2\lambda x}-1} = \frac{1}{\lambda} - b$ and $\omega_k = kx$ for $k = 1, 2, \ldots$. For all k, the advisor of type $v \in [\omega_{k-1}, \omega_k)$ sends message $m_k = \mathbb{E}[v|[\omega_{k-1}, \omega_k)]$ and the bidder submits a bid equal to the message received.*

[16] The bias toward underbidding is also relevant in takeover contests when the management has the "quiet life" preference (Bertrand and Mullainathan, 2003): incorporating additional business requires additional effort from managers and managers prefer not to increase the size of the firm.

We now turn to dynamic auctions. When $b < 0$, if the bidder knew the value, then she would submit a higher bid than the advisor. Then the English auction does not have an advantage over static auctions, as it only restricts the bidder to submit bids lower than the running price. However, now the Dutch auction can allow for a better information transmission, as it restricts the bidder from submitting bids above the running price. A relevant statistics of the distribution is the *mean-advantage-over-inferiors* MAI defined as

$$MAI(s) \equiv s - \mathbb{E}[v|v \leq s].$$

Most of the commonly used distributions have strictly increasing MAI.[17] The next theorem constructs a partially revealing PBEM of the Dutch auction.

Theorem 6. *Suppose $b < 0$ and MAI is strictly increasing. Let v^* be the solution to*

$$\mathbb{E}[v|v < v^*] = v^* + b, \tag{11}$$

where $v^* = \overline{v}$ if equation (11) does not have a solution. There exists PBEM of the Dutch auction characterized by $\{\sigma(\cdot), v^*\}$ as follows. The advisor of type $v > v^*$ sends message "stop" when the running price p reaches $\sigma(v) \equiv \mathbb{E}[\max\{\hat{v}, v^*\} + b|\hat{v} < v]$ and the advisor of type $v < v^*$ sends "stop" when the running price p reaches $\sigma(v^*)$. The bidder immediately stops the auction after she receives the message "stop" or when the running price p reaches $\sigma(v^*)$.

As with the overbidding bias, the equilibrium constructed in Theorem 6 is partially revealing and quite different from the partition equilibrium in static auctions. The partially perfect communication is possible because the Dutch auction restricts bidders not to increase the bids. Because of the underbidding bias of the advisor, the optimal price of stopping the auction for the bidder is higher than for the advisor. Thus, it is optimal for her to stop immediately after she gets a recommendation from the advisor. Function $\sigma(v)$ in Theorem 6 is the equilibrium bidding strategy in the Dutch auction if bids were submitted directly by advisors, and it is implemented for $v > v^*$. However, unlike the case $b > 0$, the pooling happens at the bottom of the distribution, not at the top. The reason for this is that at a certain stage, the uncertainty of the bidder about her value is sufficiently reduced. Then the bidder prefers stopping the auction immediately to guarantee the victory, rather than trying to win at a lower price, but facing the risk of loosing the auction.

We next turn to the comparison of auction formats.

Theorem 7. *Suppose $b < 0$. The PBEM of the Dutch auction in Theorem 6 is more efficient than any conditionally efficient equilibrium of any static auction.*

As in the case of overbidding bias, because of the superior information transmission, the Dutch auction is more efficient that any static auction. This is true despite that there is a pooling at the bottom in the Dutch auction. We show that generally the pooling region

[17] In particular, all distributions with a strictly concave log of c.d.f. have increasing MAI (see Bagnoli and Bergstrom (2005) for related results and a list of distributions with log-concave c.d.f.).

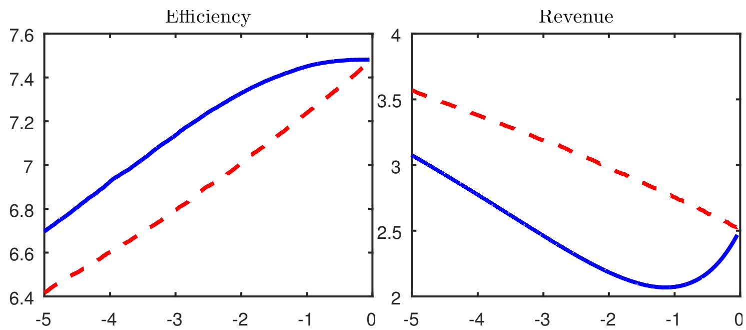


Figure 4: Efficiency and revenue comparison for $b < 0$: the Dutch auction (solid line) and the second-price auction (dashed line). Bias b is plotted on the horizontal axis.

below v^* is always smaller than any first element of the partition equilibrium in any static auction. As an illustration, in the exponential example, equation (11) can be simplified to

$$\frac{v^*}{1 - e^{-\lambda v^*}} = \frac{1}{\lambda} - b,$$

from which one can see that $v^* < x$ where x is defined in Proposition 4.[18] The left panel of Figure 4 depict the expected value of the winner in the two auction formats when $\lambda = \frac{1}{5}$. The Dutch auction is more efficient than the second-price auction and the gap in the efficiency increases as the size of the bias increases.

Unlike the case $b > 0$, the revenue comparison is ambiguous when the advisor is biased toward underbidding. We can proceed as in Theorem 5 to break down the expected revenue into two parts as in (10): the part increasing with the auction efficiency and the part decreasing with the rent of the lowest type. Because of the higher efficiency the first term in (10) is higher in the Dutch auction. However, the rent of the lowest type is lower in the second-price auction. Indeed, because the first element of the partition in the second-price auction is larger than the pooling region in the Dutch auction ($\omega_1 > v^*$), type 0 has a lower probability of winning in the Dutch auction and he also pays a lower price conditional on winning.[19] Therefore, the second-price auction may bring higher revenue than the Dutch auction, if the second term in (10) dominates the first term. In fact, in our exponential example, the revenue of the Dutch auction is lower for all $b < 0$ (see the right panel of Figure 4).

Finally, notice that the NITS condition does not play any role in the revenue comparison. When $b < 0$, if the bidder asked value v and thought that the advisor tells the truth, then all types of the advisor would bias their reports downwards and nobody would prefer to tell $\overline{v}$. Thus, when $b < 0$, the weakest type of advisor v_w^p is the highest type remaining in the game at stage p. With this change in the specification of the weakest type, the definition of the NITS condition for $b < 0$ is the same as in Definition 2. Since NITS puts restrictions

[18] One needs to show that $v\dfrac{e^{\lambda v}}{e^{\lambda v} - 1} > v\dfrac{e^{2\lambda v} + 1}{e^{2\lambda v} - 1}$, which indeed holds.

[19] The advisor of type $\underline{v}$ gets negative profit in both auctions, and pays $m_1 = \mathbb{E}[v|v < \omega_1]$ in the second-price auction, and $v^* + b = \mathbb{E}[v|v \leq v^*]$ in the Dutch auction. Since $\omega_1 > v^*$, the latter price is smaller.

on the utility of the highest type, while in the revenue comparison, the utility of the lowest type matters, the NITS condition does not play as important role as in the case of $b > 0$.

6 Quantitative Example: Auctions of Companies

In this section, we assess the quantitative implications of our analysis, applying the model to auctions of companies. The analysis of Section 4 proves that the ascending-bid auction dominates static mechanisms in both efficiency and revenues. However, these results do not imply that the difference is meaningful quantitatively. Suppose that each bidder i is a firm, consisting of the board of directors and the manager. The board has formal authority over submitting bids but has no information about firm's valuation of the target v_i, except for the prior distribution. The manager knows v_i, but has a bias $b > 0$ for overpayment.

To get a plausible value of b, we use the following argument. There is a strong empirical evidence that the compensation of CEO and other top executives is increasing in the absolute size of the firm. This dependence leads to their bias for overpaying for the target. On the other hand, overpaying for the target results in the destruction of firm value and ultimately in a poor performance of the acquirer's stock price. Since the wealth of top managers is sensititve to their company's stock price, there is a limit to which they are willing to overpay for the target. Bias b is the point at which the positive effect on compensation of higher firm size exactly is exactly offset by the negative effect on compensation due to firm value destruction. To get the estimate of b, we use CEO compensation regressions from Harford and Li (2007) and the characteristics of the typical deal from Betton, Eckbo, and Thorburn (2008). Since the market leverage ratio of the median target is 13.1% and the median ratio of the deal size to the acquirer's assets is 31%, the ratio of the deal size to the acquirer's equity for a typical deal is $31\% \times \frac{1}{0.869} = 35.67\%$. Since the median acquisition premium is 39%, the ratio of the pre-deal target's equity to the acquirer's equity is $35.67\% \times \frac{1}{1.39} = 25.66\%$. Assume that after the acquisition, the sales of the combined company increase by the same amount, i.e., by 25.66% in perpetuity. Using the estimate of Harford and Li (2007), this increase in sales leads to an increase in the acquirer's CEO compensation by $0.435 \times \log(1 + 0.2566) = 4.32\%$ every year. In addition, acquiring the target is associated with an increase in the CEO compensation of 3.7%, irrespectively of the increase in sales, in the year of the acquisition. Thus, the positive effect of acquiring the target on CEO compensation for a typical deal is 8.02% in the year of the deal and 4.32% in eveyr subsequent year. Using the expected tenure of 6 years and discounting at 10%, the present value of the positive effect is 22.52% of the CEO's annual compensation. On the other hand, overbidding by b (normalizing the pre-acquisition value of the target's equity to one), reduces the acquirer's equity value by $b \times 0.2566$. Since the porftolio value of equity incentives is 9.5 times the CEO annual pay (Table II in Harford and Li, 2007), the negative effect on the CEO wealth is $9.5 \times b \times 25.66\%$ of the CEO's annual compensation. The estimate of overpayment bias b is thus $b = \frac{0.2252}{9.5 \times 0.2566} = 9.2\%$.[20] For example, if the value of target under its current ownership if \$1billion and the true value of the target to the

[20]This value is likely an underestimate, since it ignores non-financial benefits of the acquirer's CEO, such as the preference for power and empire-building, and since the sales of the combined firm may exceed the sum of the individual companies' sales due to synergies.

	Ascending	Static	Ratio
Exp. Revenues	1.49	1.21	1.23
Exp. Valuation of Winner	1.65	1.57	1.05
Exp. Payoff of Bidder	0.04	0.16	0.25

Table 1

acquirer is $1.4 billion, the maximum willingness to pay for the target by the CEO is $1.492 billion.

To get a plausible distribution of valuations, we use the estimates from Gorbenko and Malenko (2014). We normalize the value of the target under its current management to one. Using data on bids and assuming lognormal distribution, Gorbenko and Malenko (2014) estimate that the valuations of strategic bidders are distributed with parameters $\mu = 0.167$ and $\sigma = 0.258$. We use this distribution, truncated at one, for the distribution of valuations in our numerical example. We assume that there are $N = 4$ bidders.

The results are presented in Table 1. Firs, consider the ascending-bid auction. The unique equilibrium satisfying the NITS criterion is that each bidder increases her bid until her advisor recommends to stop doing so. In other words, the estimated value of b is low enough to imply informative communication. Second, consider static auctions (for concreteness, the second-price auction). The most informative equilibrium in this case consists of three partitions, $[1, 1.12]$, $[1.12, 1.38]$, and $[1.38, \infty]$. The corresponding expected valuations are 1.06, 1.24, and 1.64. In the second-price auction, each advisor communicates that the valuation is in one of the three partitions, and the bidder submits one of the three expected values. The comparison of expected revenues is striking. The expected takeover premium is 49% in the ascending-bid auction, which is 23% higher than the expected takeover premium in static auctions (21%). The comparison of efficiency is less striking, but the difference is also sizable: The expected valuation of the winning bidder is 1.65 in the ascending-bid auction, but 1.57 in the static auction. As the comparison of expected bidders' payoffs illustrates, an increase in revenues largely occurs because of the more aggressive bidding among bidders. Overall, we conclude that the result that the ascending-bid auction is more efficient and brings more revenues than static auctions is quantitatively very large, at least for the application of auctions of companies.

7 Conclusion

This paper studies the interaction between the information transmission and bidding in auctions. In static auctions, the revenue-equivalence result holds giving in particular equivalence of the first- and second- price auctions. However, dynamic auctions, such as the English and the Dutch auctions, are generally more efficient than static auctions. This happens because in dynamic auctions the set of bids available to the bidder shrinks. Therefore, by sending the information later in the game, the advisor can induce the bidder to choose a more favorable action and hence, would provide a more refined information to the bidder. Moreover, the English auction also dominates static auctions in terms of revenue when advisors are biased toward overbidding, the case most relevant empirically. This paper characterize equilibria in different auction formats and shows the efficiency/revenue comparison.

A Appendix

Proofs for Section 2

Proof of Lemma 1. Specify new online strategies m' and a' as follows. Let $m'(v,h) = a(h, m(v,h))$ and $a'(h,\tilde{a}) = \tilde{a}$ for all $h \in \mathcal{H}$ and all $\tilde{a}$ in the image of $m'(\cdot,h)$. For any h, fix an action $\tilde{a}(h)$ in the image of $m'(\cdot,h)$. For any recommendation that does not belong to the image of $m'(\cdot,h)$, the bidder interprets this deviation as a recommendation of action $\tilde{a}(h)$. Hence, it is sufficient to guarantee that advisors do not deviate to on-path recommendations. Clearly, strategy profiles m' and a' generate the same outcome. The proof that they constitute an equilibrium is provided in the text. □

Proof of Proposition 1. □

Theorem 1 shows generally that advisor's strategy in the static auction takes a partition form as described in the proposition. Here, we simply derive this strategy. Given the exponential assumption, we can compute messages m_k explicitly as functions of thresholds ω_{k-1} and ω_k:

$$m_k = \mathbb{E}\left[v|v \in [\omega_{k-1}, \omega_k)\right] = \frac{1}{\lambda} + \frac{\omega_{k-1}e^{-\lambda\omega_{k-1}} - \omega_k e^{-\lambda\omega_k}}{e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_k}}. \tag{12}$$

Threshold types ω_k of advisor must be indifferent between sending messages m_k and m_{k+1}. This gives us

$$\frac{1}{2}\mathbb{P}(v \in [\omega_{k-1}, \omega_k))(\omega_k + b - m_k) = -\frac{1}{2}\mathbb{P}(v \in [\omega_k, \omega_{k+1}))(\omega_k + b - m_{k+1}), \tag{13}$$

which implies

$$-\left(e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_k}\right)(\omega_k + b - m_k) = \left(e^{-\lambda\omega_k} - e^{-\lambda\omega_{k+1}}\right)(\omega_k + b - m_{k+1}).$$

Combining this equation with (12) yields the recursive equation (3), where the initial value condition is $\omega_0 = 0$ and ω_k must be an increasing sequence. When $b > \frac{1}{\lambda}$, there is no solution to this recursion, which implies that the unique equilibrium is babbling: $m = \mathbb{E}[v] = \frac{1}{\lambda}\ \forall v$. Consider $b \leq \frac{1}{\lambda}$. Denoting by $x_k \equiv \omega_k - \omega_{k-1}$, we can rewrite the recursion (3) in terms of x_{k+1} and x_k as

$$x_{k+1} + x_k = \left(\frac{1}{\lambda} - b - x_k\right)\left(e^{\lambda(x_k + x_{k+1})} - 1\right). \tag{14}$$

The requirement that ω_k is increasing translates into x_k being positive. Define function ψ to be $x_{k+1} = \psi(x_k)$, such that x_k and x_{k+1} satisfy recursion (14). The following auxilliary claim will be helpful for proving the proposition:

Claim 1. ψ is well defined. For $x \in \left(0, \frac{1}{\lambda} - b\right]$, it holds $\psi'(x) \geq 1$, $\lim_{x\to 0}\psi(x) > 0$, and $\lim_{x\to\frac{1}{\lambda}-b}\psi(x) = \infty$.

Proof. **Proof:** First, we show that φ is well-defined. Since $x_k > 0$, we car rewrite (14) as

$$\frac{x_{k+1} + x_k}{e^{\lambda(x_{k+1}+x_k)} - 1} = \frac{1}{\lambda} - b - x_k.$$

Let $h(x)$ be the function implicitly defined by the solution to the equation

$$\frac{h}{e^{\lambda h}-1}=\frac{1}{\lambda}-b-x. \tag{15}$$

The left-hand side of (15) is decreasing in h and it takes values in $\left(0,\frac{1}{\lambda}\right]$ for $h\geq 0$. Therefore, the solution $h(x)$ always exists whenever $x\in\left(0,\frac{1}{\lambda}-b\right]$. Next, we show that $\psi'(x)\geq 1$ for $x\in\left(0,\frac{1}{\lambda}-b\right]$. The derivative of h is given by

$$h'(x)=\frac{(e^{\lambda h(x)}-1)^2}{e^{\lambda h(x)}\left(e^{-\lambda h(x)}-1+\lambda h\left(x\right)\right)}\geq 0,$$

which follows from $e^{-z}>1-z$ for $z>0$. Then $\psi(x)=h(x)-x$ and

$$\psi'(x)=\frac{e^{\lambda h(x)}-1-\lambda h(x)}{e^{-\lambda h(x)}-1+\lambda h(x)}.$$

We have $\psi'(x)\geq 1$ if and only if $e^{\lambda h}-e^{-\lambda h}-2\lambda h\geq 0$ for $h\geq 0$. This is implied by the fact that $e^{\lambda h}-e^{-\lambda h}-2\lambda h$ is increasing in h and equals zero at 0.[21] Finally, from (14) it follows that

$$\lim_{x\to 0}\left(1-\lambda b\right)\frac{e^{\lambda\psi(x)}-1}{\lambda\psi(x)}=1$$

and so, $\lim_{x\to 0}\psi(x)>0$. Also from (14) it follows that $\psi(x)\to\infty$ as $x\to\frac{1}{\lambda}-b$.

By Claim 1, $\psi(\cdot)$ is strictly above the diagonal line for $b>0$, as depicted in Figure 5. It is must be that $x_{K+1}=\infty$ for some K, since otherwise x_k eventually becomes negative). Then, we can construct any equilibrium working backwards: $x_K=\frac{1}{\lambda}-b$, $x_{K-1}=\psi^{-1}(x_K)$ and so on until we reach x_1. The most informative equilibrium corresponds to K such that $\psi^{-1}(x_1)\leq 0$. Since $\lim_{x\to 0}\psi(x)>0$, K is finite. From sequence $(x_k)_{k=1}^K$, we reconstruct threshold types $\omega_k=\omega_{k-1}+x_k$ and $\omega_0=0$. □

Proofs for Section 3

Proof of Lemma 2. Denote by $q_i(\theta_i)=\mathbb{E}[q_i|\theta_i]$ the expected probability of allocation for type θ_i and by $t_i(\theta_i)=\mathbb{E}[t_i|\theta_i]$ the expected transfer from type θ_i given that other bidders use their equilibrium strategies. Necessary conditions for q_i and t_i to be part of equilibrium are the following for all $i=1,\ldots,N$:

$$q_i(\theta_i)\theta_i-t_i(\theta_i)\geq q_i(\theta_i')\theta_i-t_i(\theta_i')\qquad\text{for all }\theta_i'\in[\underline{v},\overline{v}], \tag{16}$$

$$q_i(\theta_i)\theta_i-t_i(\theta_i)\geq 0\qquad\text{for all }\theta_i\in[\underline{v},\overline{v}]. \tag{17}$$

Denote $U_i(\theta_i)\equiv q(\theta_i)\theta_i-t_i(\theta_i)$. Lemma 2 in Myerson (1981) gives the following integral formula for U_i.

Lemma 3 (Myerson (1981)). *Conditions* (16) *and* (17) *imply that for all* $i=1,\ldots,N$ *and all* $\theta_i\in[\underline{v},\overline{v}]$*:*

$$U_i(\theta_i)=U_i(\underline{v})+\int_{\underline{v}}^{\theta_i}\left(\int_{[\underline{v},\overline{v}]^{N-1}}q_i(\theta,\theta_{-i})dF_\theta(\theta_{-i})\right)d\theta.$$

[21] Its derivative is $\lambda e^{\lambda h}+\lambda e^{-\lambda h}-2\lambda=\lambda e^{-\lambda h}(e^{\lambda h}-1)^2\geq 0$.

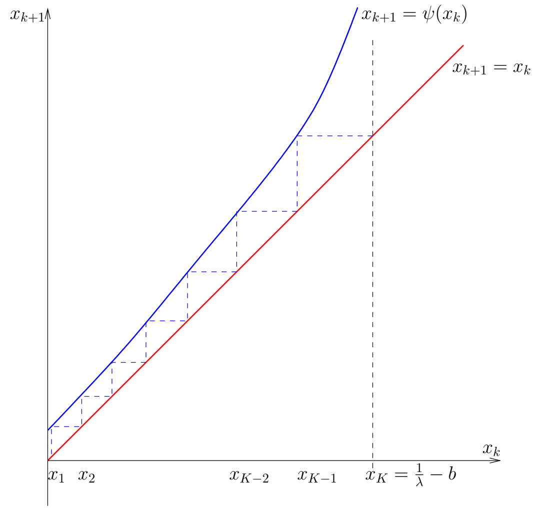


Figure 5: Recursion $x_{k+1} = \psi(x_k)$ for $\frac{1}{\lambda} > b > 0$.

Consider a strategy m_i of the advisor i in the efficient equilibrium of a static auction $\mathcal{A}$ and corresponding probability distribution F_θ generated by strategy m_i. The expected probability of allocation from following the equilibrium strategy for type θ_i is $\int_{[\underline{v},\overline{v}]^{N-1}} q_i(\theta_i, \theta_{-i}) dF_\theta(\theta_{-i})$, and by Lemma 3, the expected transfer from reporting type θ_i is $P(\theta_i)\theta_i - U_i(\mathcal{M}, \theta_i)$. Both quantities depend only on function q_i and $U_i(\underline{v})$ and hence are the same for the auction $\mathcal{A}$ and the second price auction. This implies that the strategy m_i is also constitutes equilibrium in the second-price auction. □

Proof of Theorem 1. Clearly, the second-price auction is an efficient mechanism. By Theorem 2 it is sufficient to analyze equilibria of the second-price auction.

To any profile of bids $\bar{a} = (a_i)_{i \in N}$ corresponds an allocation $(q_1(\bar{a}), \ldots, q_N(\bar{a}))$ such that $\sum_{i=1}^{N} q_i(\bar{a}) = 1$ and transfers $(t_1(\bar{a}), \ldots, t_N(\bar{a}))$. Denote by $q(a_i) \equiv \mathbb{E}[q_i(a_i, a_{-i})]$ and $t(a_i) \equiv \mathbb{E}[t_i(a_i, a_{-i})]$ the expected probability of allocation and transfer, respectively, from action a_i, where expectations are taken fixing strategies of other bidders and advisors m_{-i} and a_{-i}. Bidder i chooses a bid from A given that her expected value is $\theta_i = \mathbb{E}[v_i | a_i]$. Let where $Q = \{q(a_i), a_i \in A\}$ and $t(q) = \min_{a_i : q = q(a_i)} t(a_i)$. Then the bidder and the advisor play the cheap-talk game with payoffs given by

$$\text{Bidder} \quad : \quad qv - t(q), \tag{18}$$
$$\text{Advisor} \quad : \quad q(v + b) - t(q). \tag{19}$$

Since the mixed derivatives of (18) and (19) are positive, the set of types of the advisor that induce the same probability of allocation q is an interval. Therefore, to characterize equilibria of the second-price auction, we need to determine incentives of threshold types of the advisor ω_k. Consider any such type ω_k. In the second-price auction, a message is simply an expected value of the bidder m_k. Let $\hat{m}$ be the message of the highest bidder among $N - 1$ opponents of the bider. From submitting a message m_k, type ω_k gets utility

$$\mathbb{E}[\omega_k + b - \hat{m} | \hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)\Lambda_k(\omega_k + b - m_k).$$

From submitting a message m_{k+1}, type ω_k gets utility

$$\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) + G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}).$$

Type ω_k should be indifferent between the two which gives equation (5).

Claim 2. If $\omega_{k+1} = \omega_k$, then either $k = 0$ or $k = K$.

Proof: Suppose to contradiction that for some $0 < k < K$, $\omega_{k+1} = \omega_k$. This implies that $H(\omega_k, \omega_{k+1}) = 0$ and so, from (5), $H(\omega_{k-1}, \omega_k)(1-\Lambda_{k-1})(\omega_k + b - m_k) = 0$ and $H(\omega_{k+1}, \omega_{k+2})\Lambda_{k+2}(\omega_{k+1} + b - m_{k+2}) = 0$. This implies that $\omega_k + b = m_k$ and $\omega_{k+1} + b = m_{k+2}$. But only the first equality can hold if $b < 0$ and only the second equality can hold if $b > 0$, contradiction. **q.e.d.**

Claim 3. There exists $\varepsilon > 0$ such that for all k, either $\omega_{k+1} - \omega_k > \varepsilon$ for $0 < k < K$.

Proof: It follows from (5) that whenever $\omega_{k-1} < \omega_k < \omega_{k+1}$, we have

$$\omega_k + b > \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \tag{20}$$

and

$$\omega_k + b < \mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]. \tag{21}$$

First, consider $b > 0$. If for any $\varepsilon > 0$, there exists and equilibrium such that $\omega_{k+1} - \omega_k < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \le \omega_k + \varepsilon$ which contradicts (21) for sufficiently small ε. Now, consider $b < 0$. If for any $\varepsilon > 0$, there exists and equilibrium such that $\omega_k - \omega_{k-1} < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \ge \omega_k - \varepsilon$ which contradicts (20) for sufficiently small ε. **q.e.d.**

The fact that there exists $\overline{K}$ such that there is an equilibrium with K segments for any $1 \le K \le \overline{K}$, but not for $K > \overline{K}$ can be proven by the same argument as in the proof Theorem 1 in Crawford and Sobel (1982). □

Proof of Corollary 1. Sine there is an equilibrium partition for any $K \le \overline{K}$, it is sufficient to show that there is an equilibrium with two intervals in the partition. For $K = 2$, we can use the same argument as in the proof of Theorem 4 to show that (5) implies

$$\omega_1 + b - \mathbb{E}[v|v \in [\omega_0, \omega_2)] \le 0. \tag{22}$$

Since in the equilibrium with $K = 2$, $\omega_0 = \underline{v}$ and $\omega_2 = \overline{v}$, and $\omega_1 \ge \underline{v}$, we get the desired conclusion. It is easy to check that for $N = 2$, the inequality in (22) is an equality and an equilibrium with two segments exists whenever equation $\omega_1 + b - \mathbb{E}[v] = 0$ has a solution. Since $b > 0 \ge \mathbb{E}[v] - \underline{v}$, whenever $b \le \mathbb{E}[v] - \underline{v}$, such solution exists by continuity which proves the sufficiency of condition in the corollary. □

Proof of Theorem 2. In this proof, it is useful to introduce the following notations:

$$\begin{aligned}
\Phi(\omega_{k-1}, \omega_k) &\equiv \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{1}{n+1}, \\
\Psi(\omega_{k-1}, \omega_k) &\equiv \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{n}{n+1}, \\
m(\omega_{k-1}, \omega_k) &\equiv \mathbb{E}[v|v \in (\omega_{k-1}, \omega_k)].
\end{aligned}$$

Define function

$$H(\omega_{k-1}, \omega_k, \omega_{k+1}) \equiv \Psi(\omega_{k-1}, \omega_k)(\omega_k + b - m(\omega_{k-1}, \omega_k)) + \Phi(\omega_k, \omega_{k+1})(\omega_k + b - m(\omega_k, \omega_{k+1})). \tag{23}$$

It is easy to check that function H coincides with the left-hand side of equation (5). By Theorem 1, any equilibrium of the second-price auction is outcome-equivalent to an equilibrium having the partition structure with thresholds $(\tilde{\omega}_k)_{k=1}^K$ solving the recursion

$$H(\tilde{\omega}_{k-1}, \tilde{\omega}_k, \tilde{\omega}_{k+1}) = 0 \tag{24}$$

with $\tilde{\omega}_0 = \underline{v}$ and $\tilde{\omega}_{K+1} = \overline{v}$. We show that if the NITS condition fails, then for any such solution, there exists a different solution to the recursion (24) with $K+1$ partition cells. Since there are at most $\overline{K}$ partition cells, this implies that there exists an equilibrium satisfying NITS, and in particular, the most informative equilibrium satisfies NITS. We consider separately cases $b > 0$ and $b < 0$.

Case $b > 0$. If type $\underline{v}$ reveals herself to the bidder, then the bidder prefers to submit a loosing bid. Suppose to contradiction that NITS fails and $\underline{v} + b < m(\underline{v}, \tilde{\omega}_1)$. We show by induction that for any $k \leq K+1$, there exists another solution $(\omega_j^k)_{j=1}^{K_j}$ to (24) such that $\omega_0^k = \underline{v}$, $\omega_k^k > \tilde{\omega}_{k-1}$, and $\omega_{k+1}^k = \tilde{\omega}_k$. Theorem 2 follows from the claim applied to $k = K+1$.

For $k = 1$, the failure of NITS implies that $H(\underline{v}, \underline{v}, \tilde{\omega}_1) = \Phi(\underline{v}, \tilde{\omega}_1)(\underline{v} + b - m(\underline{v}, \tilde{\omega}_1)) < 0$. At the same time, $H(\underline{v}, \tilde{\omega}_1, \tilde{\omega}_1) = \Psi(\underline{v}, \tilde{\omega}_1)(\tilde{\omega}_1 + b - m(\underline{v}, \tilde{\omega}_1)) > 0$, as $\tilde{\omega}_1 = m(\tilde{\omega}_1, \tilde{\omega}_1) \geq m(\underline{v}, \tilde{\omega}_1)$. By continuity, there exists $x \in (\underline{v}, \tilde{\omega}_1)$ such that $H(\underline{v}, x, \tilde{\omega}_1) = 0$ proving the claim for $k = 1$.

Suppose the statement is true for k and we next prove it for $k+1$. Since $\tilde{\omega}_k$ solves (24), $H(\tilde{\omega}_{k-1}, \tilde{\omega}_k, \tilde{\omega}_{k+1}) = 0$ or

$$\Psi(\tilde{\omega}_{k-1}, \tilde{\omega}_k)(\tilde{\omega}_k + b - m(\tilde{\omega}_{k-1}, \tilde{\omega}_k)) + \Phi(\tilde{\omega}_k, \tilde{\omega}_{k+1})(\tilde{\omega}_k + b - m(\tilde{\omega}_k, \tilde{\omega}_{k+1})) = 0. \tag{25}$$

Let $\omega_k^k > \tilde{\omega}_{k-1}$ and $\omega_{k+1}^k = \tilde{\omega}_k$ as in the inductive hypothesis and consider $H(\omega_k^k, \tilde{\omega}_k, \tilde{\omega}_{k+1})$:

$$\Psi(\omega_k^k, \tilde{\omega}_k)(\tilde{\omega}_k + b - m(\omega_k^k, \tilde{\omega}_k)) + \Phi(\tilde{\omega}_k, \tilde{\omega}_{k+1})(\tilde{\omega}_k + b - m(\tilde{\omega}_k, \tilde{\omega}_{k+1})),$$

which differs from (25) only in the first term. Since $\omega_k^k > \tilde{\omega}_{k-1}$, $m(\omega_k^k, \tilde{\omega}_k) > m(\tilde{\omega}_{k-1}, \tilde{\omega}_k)$. Moreover, the binomial distribution with probability of success $\frac{F(\tilde{\omega}_{k-1}, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}$ first-order stochastic dominance the binomial distribution with probability of success $\frac{F(\omega_k^k, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}$. Hence,

$$\begin{aligned}
\frac{\Psi(\omega_k^k, \tilde{\omega}_k)}{F(\tilde{\omega}_k)^N} &= \sum_{n=1}^{N-1} \binom{N-1}{n} \left(\frac{F(\omega_k^k, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}\right)^n \left(\frac{F(\omega_k^k)}{F(\tilde{\omega}_k)}\right)^{N-1-n} \frac{n}{n+1} \\
&< \sum_{n=1}^{N-1} \binom{N-1}{n} \left(\frac{F(\tilde{\omega}_{k-1}, \tilde{\omega}_k)}{F(\tilde{\omega}_k)}\right)^n \left(\frac{F(\tilde{\omega}_{k-1})}{F(\tilde{\omega}_k)}\right)^{N-1-n} \frac{n}{n+1} = \frac{\Psi(\tilde{\omega}_{k-1}, \tilde{\omega}_k)}{F(\tilde{\omega}_k)^N},
\end{aligned}$$

as and $\frac{n}{n+1}$ is increasing in n. Therefore, $H(\omega_k^k, \tilde{\omega}_k, \tilde{\omega}_{k+1}) < 0$.

On the other hand, since $\omega_{k+1}^k = \tilde{\omega}_k$,

$$H(\omega_k^k, \tilde{\omega}_k, \omega_{k+1}^k) = \Psi(\omega_k^k, \tilde{\omega}_k)(\tilde{\omega}_k + b - m(\omega_k^k, \tilde{\omega}_k)) > 0.$$

By continuity, there exists $x \in (\tilde{\omega}_k, \tilde{\omega}_{k+1})$ such that $H(\omega_k^k, \omega_{k+1}^k, x) = 0$. By continuity, we can find solution $(\omega_j^{k+1})_{j=1}^{K_{k+1}}$ to (24) with $\omega_{k+1}^{k+1} > \tilde{\omega}_k$ and $\omega_{k+2}^{k+1} = \tilde{\omega}_{k+1}$, which completes the proof of the inductive step.

Case $b < 0$. If type $\overline{v}$ reveals herself to the bidder, then the bidder prefers to submit a bid that is guaranteed to win. Type $\overline{v}$ does not want to reveal herself if and only if $\overline{v} + b - m(\tilde{\omega}_K, \overline{v}) \leq 0$. Suppose to contradiction that NITS fails and $\overline{v} + b > m(\tilde{\omega}_K, \overline{v})$. We show by induction that for any $k \leq K+1$, there exists another solution $(\omega_j^k)_{j=1}^{K_j}$ to (24) such that $\omega_{K_j+1}^k = \overline{v}$, $\omega_{K_j-k}^k < \tilde{\omega}_{K-k+1}$, and $\omega_{K_j-k-1}^k = \tilde{\omega}_{K-k}$. Theorem 2 follows from the claim applied to $k = K$.

For $k = 1$, the failure of NITS implies that $H(\tilde{\omega}_K, \overline{v}, \overline{v}) = \Psi(\tilde{\omega}_K, \overline{v})(\overline{v} + b - m(\tilde{\omega}_K, \overline{v})) > 0$. At the same time, $H(\tilde{\omega}_K, \tilde{\omega}_K, \overline{v}) = \Phi(\tilde{\omega}_K, \overline{v})(\tilde{\omega}_K + b - m(\tilde{\omega}_K, \overline{v})) > 0$, as $\tilde{\omega}_K$ satisfies (24). By continuity, there exists $x \in (\tilde{\omega}_K, \overline{v})$ such that $H(\tilde{\omega}_K, x, \overline{v}) = 0$ proving the claim for $k = 1$.

Suppose the statement is true for k and we next prove it for $k+1$. Since $\tilde{\omega}_k$ solves (24), $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1}) = 0$ or

$$\Psi(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})) + \\ \Phi(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})) = 0. \quad (26)$$

Let $\omega_{K_j-k}^k < \tilde{\omega}_{K-k+1}$ and $\omega_{K_j-k-1}^k = \tilde{\omega}_{K-k}$ as in the inductive hypothesis and consider $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \omega_{K_j-k}^k)$:

$$\Psi(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})) + \\ \Phi(\tilde{\omega}_{K-k}, \omega_{K_j-k}^k)(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \omega_{K_j-k}^k)),$$

which differs from (26) only in the second term. Since $\omega_{K_j-k}^k < \tilde{\omega}_{K-k+1}$, $m(\tilde{\omega}_{K-k}, \omega_{K_j-k}^k) < m(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})$. Moreover,

$$\begin{aligned} \Phi(\tilde{\omega}_{K-k}, \omega_{K_j-k}^k) &= \sum_{n=1}^{N-1} \binom{N-1}{n} F(\tilde{\omega}_{K-k}, \omega_{K_j-k}^k)^n F(\tilde{\omega}_{K-k})^{N-1-n} \frac{1}{n+1} + G(\omega_{K_j-k}^k, \tilde{\omega}_{K-k}) \cdot 0 \\ &< \sum_{n=1}^{N-1} \binom{N-1}{n} F(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})^n F(\tilde{\omega}_{K-k})^{N-1-n} \frac{1}{n+1} \\ &= \Phi(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1}). \end{aligned}$$

Hence, $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \omega_{K_j-k+1}^k) > 0$. On the other hand, since $\omega_{K_j-k-1}^k = \tilde{\omega}_{K-k}$,

$$H(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k}, \omega_{K_j-k+1}^k) = \Phi(\tilde{\omega}_{K-k}, \omega_{K_j-k+1}^k)(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \omega_{K_j-k+1}^k)) < 0.$$

Therefore, there exists $x \in (\tilde{\omega}_k, \tilde{\omega}_{k+1})$ such that $H(x, \tilde{\omega}_{K-k}, \omega_{K_j-k+1}^k) = 0$. By continuity, we can find solution $(\omega_j^{k+1})_{j=1}^{K_{k+1}}$ to (24) with $\omega_{K_j-k-1}^k < \tilde{\omega}_{K-k}$ and $\omega_{K_j-k-2}^k = \tilde{\omega}_{K-k-1}$, which completes the proof of the inductive step. □

Proofs for Section 4

By Lemma 1, the strategy of the advisor can be described by a function $m(v)$ which specifies at what price the advisor sends message "quit" to the bidder. The following lemma shows that in the English auction, types of advisor either perfectly reveal themselves to the bidder or pool with neighboring types.

Lemma 4. *Function $m(v)$ is increasing on a subset of $[\underline{v}, \overline{v}]$ of Lebesgue measure $\overline{v} - \underline{v}$.*

Proof of Lemma 4. Suppose to contradiction that $m(v)$ is strictly decreasing on a set of positive measure I. Let $v = \inf I$ and $v' = \sup I$. Then $m(v) > m(v')$ and $G(v') - G(v) > 0$. Let q and t be the probability of winning and expected price paid conditional on using strategy $m(v)$ and q' and t' be the probability of winning and expected price paid conditional on using strategy $m(v')$. Then

$$\begin{aligned} qv - t &\geq q'v - t', \\ q'v' - t' &\geq qv' - t, \end{aligned}$$

implies that $q' \geq q$. By quitting at price $m(v)$ instead of $m(v')$, the advisor increases the probability of winning by at least $G(v') - G(v) > 0$ and so, $q > q'$ which is a contradiction. □

Lemma 5. *m is strictly increasing on $[\underline{v}, v^*)$ and is constant almost everywhere on $[v^*, \overline{v}]$ where v^* satisfies* (6) *when $v^* > \underline{v}$ and $v^* + b \geq \mathbb{E}[v|v \geq v^*]$ when $v^* = \underline{v}$.*

Proof of Lemma 5. Any equilibrium generates a partition Π of $[\underline{v}, \overline{v}]$ satisfying for any $\pi \in \Pi$, $v, v' \in \pi \iff m(v) = m(v')$. We say that types in $\pi \in \Pi$ pool if $m(v)$ is constant on π, i.e. these types send message "quit" at the same price. We say that types in $[v', v'']$ separate, if $m(v)$ is strictly increasing on $[v', v'']$, i.e. all these types send message "quit" at different prices. Define by Π^P the closure of the set of all types that pool with some other type. Then $\Pi^S = [\underline{v}, \overline{v}] \backslash \Pi^P$ is the set of all types that separate and denote by $\partial\Pi^P$ the boundary of Π^P.

Notice that the babbling equilibrium is an equilibrium of the English auction and it satisfies NITS if and only if $\mathbb{E}[v] \leq \underline{v} + b$. So we focus on the case when there is a non-trivial information transmission in equilibrium, i.e. $\Pi^S \neq \emptyset$.

We first show that whenever an interval of types perfectly reveals their value to the bidder in the auction, then these types quit at the optimal time.

Claim 4. *If m is strictly increasing on a subset S of (v', v'') of (Lebesgue) measure $|v'' - v'|$, then $m(v) = v + b$ on (v', v'').*

Proof: There exists at most countable number of discontinuities of m on S. Consider a type v at which m is continuous, i.e. there exist sequences $v_j^- \to v - 0$ and $v_j^+ \to v + 0$ such that $m(v_j^-) \to m(v) - 0$ and $m(v_j^+) \to m(v) + 0$. We show that $m(v) = v + b$. Suppose to contradiction that $m(v) < v + b$. Choose j large so that $m(v) < m(v_j^+) < v + b$. If type v sends "quit" at price $m(v_j^+)$ instead of $m(v)$, then she can additionally win against types in $[v, v_j^+)$ and pay at most $m(v_j^+) < v + b$. Therefore, her utility is higher contradicting the rationality of type v. Now, suppose to contradiction that $m(v) > v + b$. Choose j large so that $v + b < m(v_j^-) < m(v)$. If type v sends "quit" at price $m(v)$ instead of $m(v_j^-)$, then she additionally wins against types in $[v_j^-, v)$ and pays at least $m(v_j^-) > v + b$. Therefore, she strictly gains from sending "quit" at price $m(v_j^-)$ contradicting the rationality of type v. Therefore, $m(v) = v + b$ for all continuity points of S. Since $m(v) = v + b$ on a dense subset of S, it is also true on the whole S. Since S has measure $|v'' - v'|$, set S is dense in (v', v'') and so, $m(v) = v + b$ on (v', v''). **q.e.d.**

Claim 5. $\Pi^P = [v^*, \overline{v}]$ for some $v^* \geq \underline{v}$.

Proof: Consider $v \in \partial\Pi^P$. Type v is indifferent between pooling with some interval of types $\pi \ni v$ and separating. Indeed, since $\pi \in \partial\Pi^P$ and $\Pi^S \neq \emptyset$, there exists a sequence $v_j \to v$ such that $m(v_j) = v_j + b$ by Claim 1. Type v can mimic type v_j and for large j get utility arbitrarily close to her maximal utility. Therefore,

$$m(v) = v + b = \mathbb{E}[v|v \in \pi]. \tag{27}$$

Suppose to contradiction to Claim 2 there exists a sequence $v_j \to v$ such that $v_j \in \Pi^P$ and $v_j < v$. Then $\mathbb{E}[v|v \in \pi] < v + b$ which is a contradiction. **q.e.d.**

Next, we show that all types in Π^P send "quit" at the same price.

Claim 6. $\Pi^P = \pi$ for some $\pi \in \Pi$.

Proof: Suppose to contradiction that there are two adjacent intervals of types π and π' such that types in π send "quit" at price m and types in π' send "quit" at price $m' > m$. Consider type v that is at the boundary of π and π'. By continuity, type v is indifferent between sending "quit" at price m and m'. The benefit of quitting at m' rather than m is that type v wins against types in π, but there is a risk that she will tie with types in π'. The indifference of type v implies that $m' > v + b$. But then consider a time when the running price reaches m'. Type v is the lowest type. However, she gets a negative utility from pooling with types in π'. This contradicts the NITS condition. **q.e.d.**

Finally, equation (6) follows from (27) and Claim 3. □

Proof of Theorem 3. By Lemma 5 condition (6) is a necessary condition. For any v^* satisfying in addition (9), we construct an equilibrium in online strategies satisfying NITS for dynamic auctions. Then we show that if v^* fails (9), then it cannot be part of equilibrium.

Consider strategies described in the theorem. The optimality of the advisor and the bidder after she receives the message "quit" is verified in the text. Let us check the optimality of the bidder. Let N_p be the number of bidders remaining in the game at price p and v_p be the lowest type remaining in the game at price p. The utility of the bidder from following the recommendation of the advisor starting from running price p is equal to

$$V(N_p, v_p) = \frac{1}{(1-F(v_p))^{N_p} F(v_p)^{N-1-N_p}} \left(\int_{v_p}^{v^*} (1-F(s))(\mathbb{E}[v|v>s] - s - b) dG_{N_p}(s) \right) + \frac{1}{(1-F(v_p))^{N_p} F(v_p)^{N-1-N_p}} \sum_{n=1}^{N_p-1} \binom{N_p-1}{n} (1-F(v^*))^{n+1} (F(v^*) - F(v_p))^{N_p-1-n} \frac{1}{n} (\mathbb{E}[v|v \geq v^*] - v^* - b). \tag{28}$$

By the definition of v^*, the last term is zero and so,

$$V(N_p, v_p) = \frac{1}{(1-F(v_p))^{N_p} F(v_p)^{N-1-N_p}} \left(\int_{v_p}^{v^*} (1-F(s))(\mathbb{E}[v|v>s] - s - b) dG_{N_p}(s) \right). \tag{29}$$

The bidder prefers to quit immediately at the first time $V(N_p, v_p)$ becomes negative. Moreover, (9) implies for all $v_p \leq v^*$, $V(N_p, v_p) \geq 0$ which proves the optimality of the bidder's strategy. □

Proof of Corollary 2. To show that $v^* < \overline{v}$, notice that the left-hand side of (6) is greater than the right-hand side for v^* sufficiently close to $\overline{v}$. Therefore, $v_1^* < \overline{v}$ and for $\tilde{v} \in (v_1^*, v_0^*)$, $\mathbb{E}[v|v > \tilde{v}] - \tilde{v} - b < 0$. This implies that $v^* < \overline{v}$. □

Proof of Proposition 3. The profit of the seller in the English auction is given by

$$\int_{\underline{v}}^{\overline{v}} (\min\{\hat{v}, v^*\} + b) dG(\hat{v}) = b + \int_{\underline{v}}^{v^*} \hat{v} dG(\hat{v}) + (1 - G(v^*))v^*. \tag{30}$$

The derivative of (30) with respect to b equals $1 + (1 - G(v^*))\frac{d}{db}v^*$.We can find $\frac{d}{db}v^*$ by the implicit function theorem from (6),

$$\tfrac{d}{db}v^* = -\left(1 - \tfrac{f(v^*)}{1-F(v^*)} b\right)^{-1}.$$

Since MRL is strictly decreasing, $\frac{d}{db}v^* < 0$ for $b \in [0, \overline{b}]$. Thus, the derivative of the profit with respect to b equals

$$\frac{G(v^*) - \frac{f(v^*)}{1-F(v^*)}b}{1 - \frac{f(v^*)}{1-F(v^*)}b}. \tag{31}$$

When b is close to 0, v^* is close to $\overline{v}$. Using (6) and L'Hospital's Rule, we get

$$\begin{aligned} \lim_{v^* \to \overline{v}} \frac{f(v^*)}{1-F(v^*)}b &= \lim_{v^* \to \overline{v}} \frac{f(v^*) \int_{\underline{v}}^{v^*} (v - v^*) dF(v)}{(1-F(v^*))^2} \\ &= f(\overline{v}) \lim_{v^* \to \overline{v}} \frac{1}{2f(v^*)} = \frac{1}{2}. \end{aligned}$$

Since $G(\overline{v}) = 1$, expression (31) is positive for small b. When b is approaches $\overline{b}$, v^* approaches $\underline{v}$ and so, $\frac{f(v^*)}{1-F(v^*)}b \to f(\underline{v})\overline{b}$ while $G(v^*) \to 0$. Therefore, expression (31) is negative for sufficiently large b, which completes the proof of the first statement. To prove the second statement, notice that $\lim_{N\to\infty} G(v^*) = 0$. Therefore, there exists $N(b)$ such that $G(v^*) - \frac{f(v^*)}{1-F(v^*)}b < 0$ for all $N > N(b)$. Since $G(v^*)$ decreases monotonically in N, there is $\varepsilon(b) > 0$ such that a decrease in b to $b - \varepsilon(b)$ is beneficial for the seller for sufficiently large N. The last statement follows from Corollary 2. □

Proof of Theorem 4. We will show that for any equilibrium of the second-price auction, there is no $\omega_k \in (v^*, \overline{v})$. This implies that the partition generated by the second-price auction is cruder, and so the English auction is more efficient. Suppose to contradiction that there is $\omega_k \in (v^*, \overline{v})$ such that $\omega_{k-1} \le v^*$. Notice that in equation (5) $\Lambda_{k+1} \le \frac{1}{2}$,$1 - \Lambda_k \ge \frac{1}{2}$, $\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]] \ge 0$ and $\omega_k + b - \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \le 0$. Therefore, equation (5) implies

$$G(\omega_{k-1}, \omega_k)(\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]) + G(\omega_k, \omega_{k+1})(\omega_k + b - \mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]) \le 0$$

or

$$\omega_k + b - \frac{G(\omega_{k-1}, \omega_k)}{G(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] - \frac{G(\omega_k, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \le 0.$$

Observe that[22]

$$\frac{G(\omega_k, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})} \ge \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}.$$

Since $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \ge \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$,

$$\omega_k + b - \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] - \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \le 0$$

or

$$\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1})] \le 0.$$

[22] Indeed

$$\frac{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_k)}{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_{k-1})} \ge \frac{F(\omega_{k+1}) - F(\omega_k)}{F(\omega_{k+1}) - F(\omega_{k-1})}$$

if and only if

$$\gamma F^{N-1}(\omega_{k+1}) + (1-\gamma)F^{N-1}(\omega_{k-1}) \ge F^{N-1}(\omega_k)$$

for γ satisfying $\gamma F(\omega_{k+1}) + (1-\gamma)F(\omega_{k-1}) = F(\omega_k)$ which holds by Jensen's inequality.

Then

$$\begin{aligned}\omega_k - b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1}]] &\geq \omega_k - b - \mathbb{E}[v|v \in [v^*, \omega_{k+1}]] \\ &\geq \omega_k - b - \mathbb{E}[v|v \geq v^*] \\ &= \omega_k - v^* > 0,\end{aligned}$$

which is a contradiction. □

Proofs for Section 5

See Online Appendix for the proof of Proposition 4 and the construction of partially revealing PBEM in the Dutch auction for the exponential distribution.

Proof of Theorem 6. We first show that strategies described in Theorem 6 constitute an equilibrium of the Dutch auction.

Indeed, the left-hand side of equation (11) is greater than $\underline{v}$ and bounded from above by $\mathbb{E}[v]$. The right-hand side of equation (11) is less than $\underline{v}$ for small v^* and is greater than $\mathbb{E}[v]$ for sufficiently large v^*. By continuity, equation (11) has a solution.

To prove that conjectured strategies constitute an equilibrium, we need to show that the advisor sends the message "stop" at the optimal time given that bidder follows her recommendation, and that the bidder prefers to follow recommendations of the advisor.

Optimality of the advisor First, we show that strategy $\sigma(\cdot)$ is optimal for the advisor. The advisor of type v solves the following problem

$$\max_{\sigma}(v + b - \sigma)G(\sigma^{-1}(\sigma)), \tag{32}$$

for which the first-order condition is

$$g(v)(v + b) = (G(v)\sigma(v))' \tag{33}$$

with the initial condition $\sigma(v^*) = v^* + b$. From (33),

$$\begin{gathered}\sigma(v) = \frac{G(v^*)}{G(v)}(v^* + b) + \frac{1}{G(v)}\int_{v^*}^{v} g(\hat{v})(\hat{v} + b)d\hat{v} = \\ \frac{G(v^*)}{G(v)}(\mathbb{E}[v|v < v^*]) + \frac{G(v) - G(v^*)}{G(v)}(\mathbb{E}[\hat{v}|\hat{v} \in [v^*, v]] + b) = \\ b\frac{G(v) - G(v^*)}{G(v)} + \mathbb{E}[\hat{v}|\hat{v} < v] - (\mathbb{E}[\hat{v}|\hat{v} < v^*] - \mathbb{E}[v|v < v^*])\,\frac{G(v^*)}{G(v)} = \\ b + \mathbb{E}[\hat{v}|\hat{v} < v] + (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*])\,\frac{G(v^*)}{G(v)}.\end{gathered} \tag{34}$$

The equilibrium bid is equal to expectation of $\max\{v^*, \hat{v}\} + b$ conditional on $\hat{v} < v$. Given (34), the utility of the advisor from winning the auction equals

$$\begin{gathered}v - \mathbb{E}[\hat{v}|\hat{v} < v] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*])\frac{G(v^*)}{G(v)} = \\ \frac{1}{G(v)}\left(G(v)v - G(v^*)v^* - \mathbb{E}[\hat{v} : \hat{v} \in [v^*, v]]\right) = \\ \int_{v^*}^{v} \frac{G(\omega)}{G(v)}d\omega > 0.\end{gathered} \tag{35}$$

Hence, if the bidder follows her strategy, then it is optimal for the advisor to follow her strategy.

Optimality of the bidder By the single-crossing property of payoffs, when the bidder knows v, the bidder prefers to stop the auction earlier. Hence, having received the message "stop" from the advisor, the bidder prefers to stop immediately. It remains to check that the bidder does not want to quit the auction before she gets a recommendation from the advisor. By (34), if the bidder quits at time t, then her payoff equals

$$\mathbb{E}[v|v < v_p] - \sigma(v_p) = \mathbb{E}[v|v < v_p] - b - \mathbb{E}[\hat{v}|\hat{v} < v_p] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}. \tag{36}$$

On the other hand, if the bidder follows her equilibrium strategy, then her expected utility is given by

$$\begin{gathered} \mathbb{E}\left[(v - \sigma(v))1\{v > \hat{v}\}|\hat{v}, v < v_p\right] = \\ \mathbb{E}\left[(v - \sigma(v))1\{v > \hat{v}\}|v^* < v < v_p, \hat{v} < v_p\right] \frac{F(v_p) - F(v^*)}{F(v_p)} = \\ \frac{1}{F(v_p)^2} \int_{v^*}^{v_t} \left(v - b - \mathbb{E}[\hat{v}|\hat{v} < v] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v)}\right) F(v) dF(v), \end{gathered} \tag{37}$$

where the first equality is by the fact that $\mathbb{E}\left[(v - \sigma(v))1\{v > \hat{v}\}|v < v^*, \hat{v} < v_p\right] = 0$, the second equality is by (34). We need to show that (36) is less than(37). We evaluate the difference

$$\begin{gathered} \int_{v^*}^{v_p} \left(v - b - \mathbb{E}[\hat{v}|\hat{v} < v] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v)}\right) F(v) dF(v) - \\ F^2(v_p) \left(\mathbb{E}[v|v < v_p] - b - \mathbb{E}[\hat{v}|\hat{v} < v_p] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}\right). \end{gathered} \tag{38}$$

The derivative of (38) divided by $f(v_p)F(v_p)$ is equal to

$$\begin{gathered} v_p - b - \mathbb{E}[\hat{v}|\hat{v} < v_t] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)} - \\ 2\left(\mathbb{E}[v|v < v_p] - b - \mathbb{E}[\hat{v}|\hat{v} < v_p] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}\right) - \\ F(v_p)\left(\frac{1}{F(v_p)}(v_t - \mathbb{E}[v|v < v_p]) - \frac{(N-1)F^{N-2}(v_p)}{G(v_p)}\left(v_p - \mathbb{E}[\hat{v}|\hat{v} < v_p] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}\right)\right) = \\ v_p - 2\mathbb{E}[v|v < v_p] + b + \mathbb{E}[\hat{v}|\hat{v} < v_p] + (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)} - \\ \left(v_p - \mathbb{E}[v|v < v_p] - (N-1)(v_p - \mathbb{E}[\hat{v}|\hat{v} < v_p]) + (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}(N-1)\right) = \\ (N-1)(v_p - \mathbb{E}[\hat{v}|\hat{v} < v_p]) + (\mathbb{E}[\hat{v}|\hat{v} < v_p] - \mathbb{E}[v|v < v_p]) + b - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}(N-2) = \\ v_p - \mathbb{E}[v|v < v_p] + b + (N-2)\left(v_p - \mathbb{E}[\hat{v}|\hat{v} < v_p] - (v^* - \mathbb{E}[\hat{v}|\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}\right) \end{gathered} \tag{39}$$

In (39) $v_p - \mathbb{E}[v|v < v_p] + b > 0$ by the fact that v^* is the unique solution to (11). The remaining term in (39) is positive by (35). Hence, the derivative of (38) is positive. Therefore, since at $v_p = v^*$, the expression (38) is equal to zero by (11), for $\geq v^*$ the expression (38) is non-negative. This proves that the bidder prefers to follow recommendations of the advisor rather than stop the auction earlier. □

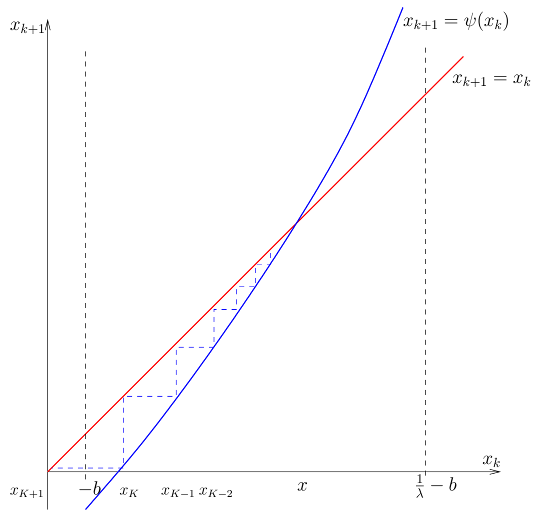


Figure 6: Recursion $x_{k+1} = \psi(x_k)$ for $b < 0$.

Proof of Theorem 7. We want to show that there is no partition $(\omega_k)_{k=1}^K$ induced by the equilibrium of the second-price auction such that $\omega_k \in [\underline{v}, v^*]$. Since v^* is the unique solution to (11) and $\underline{v} + b - \mathbb{E}[v|v \leq \underline{v}] = b < 0$, $\omega_k + b - \mathbb{E}[v|v \leq \omega_k] < 0$. Therefore,

$$\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \quad \leq \quad \omega_k + b - \mathbb{E}[v|v < \omega_k] < 0,$$

which contradicts the fact that the first term in (5) should be positive. □

B Online Appendix (Not for Publication)

Proof of Proposition 4. We can use the analysis of the case $b > 0$ in Proposition 1 to prove Proposition 4. Indeed, the derivation of the recursion (14) does not depend on the value of b. Recall the function ψ which is defined implicitly as a value of $x_{k+1} = \psi(x_k)$ that satisfies (14) for given x_k. When $b < 0$, there is a fixed point of ψ that gives the PBEM described in Proposition 4 (see Figure 6).

Since $\overline{v}_w = \infty$, it is not clear in what sense the PBEM in Proposition 4 satisfies the NITS condition. However, we can construct a sequence of equilibria with $\overline{v} < \infty$ that satisfies NITS and converges to the equilibrium in Proposition 4 as $\overline{v} \to \infty$. Indeed, fix an integer K. Let $x_{K+1} = 0$ and recursively define $x_k = \psi^{-1}(x_{k+1})$. It is easy to verify that these strategies constitute an equilibrium when F is the exponential distribution with parameter λ truncated at $\overline{v} = \sum_{k=1}^K x_k \to \infty$ as $K \to \infty$. Since $x_{K+1} = 0$, type $\overline{v}$ perfectly reveals herself and so, this equilibrium satisfies NITS. Moreover, for any $\varepsilon > 0$ there exists $\overline{K}$ such that for any K, $x > x_k > x - \varepsilon$ for all but $\overline{K}$ indexes k. This way, even though the equilibrium in Proposition 4 cannot be verified to satisfy the NITS condition, it is a limit of equilibria satisfying NITS. □

Proposition 5. *Suppose $b < 0$. There exists an equilibrium of the Dutch auction described by a tuple $\{v^*, \sigma(\cdot)\}$ as follows. Types of advisor $v \leq v^*$ send message "stop" when $p = v^* + b$. Any type of advisor $v \geq v^*$ sends message "stop" at time t when $p = \sigma(v)$. The bidder follows the recommendation of advisor when the running price is above $v^* + b$ and stops the auction if the* *running price is $v^* + b$. Threshold v^* is the solution to*

$$\frac{v^*}{1 - e^{-\lambda v^*}} = \frac{1}{\lambda} - b \tag{40}$$

and bidding strategy $\sigma(\cdot)$ is given by

$$\sigma(v) = \mathbb{E}[\max\{v^*, \hat{v}\} + b | \hat{v} < v], \text{ for } v \geq v^*. \tag{41}$$

Proof. First, observe that (40) is the equation $\mathbb{E}[v|v < v^*] = v^* + b$ for the exponential distribution. The left-hand side of (40) is a strictly increasing function[23] which is $\frac{1}{\lambda}$ at $v^* = 0$ and converges to infinity as $v^* \to \infty$, while the right-hand side is greater than $\frac{1}{\lambda}$. Hence, there is a unique solution to (40).

We first verify that the advisor does not have incentives to deviate from her strategy. As a preliminary step, we derive the equilibrium of the first-price auction where bids are submitted directly by advisors and the lowest participating bidder has type v^* and simply bids her value $v^* + b$. The advisor with type v solves the following problem

$$\max_{\sigma} (v + b - \sigma) F(\sigma^{-1}(\sigma)), \tag{42}$$

for which the first-order condition is

$$f(v)(v + b) = (F(v)\sigma(v))' \tag{43}$$

with the initial condition $\sigma(v^*) = \mathbb{E}[v|v < v^*] = v^* + b$. From (43),

$$\sigma(v) = \sigma(v^*) \frac{F(v^*)}{F(v)} + \frac{1}{F(v)} \int_{v^*}^{v} f(\hat{v})(\hat{v} + b) d\hat{v} = \mathbb{E}[\hat{v}|\hat{v} < v] + \frac{F(v) - F(v^*)}{F(v)} b, \tag{44}$$

which gives equation (41). Let $p^* = \sigma(v^*)$. The utility of the advisor from winning the auction is

$$v - \mathbb{E}[\hat{v}|\hat{v} < v] + b \frac{F(v^*)}{F(v)} \geq v + b - \mathbb{E}[\hat{v}|\hat{v} < v].$$

Since v^* solves (40), the advisor gets a positive utility from the auction for $v > v^*$.

If the bidder follows the recommendation of the advisor, then the strategy to stop when $p = \sigma(v)$ is optimal for the advisor when $v > v^*$, as it is an equilibrium strategy in the Dutch auction where the advisor decides when to stop. For $v \leq v^*$, the advisor gets utility $\frac{1}{N}(v - v^*) \leq 0$ if she follows the strategy and $v + b - \sigma(v_p)$ if she stop at a price above p^*. Since

$$v + b - \sigma(v_p) \leq v + b - \sigma(v^*) = v - v^* \leq \frac{1}{N}(v - v^*),$$

sending the message "stop" at price p^* is optimal for the advisor.

Notice that the mixed derivative in b and σ of the maximized function (42) is positive. Hence, if the bidder submits the bid, then she chooses a higher bid. Therefore, it is optimal for her to stop when she gets the message from the advisor with type $v > v^*$.

[23] Indeed, its derivative is equal to

$$\frac{e^{-\lambda v}}{(1 - e^{-\lambda v})^2} (e^{\lambda v} - (1 + \lambda v)) > 0$$

To finish the proof, we show that the bidder does not want to stop the auction earlier. Let $v_p \equiv \sigma^{-1}(p)$ for all $p > p^*$. Denote by $\hat{v}$ the value of the opponent bidder. The expected utility of the bidder at time t from following the recommendation of the advisor is

$$\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v,\hat{v}<v_p\right] = \mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v^*<v<v_p;\hat{v}<v_p\right]\frac{F(v_p)-F(v^*)}{F(v_p)},$$

where we used the fact that at stage p^*, the bidder gets utility zero from winning. We need to compare this utility with the utility that the bidder gets if she quits before the advisor's message

$$\mathbb{E}[v|v<v_p]-\sigma(v_p) = -b\frac{F(v_p)-F(v^*)}{F(v_p)},$$

which boils down to showing that

$$\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v^*<v<v_p;\hat{v}<v_p\right]+b \tag{45}$$

is non-negative. Using (44) and $\mathbb{E}[\hat{v}|\hat{v}<v] = \frac{1}{\lambda}\left(1-\frac{vf(v)}{F(v)}\right)$ for the exponential distribution, we can re-write (45) as follows

$$\int_{v^*}^{v_p}\left(-b-\frac{1}{\lambda}+\frac{v+bF(v^*)}{F(v)}\right)\frac{F(v)}{F(v_p)}\frac{dF(v)}{F(v_p)-F(v^*)}+b$$

or rearranging terms

$$\int_{v^*}^{v_p}\left(F(v)\left(-b-\frac{1}{\lambda}\right)+v+bF(v^*)\right)dF(v)+bF(v_p)(F(v_p)-F(v^*)). \tag{46}$$

We will show that (46) is increasing in v_p. Since (46) is zero at $v_p = v^*$, this would imply that (46) is non-negative for all $v_p > v^*$. The derivative of (46) is equal to

$$f(v_p)\left(F(v_p)\left(-b-\frac{1}{\lambda}\right)+v_p+bF(v^*)+b(2F(v_p)-F(v^*))\right) = f(v_p)F(v_p)\left(b-\frac{1}{\lambda}+\frac{v_p}{F(v_p)}\right)>0$$

where the inequality follows from the fact that v^* is the unique solution to (40). □

References

[1] An, M. (1998): "Logconcavity versus Logconvexity: a Complete Characterization," *Journal of Economic Theory* 80, 350-369.

[2] Aumann, R., and S. Hart (2003): "Long Cheap Talk," *Econometrica* 71, 1619-1660.

[3] Bagnoli, M., and T. Bergstrom (2005): "Log-concave Probability and its Applications," *Economic Theory* 26, 445-469.

[4] Bertrand, Marianne, and Sendhil Mullainathan (2003): "Enjoying the Quiet Life? Corporate Governance and Managerial Preferences," *Journal of Political Economy* 111, 1043-1075.

[5] Betton, Sandra, B. Espen Eckbo, and Karin S. Thorburn (2008): "Corporate Takeovers," *Handbook of Empirical Corporate Finance*, Vol. 2, Elsevier B.V.

[6] Burkett, Justin. (2015): "Endogenous Budget Constraints in Auctions," *Journal of Economic Theory* 158, 1-20.

[7] Che, Y., and I. Gale (1998): "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies*, Vol. 65, 1-21.

[8] Chen, Y., N. Kartik, and J. Sobel (2008): "Selecting cheap-talk equilibria," *Econometrica*, Vol. 76, 117-136.

[9] Compte, O., and P. Jehiel (2007): "Auctions and Information Acquisition: Sealed Bid or Dynamic Formats?" *Rand Journal of Economics*, Vol. 38, 355-372.

[10] Crawford, V., and J. Sobel (1982): "Strategic Information Transmission," *Econometrica*, Vol. 50, 1431-1451.

[11] Golosov, M., Skreta, V., Tsyvinski, A., and A. Wilson (2014): "Dynamic Strategic Information Transmission," *Journal of Economic Theory*, Vol. 151, 304-341.

[12] Gorbenko, Alexander S., and Andrey Malenko (2014): "Strategic and Financial Bidders in Takeover Auctions," *Journal of Finance* 69, 2513-2555.

[13] Grenadier, Steven R., Andrey Malenko, and Nadya Malenko (2015): "Timing Decisions in Organizations: Communication and Authority in a Dynamic Environment," *Working Paper.*

[14] Harford, Jarrad, and Kai Li (2007): "Decoupling CEO Wealth and Firm Performance: The Case of Acquiring CEOs," *Journal of Finance* 62, 917-949.

[15] Inderst, Roman, and Marco Ottaviani (2013): "Sales Talk, Cancellation terms and the Role of Consumer Protection," *Review of Economic Studies*, 1-25

[16] Kim, Kyungmin, and Philipp Kircher (2015): "Efficient Competition through Cheap Talk: The Case of Competing Auctions," *Econometrica* 83, 1849-1875.

[17] Kos, N. (2012): "Communication and Efficiency in Auctions," *Games and Economic Behavior,* Vol. 75, 233-249.

[18] Krishna V. and J. Morgan (2004): "The Art of Conversation: Eliciting Information from Experts through Multi-Stage Communication," *Journal of Economic Theory*, Vol. 117, 147-179.

[19] Maskin, E., and J. Riley (2000): "Asymmetric Auctions," *Review of Economic Studies*, 413-438.

[20] Milgrom, P., and R. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 1089-1122.

[21] Morris, S. (2001): "Political Correctness," *Journal of Political Economy*, Vol. 109, 231-265.

[22] Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, Vol. 6, 58-73.

[23] Ottaviani, M. and P. Sorensen (2006a): "Professional Advice," *Journal of Economic Theory*, Vol. 126, 120-142.

[24] Ottaviani, M. and P. Sorensen (2006b): "Reputational Cheap Talk," *RAND Journal of Economics*, Vol. 37, 155-175.

[25] Riley, J., and W. Samuelson (1981): "Optimal Auctions," *American Economic Review*, 381-392.

[26] Savage, L. (1954): *The Foundations of Statistics*, John Wiley & Sons Inc., New York.

[27] Simon, Leo K., and Maxwell B. Stinchcombe (1989). Extensive form games in continuous time: Pure strategies, *Econometrica* 57, 1171–1214.

[28] Sobel, J. (1985): "A Theory of Credibility," *Review of Economic Studies*, Vol. 52, 557-573.

[29] Ye, Lixin (2007). "Indicative bidding and a theory of two-stage auctions." *Games and Economic Behavior* 58, 181-207.