Decentralized Exchange*

Semyon Malamud† and Marzena Rostek‡

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Abstract

This paper develops an equilibrium model of decentralized trading that accommodates any coexisting exchanges including networks and more general, common market structures represented by hypergraphs. The model allows for any number of strategic traders and multiple divisible assets. We characterize equilibrium and welfare, and develop comparative statics with respect to preferences, assets, and market structures. Changes in market structure that increase price impact may increase utility of every agent. Equilibrium utility in a decentralized market can be strictly higher in the Pareto sense than in a centralized market with the same traders and assets. Agents with larger price impact may have higher equilibrium utility. Asset substitutability, or complementarity, is not determined by the primitive payoff covariance, but is endogenous and may differ across agents, depending on their participation in the exchanges.

JEL Classification: D53, G11, G12

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1 Introduction

In the classical theory, markets are centralized. All units are exchanged through a single market clearing at terms of trade that apply to all agents equally. In contemporaneous markets, many types of assets are traded increasingly, or exclusively, *over the counter* (OTC). Trade away from open exchanges is common not only for assets with units as heterogeneous as real estate, but most bonds – government, municipal, and corporate – are traded over the counter, as are asset-backed securities, derivatives, loans, and foreign exchange (e.g., Duffie (2012)). Likewise, stock trading occurs in many different trading venues. Over the past decade and a half, new electronic markets have also emerged that offer different types of market clearing (e.g., direct matching with an intermediary, trading in

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†Swiss Finance Institute, EPF Lausanne; E-mail: semyon.malamud@epfl.ch.

‡University of Wisconsin-Madison, Department of Economics; E-mail: mrostek@ssc.wisc.edu.

a dealer network, or an online centralized exchange) to different types of traders, institutional and retail.[1] Moreover, markets are decentralized simply because different participants trade different assets.[2] To the extent that more types of trading venues other than open exchanges are being created and trading volumes outside of open exchanges have increased several-fold over the past couple of decades, this suggests efficiency gains associated with decentralization of trade.[3] The goal of this paper is to examine the potential of market decentralization to improve efficiency. We aim to identify economic mechanisms that arise in decentralized- (but not centralized-) trade settings.

The growing literature on decentralized markets has emphasized frictions due to decentralization of trade: a search friction in the time needed to find a trading partner, a friction in aggregating private information, or counterparty risk externality. We develop an equilibrium model of decentralized trading for arbitrary market structures with coexisting exchanges and any form of limited participation that encompasses centralized markets, networks, and more general common market structures. An arbitrary number of strategic traders with heterogeneous preferences and endowments trade multiple divisible risky assets in a CARA-Normal setting. The sole assumption of the centralized frictionless model which we relax is that a single market clearing determines all agents' allocations. Agents can trade assets in many exchanges, each with a separate clearing price. An *exchange*, defined by the subset of agents who trade and the subset of assets traded, operates as the classical uniform-price double auction (e.g., Kyle (1989); Vives (2011)). Agents submit (net) demand schedules in the exchanges in which they participate. A set of exchanges defines the *market structure* and corresponds to a hypergraph.[4] Our results hold under complete and incomplete information about trader values and initial holdings with independent private values.[5] Next, we describe the main predictions.

In general, agents have strictly positive price impact and, thus, are nonnegligible in trade. In particular, if the market includes a perfectly liquid exchange, traders' equilibrium price impact is,

[1] E.g., BondDesk, BrokerTec, eSpeed, MarketAxess, and TradeWeb.

[2] Even large financial institutions typically participate in only a few trading venues and trade a small subset of the existing securities; e.g., pension funds cannot trade many types of derivatives and most hedge funds have a clear specialization in trading a limited number of securities.

[3] In the U.S. bond market, the outstanding debt nearly tripled over the last two decades, reaching approximately \$3.7 trillion in 2012 (*SIFMA, U.S. Bond Market Statistics, June 2012*). The outstanding gross notional of OTC derivatives was \$639 trillion as of June 2012, with a gross market value of \$25 trillion (`http://www.bis.org/statistics/derstats.htm`).

The U.S. equity market structure has also changed significantly in recent years. Today, the NYSE executes less than a quarter of the volume in its listed stocks; the remaining volume is created in over 10 public exchanges, more than 30 liquidity pools, and through over 200 broker-dealers. About 30% of the volume of U.S.-listed equities is executed in non-public markets that do not display their liquidity; this percentage has increased nearly monthly. Over the past few years, liquidity pool trading in the U.S. has grown by more than 50%, while in Europe, trading in liquidity pools relative to order-book activity has more than doubled (Schapiro (2010)). Similarly, while prior to 2007, equity markets in Europe were characterized by dominant exchanges in each domestic market, a regulatory reform of MiFID in 2007 led to the creation of more than 200 new trading venues in which equities, bonds, and even derivatives are traded. As of 2012, these alternative venues accounted for at least 30% of the total equity turnover.

[4] A hypergraph generalizes a graph by allowing an edge to connect any number of nodes, not just two (see Berge (1973)). Agents can participate in many different types of trading venues for possibly non-disjoint subsets of traders (such as a centralized exchange, a dealer network, liquidity pools), some of which may involve multilateral clearing. Nodes may reside on many graphs (e.g., traders may participate in different types of trading venues), and edges may originate and point to different graphs (e.g., there can be intermediaries between different types of trading venues).

[5] With divisible goods, equilibrium is invariant to the distribution of independent private uncertainty.

in general, positive in other exchanges in which they participate. In decentralized markets with divisible assets, traders' equilibrium schedules and price impact – and hence, prices and allocations – in any exchange depend on the preferences and assets in all exchanges. The same assets may trade at different prices in different exchanges. We give necessary and sufficient conditions on the market structure for an asset to trade at the same price in different exchanges. A decentralized market CAPM holds: In each exchange, expectations of asset payoffs lie on a security-market line defined by an agent-specific portfolio. With many assets, agents diversify risk through different (and multiple) funds, which depend on the agents' participation in exchanges. While the market structure is taken as a primitive in the analysis, we show that, in any decentralized market, equilibrium is as if the hypergraph were a forest (in graph-theoretic terms, a disjoint union of trees) with respect to agents and assets.[6]

Changes in the market structure that lower price impact (i.e., improve liquidity) in an exchange lower the price impact in other exchanges, even if the exchanges are linked only indirectly through a sequence of counterparties. In any decentralized market, creating a new exchange (weakly) lowers the price impacts in all exchanges, whereas breaking up an exchange increases price impact. Thus, creating private exchanges separately from public ones improves the liquidity in all exchanges.[7]

In contrast to centralized markets, no general link exists between equilibrium price impact and utility in decentralized markets. Changes in market structure that increase price impact may also increase equilibrium utility of every agent. Restricting trader participation or creating a new exchange for a subset of traders and assets may improve or reduce welfare in the Pareto sense. Equilibrium utility in a decentralized market can be strictly higher for every agent compared to that in a centralized market with the same set of traders and assets. We show that welfare implications of exchange creation, intermediation, and a trader's position in the market, as measured by participation in exchanges, depend on the distribution of aggregate and idiosyncratic risk in traders' initial holdings. Apart from improving liquidity of the market, private exchanges create value when some market participants have idiosyncratic risks that cannot be fully hedged by the assets available in public exchanges, which serve only "sufficiently standardized" assets, or when trade involves products that are standardized but illiquid.

Agents' participation in the market alters the residual payoff riskiness of assets traded in an exchange. In equilibrium, agents trade as if the riskiness of assets differed from the primitive payoff covariance and were heterogeneous across agents, even if information is complete and all primitives

[6] Forests encompass markets with various forms of intermediation, including brokers, dealers, or specialists who trade different assets as well as non-specialist intermediaries, and the empirically common core-periphery and hub-and-spoke architectures (e.g., the U.S. Federal Funds Market, the interbank markets, the U.S. municipal bonds market; see Bech and Atalay (2010); Craig and Peter (2010); Afonso, Kovner, and Schoar (2012)).

[7] "The competition induced by the proliferation of alternative trading venues such as dark pools has been a mostly positive development, as bid-offer spreads and trading costs have fallen" (*Financial Times*, Jan 6, 2013, Rhodri Preece, Director of Capital Markets Policy at CFA Institute). Competition among exchanges has substantially decreased liquidity costs in many markets. For example, in the U.S. stock market, after electronic trading platforms decreased the liquidity costs of trading NASDAQ stocks, the SEC adopted Regulation National Market System (NMS) in 2005, which cleared regulatory impediments to electronic trading, further increasing competition among exchanges. Similar decreases in trading costs occurred in Canada, Europe, and Asia, where different regulatory environments allowed electronic exchanges to develop earlier than those in the United States (Knight Capital Group (2010); Angel, Harris and Spatt (2011); and O'Hara and Ye (2011)).

of the market, including the distribution of assets' payoffs, are commonly known. In particular, payoff complementarity or substitutability of the same assets traded in different exchanges is endogenous.

Our analysis identifies the non-commutativity property of equilibrium as the key to the new predictions in decentralized markets. In centralized markets, the equilibrium price impact matrices of traders always commute (across agents and with the covariance matrix). Intuitively, commutativity captures a certain symmetry of centralized market equilibria that naturally does not hold in decentralized markets. Essentially, the available trading opportunities for various assets are the same for different market participants.[8] Not requiring that the trading opportunities – and, thus, terms of trade – be the same across agents is precisely what defines a *decentralized* market in our model. We show that the commutativity of equilibrium is nongeneric in decentralized markets.

Our model is part of the growing literature on decentralized markets. We study trading environments without frictions, which are complementary in the following respects. Most modern models of decentralized markets adopt one of two approaches. The first assumes that trade occurs through random matching in large markets among a continuum of nonstrategic traders (e.g., Gale (1986a,b); Duffie, Garleanu and Pedersen (2005, 2007); Weill (2008); Vayanos and Weill (2008); Duffie, Malamud and Manso (2009, 2013); Lagos and Rocheteau (2009); Lagos, Rocheteau and Weill (2011); and Afonso and Lagos (2012)). Thus, the sets of trading counterparties of any two agents are disjoint with probability one. We consider markets with any number of traders, all of whom are strategic. Empirically, while some markets are best described by random meetings, in others, relationships are not random,[9] and dealing with strategic behavior and price impact often serves as a primary motivation to create an OTC exchange.[10]

In addition, the random matching approach considers markets in which all transactions are bilateral. This is also the case in the strand of literature that views agents as interacting on a fixed network (e.g., Kranton and Minehart (2001); Gale and Kariv (2007); Blume, Easley, Kleinberg and Tardos (2009); Manea (2011); Nava (2011); Condorelli and Galeotti (2012); Elliott (2012); Fainmesser (2012); Babus and Kondor (2013); Bramoullé, Kranton, and D'Amours (2013); and Rahi and Zigrand (2013)).[11] A number of empirically common market structures are not mathematically

[8] Matrices A and B commute if $AB = BA$. Diagonalizable matrices A and B commute if, and only if, they are simultaneously diagonalizable (e.g., Horn and Johnson (2013), Theorem 1.3.12).

[9] It is well documented that dealers or brokers trade via an established network structure and trading relationships exist between banks. An average bank trades with a small number of counterparties and most banks form stable relationships with at least one lending counterparty; e.g., the U.S. Federal Funds market (Bech and Atalay (2010); Afonso, Kovner, and Schoar (2012)), interbank markets (Craig and Peter (2010); Cocco, Gomes and Martins (2009)), and the U.S. municipal bonds market (Li and Schürhoff (2012)).

[10] E.g., Knight Capital Group (2010), Angel, Harris and Spatt (2011). In the study of the competition between Island and Nasdaq, Biais, Bisière, and Spatt (2010) conclude that, even when trading platforms actively compete for order flow and quasi free entry exists, the outcomes are not perfectly competitive. Although retail traders account for a significant fraction of the number of orders placed, only a limited number of large professional traders monitored Island and Nasdaq, and strategically undercut Nasdaq quotes to earn trading profits. Biais and Green (2007) attribute the historical shift in the U.S. municipal and corporate bond markets toward OTC trading to the increased importance of institutional investors.

[11] Analyzing subgame perfect Nash Equilibria, Corominas-Bosch (2004) and Elliott (2011) allow for multilateral bargaining with search. Some models (see Duffie, Garleanu and Pedersen (2005); Lagos and Rocheteau (2009); and Lagos, Rocheteau and Weill (2011)) assume that trade can only happen through special intermediaries (dealers) who

accommodated by graphs – random or fixed – without loss.[12]

Finally, both types of models typically derive the terms of trade from Nash bargaining (e.g., take-it-or-leave-it offers) or fixed prices.[13] In our model, prices are determined by the standard divisible-good double auction protocol in which agents submit demand or supply schedules (or limit and market orders) in the exchanges where they participate; the schedules determine the prices that clear trades. Auctions are increasingly used in OTC markets. Traditionally, trading over the counter involved bilateral market clearing with a dealer over the phone. Over the past two decades, large volumes of transactions that occur outside open exchanges have shifted to new electronic auctions, which have been introduced for large institutional traders, dealers, and retail investors.[14] In terms of predictions, with Nash bargaining, surplus sharing is efficient, which is not without loss of generality for the determination of terms of trade of divisible assets when agents are strategic. In our double-auction model, traders have nonnegligible price impact through which surplus sharing in any given exchange is affected by all preferences and assets and trader participation in the market as a whole. Furthermore, models based on Nash bargaining typically assume that, upon meeting, agents reveal their private types – information is symmetric in trade.[15] Our model allows for the incorporation of interdependent values. In this paper, gains to trade come from risk sharing: endowments are heterogeneous, and agents' risk preferences and the assets they trade can be heterogeneous as well.

This paper offers two technical contributions. First, for economies with arbitrary decentralized market structures, multiple assets[16] and any, finite number of strategic agents, we characterize equilibrium and develop comparative statics of liquidity and welfare with respect to preferences, assets, and market structure. The assumed CARA utility functions and Normal distributions of asset payoffs give tractability to our model. We show that the positive semidefinite order emerges as the natural order for studying monotone comparative statics with respect to endogenous matrices that live in different subspaces, as determined by the primitive assets and decentralized market

provide liquidity. Rahi and Zigrand (2013) study trade of price-taking investors intermediated by arbitrageurs.

[12] A data set that codes which pairs of agents trade but offers no information on whether the group of agents who define the pairs trades or whether the group trades with other groups will, in general, introduce biases in inference.

[13] Like this paper, Babus and Kondor (2013) use a double auction. Traditional network optimization focuses on a network populated by nonstrategic ("obedient") users; more recently, pricing (e.g., Kranton and Minehart (2001); Blume, Easley, Kleinberg, and Tardos (2009); Nava (2011); Jackson and Zenou (2012) and references therein; and Babus and Kondor (2013)), or bargaining on networks, has been examined.

[14] The ongoing transition in the market clearing itself is sometimes called "call to electronic"; e.g., Hendershott and Madhavan (2012) examine the changes in the U.S. corporate bonds market. Innovations in trading technology allow traders to engage in multilateral trading as opposed to sequentially contacting dealers.

[15] Within the random-matching approach, a strand of literature studies OTC markets allowing for common value uncertainty (e.g., Wolinsky (1990); Blouin and Serrano (2001); Lorenzoni, Golosov and Tsyvinski (2008); Duffie, Malamud and Manso (2013)). To avoid the complexities of Nash bargaining with incomplete information, the authors either assume exogenous prices (as in Wolinsky (1990) and Blouin and Serrano (2001)) or a single unit double auction with risk neutrality and no inventory constraints (as in Duffie, Malamud and Manso (2013)). One exception is Lorenzoni, Golosov and Tsyvinski (2008), who consider a one-shot take-it-or-leave-it bargaining game. All of these papers also assume a continuum of nonstrategic agents. Some authors define utility over holdings rather than consumption (e.g., Lagos and Rocheteau (2009)); then, the observable private type of an agent contains his endowment.

[16] Weill (2008) allows for multiple assets, while assuming that, at every point in time, each agent can hold only one asset, and holdings are restricted to one unit. Our model permits unrestricted holdings with any number of assets and determines the optimal portfolio allocations.

structures. Absent commutativity, the set of symmetric matrices equipped with this order is not a lattice, and agents' payoffs are not supermodular. For the comparative statics in centralized markets, lattice theoretic arguments can be invoked. For the comparative statics in decentralized markets, we use methods from the theory of the shorted operators (e.g., Anderson (1971)) – in particular, the concavity and monotonicity properties of shorted operators – and monotone matrix functions (e.g., Donoghue (1974)). We show that there exist unique maximal and minimal equilibria.

Second, this paper contributes to the growing literature on games on networks. More generally, to the best of our knowledge, this paper is the first to study the class of games on hypergraphs. Relative to the literature on networks, ours are games with nonlinear best responses; action space is the set of positive semidefinite matrices rather than scalars; and, in equilibrium, agents' actions are functions of actions of all players in the market, not only direct neighbors, even in the one-period model. In particular, static equilibrium prices and allocations in any exchange generally depend on the primitives in the whole market. With nonlinear and nonlocal effects, the techniques from the linear best responses are inapplicable and summary statistics, such as the smallest eigenvalue (e.g., Bramoullé, Kranton, and D'Amours (2013); the *Handbook* chapter by Jackson and Zenou (2012)) do not suffice to capture the properties of the network.

The paper is organized as follows: Section 2 presents the model of decentralized markets. Section 3 characterizes equilibrium. Section 4 introduces an equivalent characterization of equilibrium. Section 5 studies equilibrium liquidity, allocations and prices. Section 6 derives the comparative statics of equilibrium liquidity. Section 7 characterizes risk sharing and welfare in decentralized markets. Section 8 concludes. All proofs appear in the Appendix.

2 A Decentralized Market Model

2.1 Notation

Unless otherwise noted, all vectors are column vectors. We denote the transpose of a matrix A by A^T. A set and the number of elements in the set are denoted by the same symbol. $\mathbf{1}_{|M|\times|M'|}$ is the $|M| \times |M'|$ matrix with all elements equal to one. $\otimes$ is the Kronecker product; for an $n \times n$ matrix A and an $m \times m$ matrix B, $A \otimes B \equiv (a_{ij}B)_{i,j} = (a_{ij}b_{kl})_{i,j,k,l}$ is an $(nm) \times (nm)$ matrix. $\oplus$ denotes the direct sum. We write $A_{N(i)} \equiv \Pi_{N(i)} A \lceil_{\mathbb{R}^{N(i)}}$, where $\Pi_{N(i)}$ is the orthogonal projection onto space $N(i)$ and $\lceil$ is the restriction symbol. That is, $A_{N(i)}$ is the sub-matrix of A with rows and columns in $N(i)$. For symmetric matrices A and B, we write $A \geq B$ if $A - B$ is positive semidefinite, and $A > B$ if $A - B$ is positive definite. We use $\text{eig}(A) = \{\mu_1(A) \geq \cdots \geq \mu_m(A)\}$ to denote the eigenvalues of a symmetric $m \times m$ matrix A ordered to be decreasing.

2.2 Decentralized Market Setting

I classes of agents with $M_i \geq 1$ agents in each class i trade K risky assets. The payoffs $R = \{R_k\}_k$ of risky assets are jointly normally distributed with the vector of expected payoffs δ and the (symmetric and positive definite) covariance matrix Σ. In addition, a risk-free asset with a zero interest rate (a numéraire) is available.

Agents trade the K risky assets in N exchanges, each with a separate clearing price. An *exchange* $n \in N$ is identified by the subset of agents $I(n) \subset I$ who trade there and the subset of assets traded $K(n) \subset K$; we take the set of exchanges $\{(I(n), K(n))\}_n$ as a primitive and index agent classes by i and assets by k. Thus, the set (and number) of exchanges is $N \subset 2^I \times 2^K$. We assume that at least three agents participate in every exchange, $\sum_{i \in I(n)} M_i > 2$ for all n.[17]

We treat assets traded at different exchanges as different assets, regardless of whether the same asset is traded. That is, we treat a market with K assets traded at N exchanges as a market with $\bar{K} \equiv \sum_n K(n)$ (replicas of) assets. Let d and $\mathcal{V} = (\nu_{k,l})_{k,l=1}^{\bar{K}}$ be the $\bar{K} \times 1$ vector of expected payoffs and the $\bar{K} \times \bar{K}$ positive semidefinite covariance matrix of assets traded in different exchanges, induced by Σ and the set of exchanges $N = \{(I(n), K(n))\}_n$. Let $\bar{R}$ be the corresponding $R^{\bar{K}}$-dimensional random vector of payoffs. A replica of asset k is traded in exchanges $N(k)$, $\sum_k N(k) = \bar{K}$. If at least one asset k is traded at more than one exchange, matrix $\mathcal{V}$ is degenerate.

The general setting with private exchanges admits trading environments more general than the standard link geometry on the set of I classes represented by a graph of unordered pairs of the I classes (a *network*). In contrast, the set of exchanges $\{(I(n), K(n))\}_n$ generates a graph on the set of N exchanges; the set of exchanges corresponds to a hypergraph[18] on the set of I agent classes and K assets. The hypergraph $((I, K), \{(I(n), K(n))\}_n) = (I, K, N)$ represents the *market structure*.

Example 1 *The model encompasses the following canonical market structures.*
(i) The centralized market*: All I agent classes trade K assets at one exchange (a* public exchange*);* $N = \{(I, K)\}$*, and* $\mathcal{V} = \Sigma$*.*

(ii) The bilateral OTC market*: Exactly two classes trade at each exchange. The set of exchanges* N^{OTC} *can be described by a network,*

$$N^{OTC} \equiv \{\big((i,j), K(i,j)\big) : i, j \in I,\ K(i,j) \subset K,\ i \textit{ trades assets from } K(i,j) \textit{ with } j\}.$$

The number of traded (replicas of) assets is

$$\bar{K}^{OTC} = \sum_{(i,j) \in N^{OTC}} |K(i,j)|.$$

When N^{OTC} *is a complete network, each class can trade bilaterally with every other class and there are* $I(I-1)/2$ *exchanges. If* $K(i,j) = K$ *for any exchange* (i,j)*, then* $\mathcal{V} = \Sigma \otimes \mathbf{1}_{N^{OTC} \times N^{OTC}}$*.*

(iii) Coexisting OTC and centralized markets*: In addition to the centralized (public) exchange as*

[17] As is well known in centralized markets, with two traders, equilibrium with trade does not exist (e.g., Kyle (1989)).

[18] A hypergraph is defined as a pair (X, E), where X is a set of elements called *nodes* and E is a set of nonempty subsets of X called *(hyper-)edges*. Thus, E is a nonempty subset of the power set of X. In our model, $X = (I, K)$ and an edge $(I(n), K(n))$ represents exchange n with $I(n)$ agent classes and $K(n)$ assets.

in (i), (a subset of) agents can trade the exchange-listed securities over the counter. The OTC market is determined by network N^{OTC} *as in (ii);* $N = \{(I, K)\} \cup N^{OTC}$ *and* $\bar{K} = K + \bar{K}^{OTC}$.

(iv) Liquidity pools*: In addition to the centralized exchange as in (i), there are L exchanges (liquidity pools) in which only subsets of agents can trade. There are L (possibly nondisjoint) subsets* $I(1), \cdots, I(L) \subset I$ *of agents, each trading in exchange* $l \in L$ *organized as a centralized exchange among agent classes* $i \in I(l)$*;* $N = \{(I, K)\} \cup \{(I(l), K(l))\}_l\}$ *and* $\bar{K} = K + \sum_l K(l)$.

The model admits a "limited participation" interpretation: Agents in class i who participate in a subset $N(i)$ of exchanges trade $\sum_{n \in N(i)} K(n)$ assets. For convenience, we use $\mathbb{R}^{N(i)}$ to denote the space in which the initial holdings and the schedules of agent i live, which has dimension $\sum_{n \in N(i)} K(n)$. An agent from class i is endowed with wealth w_i and a vector $q_i^0 \in \mathbb{R}^{N(i)}$ of initial risky asset holdings[19] and maximizes the expected CARA utility

$$E[-e^{-\alpha_i (w_i - q_i^T p_{N(i)} + (q_i^0 + q_i)^T \bar{R}_{N(i)})}],$$

where α_i is agent i's absolute risk aversion, $q_i \in \mathbb{R}^{N(i)}$ denotes agent i's traded quantities of the risky assets and $p_{N(i)}$ is the vector of prices. Initial holdings are private information with a nondegenerate distribution independent across traders. The agent's problem is equivalent to maximizing the quasilinear-quadratic utility function of after-trade risky portfolio

$$U(q_i) = (q_i^0)^T d_{N(i)} + q_i^T (d_{N(i)} - p_{N(i)}) - \frac{\alpha_i}{2} \left(q_i^0 + q_i\right)^T \mathcal{V}_{N(i)} \left(q_i^0 + q_i\right), \tag{1}$$

where $d_{N(i)}$ is the sub-vector of the expected payoffs d and $\mathcal{V}_{N(i)}$ is the sub-matrix of the covariance matrix $\mathcal{V}$ that corresponds to the assets traded by agent i. We use $q_i \in \mathbb{R}^N$ and $q_i^0 \in \mathbb{R}^N$ to also denote the vectors $q_i \in \mathbb{R}^{N(i)}$ and $q_i^0 \in \mathbb{R}^{N(i)}$ 'completed' by zeros in their $\mathbb{R}^{N \setminus N(i)}$ coordinates. We use

$$\mathbb{M} \equiv \{\{\alpha_i, M_i\}_i, (K, \delta, \Sigma), \{(I(n), K(n))\}_n\}$$

to describe the *market* and $\{N(i)\}_i$ its *participation*. Equivalently, given the "limited participation" interpretation, $\mathbb{M} = \{\{\alpha_i, M_i, N(i)\}_i, (K, d, \mathcal{V})\}$.

Each exchange n is organized as a centralized market for trader classes $i \in I(n)$ and operates as the classical uniform-price mechanism (i.e., a double auction; e.g., Kyle (1989); Vives (2011)). Trader i submits a (net) demand schedule $q_i(p_{N(i)}) : \mathbb{R}^{N(i)} \to \mathbb{R}^{N(i)}$, which specifies demanded quantities of assets in the exchanges in which he participates, where $p_{N(i)}$ is the price vector of assets in exchanges $N(i)$; the demand is strictly downward-sloping in each exchange.[20] The aggregate net demand in exchange n determines the clearing price p_n^* in this exchange, $\sum_{i \in I(n)} q_{i,n}(p_n^*, p_{N(i)\setminus\{n\}}^*) = 0$, and the allocations; trader i receives $q_i \equiv q_i(p_{N(i)}^*)$ and pays $p_{N(i)}^* \cdot q_i$. All traders are strategic; in particular, there are no noise traders.

[19] Without loss of generality, agents in each class i have identical risk aversion and initial holdings. One can always redefine the partition into classes to achieve this within-class symmetry. Since an agent's demand coincides with the average demand in the class, we use index i to denote agent-specific variables for any agent in class i.

[20] This rules out trivial equilibria with no trade.

2.3 Equilibrium

We are interested in Nash equilibrium behavior in decentralized markets: Each agent i submits a (net) demand schedule $q_i(p_{N(i)})$ in exchanges $N(i)$, which is optimal, given the schedules of all traders $j \neq i$ in all exchanges $n \in N$, $\{q_j(p_{N(j)})\}_{j\neq i}$. As is standard in centralized markets models with quasilinear-quadratic utilities, we study the linear equilibrium that is robust to adding noise in trade (*robust Nash Equilibrium*, hereafter, *equilibrium*) which, with independent private values about endowments, coincides with the linear Bayesian Nash Equilibrium.[21]

Theorem 2.1 provides an alternative characterization of equilibrium through two conditions, which correspond to an individual optimization problem and aggregation in decentralized markets. In a (robust Nash) equilibrium, the (net) demand schedule of trader i equalizes his marginal utility with the marginal payment, for each price,

$$d_{N(i)} - \alpha_i \mathcal{V}_{N(i)} \left(q_i^0 + q_i\right) = p_{N(i)} + \Lambda_i q_i, \tag{2}$$

where Λ_i is the $N(i) \times N(i)$ Jacobian matrix of the residual inverse supply of trader i, which is defined through market clearing by the schedules submitted by the traders in exchanges $N(i)$, $\{q_j(p_{N(j)}) : \mathbb{R}^{N(j)} \to \mathbb{R}^{N(j)}\}_{j\neq i}$. Λ_i measures the *price impact* of trader i in the exchanges in which he participates (i.e., a decentralized market counterpart of 'Kyle's lambda'). Entry (k,l) represents the price change of asset l that results from a marginal increase in demanded quantity of asset k, $\frac{\partial(p_1,\cdots,p_{N(i)})}{\partial(q_1,\cdots,q_{N(i)})} = \left(\frac{\partial p_l}{\partial q_k}\right)_{k,l}$. It follows from (2) that *if* trader i knew his price impact Λ_i, which is endogenous, he could determine his equilibrium demand by equalizing his marginal utility and marginal payment pointwise: Let $q_i\left(\cdot, \Lambda_i\right) : \mathbb{R}^{N(i)} \to \mathbb{R}^{N(i)}$ be the schedule defined by the pointwise optimization (2) for all prices $p_{N(i)}$, given his assumed price impact Λ_i. If the matrix $\alpha_i \mathcal{V}_{N(i)} + \Lambda_i$ is positive definite, we have

$$q_i(p_{N(i)}, \Lambda_i) = (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}(d - p_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0); \tag{3}$$

otherwise, the requirement that the slope of the demand schedule be symmetric, positive semidefinite still uniquely determines the demand schedule as in (3) on the image of $\alpha_i \mathcal{V}_{N(i)} + \Lambda_i$, with $(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$ replaced by the Moore-Penrose generalized inverse of the matrix $(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)$.[22]

[21] Equilibrium is *linear* if schedules have the functional form of $q_i(\cdot) = \alpha_0 + \alpha_{i,q} q_i^0 + \alpha_{i,p} p$. Trader strategies are not restricted to linear schedules; rather, we analyze equilibria in which it is optimal for a trader to submit a linear demand, given that others do. In centralized market models, the approach to analyze the symmetric linear equilibrium (i.e., a profile of demand schedules in which it is optimal for each player to bid linearly, given that others do, with the strategy space not restricted to linear functions) is common in the literature following Kyle (1989); our analysis does not assume equilibrium symmetry. Equilibrium in our model is *ex post* Bayesian Nash; that is, the strategies $\left\{q_i\left(p_{N(i)}\right)\right\}_i$ are optimal even if the traders learn the types (endowments or marginal utility intercepts) of all other agents. The key to the *ex post* property is that by permitting pointwise optimization – for each price – the equilibrium demand schedules are optimal for any distribution of independent private information and are independent of agents' expectations about others' holdings and private valuations. Thus, all results in this paper hold for complete information and incomplete information with independent values.

[22] I.e., a trader's optimization problem has a solution if, and only if, in equilibrium, vector $d - p_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0$ belongs to the image of $\alpha_i \mathcal{V}_{N(i)} + \Lambda_i$. For example, if the first two rows (and columns) of $\mathcal{V}_{N(i)}$ correspond to replicas of the same asset and the first two rows of $\alpha_i \mathcal{V}_{N(i)} + \Lambda_i$ coincide, this condition requires that the replica assets trade at the same prices. Section 4 characterizes when this is the case and show that demand (3) is always well defined.

In the sequel, we use A^{-1} to denote the generalized inverse of matrix A if it is not invertible.

Equilibrium price impacts $\{\Lambda_i\}_i$ live in different assets, are of different dimensionality and, in general, are not independent across exchanges. Therefore, the market clearing condition, and hence the equilibrium condition for price impacts $\{\Lambda_i\}_i$, cannot be written exchange by exchange. We use the procedure of *lifting* to apply market clearing to all assets in all exchanges. For a given subset $N(i) \subset N$, decompose $\mathbb{R}^N = \mathbb{R}^{N(i)} \oplus \mathbb{R}^{N \setminus N(i)}$ as a direct sum of two subspaces corresponding to coordinates that agent i trades and those that he does not trade, where $\mathbb{R}^N$ is the $\bar{K}$-dimensional space of asset holdings; $\bar{K} = \sum_{n \in N} K(n)$. Any symmetric matrix X can be decomposed into a block form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix},$$

where $X_{11} = X_{N(i)}$ lives in subspace $\mathbb{R}^{N(i)}$, $X_{22} = X_{N \setminus N(i)}$ lives in the complementary subspace $\mathbb{R}^{N \setminus N(i)}$, and X_{12} is a rectangular block. For any matrix X_{11} living in $\mathbb{R}^{N(i)}$, with an abuse of notation, let $\bar{X}_{11}$ denote the *lifted* matrix, which lives in $\mathbb{R}^N$

$$\bar{X}_{11} = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

and, with an abuse of notation, denote its corresponding Moore-Penrose generalized inverse by

$$(\bar{X}_{11})^{-1} = \begin{pmatrix} X_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Treating assets traded at different exchanges as distinct assets and dealing with aggregation through lifting allows us to characterize (robust Nash) equilibria in the general model of decentralized markets by two conditions on schedules and price impacts: (i) each trader submits a schedule that equalizes his marginal utility and the marginal payment given his assumed price impact (i.e., submits $q_i(\cdot, \Lambda_i)$) and (ii) the price impact Λ_i assumed by trader i is correct (i.e., it equals the slope of the residual supply resulting from the aggregation of other traders' schedules, projected on the assets relevant for trader i).

Theorem 2.1 (Equilibrium Characterization) *A profile $\{q_i(p_{N(i)})\}_i$ is a robust Nash Equilibrium in a decentralized market $\mathbb{M} = \{\{\alpha_i, M_i, N(i)\}_i, (K, d, \mathcal{V})\}$ if, and only if,*

(i) each agent i submits $q_i(p_{N(i)}, \Lambda_i) = (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}(d_{N(i)} - p_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0)$, and

(ii)

$$\Lambda_i = \left(\left((M_i - 1)(\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i)^{-1} + \sum_{j \neq i} M_j (\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)}, \quad i \in I. \tag{4}$$

Although the matrices $\mathcal{V}_{N(j)}$ and Λ_j are for "local" exchanges, when lifted, the same aggregation condition as in a centralized market applies. Theorem 2.1, thus, allows a direct comparison of equilibrium predictions in decentralized and centralized markets.[23] With decentralized trading, the new structure of equilibrium has to do with how the (primitive and endogenous) variables from all exchanges affect bidding behavior and equilibrium in the exchanges in which a trader participates.[24]

By Theorem 2.1, despite the complexity of the considered market structures, to trade optimally, each agent needs to know little about exchanges other than those in which he participates. Theorem 2.1 shows that, if a trader knows only his *own* utility, then if he knows his own price impact in the exchanges he trades, the opacity of decentralized markets is without loss of generality for trader optimization and equilibrium in a decentralized market – in trading environments with independent private values. In particular, a trader's strategy $q_i(\cdot, \Lambda_i)$ would not be altered by knowledge of the market structure (the hypergraph), the terms of trade in exchanges $N \setminus N(i)$, or even the submitted schedules, price impacts, preferences or identities of those traders. Two aspects are central to the reduction in the sufficient statistic that player i conditions on (i.e., $\Lambda_i \in \mathbb{R}^{N(i)\times N(i)}$ being sufficient for $\{q_j(\cdot) : \mathbb{R}^{N(j)} \to \mathbb{R}^{N(j)}\}_{j\neq i}$). First, downward-sloping demand schedules (i.e., negative definite $-(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$) enable agents to make choices contingent on prices. Second, while each trader's i residual supply is defined by the schedules submitted only in exchanges $N(i)$, his equilibrium price impact Λ_i in these exchanges depends on all exchanges, through the fixed-point condition (4).

2.4 Centralized Markets

Throughout, a centralized market (Example 1 (i)) serves as a benchmark: All traders participate in a single exchange, $N(i) = N = \{(I, K)\}$ and $\mathcal{V}_{N(i)} = \Sigma = \mathcal{V}$, for all $i \in I$.

Proposition 2.1 (Centralized Market Equilibrium) *In a centralized market, equilibrium exists and is unique. Price impact Λ_i of agent i is proportional to the covariance matrix $\mathcal{V}$,*

$$\Lambda_i^{\text{CM}} = \beta_i \mathcal{V},$$

with a coefficient β_i that depends on risk aversion α_i, $\beta_i = (2 - \alpha_i b + \sqrt{(\alpha_i b)^2 + 4})/2b$, and $b \in \mathbb{R}_+$ is the unique solution to $\sum_i M_i (2 + \alpha_i b + \sqrt{(\alpha_i b)^2 + 4})^{-1} = 1/2$. Agent i's equilibrium risky holdings are

$$q_i = \frac{1}{\alpha_i + \beta_i} q^{Av} + \frac{\beta_i}{\alpha_i + \beta_i} q_i^0,$$

[23] The harmonic mean is standard and results from aggregation of individual demands; for instance, in a centralized market with I strategic traders, the price impact of each trader i is

$$\Lambda_i = \left(\sum_{j \neq i} (\alpha_j \mathcal{V} + \Lambda_j)^{-1} \right)^{-1}.$$

[24] For centralized markets, the linear equilibrium coincides with Kyle (1989, without nonstrategic traders and assuming independent values), Vayanos (1999), Vives (2011), Weretka (2011) who provides a nonstrategic characterization of the equilibrium (i.e., in terms of levels rather than demand schedules) for centralized market general equilibrium settings and Rostek and Weretka (2011) who show the relationship in Theorem 2.1 for centralized market games. The linear-in-trade price impact function is the prevalent assumption in the financial industry (the "quadratic cost model;" see, e.g., Almgren (2009)).

where the average portfolio q^{Av} *is given by* $q^{Av} \equiv b^{-1}\sum_i \frac{\alpha_i}{\alpha_i+\beta_i} Q_i^0$, *where* Q_i^0 *are the total initial holdings of class* i.

If $\alpha_i = \alpha$ *for all* i, *then* $b = M(M-2)/(\alpha(M-1))$, *where* $M = \sum_i M_i$ *is the total number of agents in the market, and* $\Lambda_i = (1/(M-2))\alpha\mathcal{V}$.[25]

Due to positive price impact, allocations retain some exposure to idiosyncratic risk. Agents' risky asset holdings comprise a combination of the (per capita) market portfolio q^{Av} and initial endowment q_i^0; the two-fund separation holds. More risk averse agents face more elastic residual supply and submit more competitive schedules; if $\alpha_i > \alpha_j$, then $\beta_i < \beta_j$ and $\alpha_i + \beta_i > \alpha_j + \beta_j$.

3 Equilibrium: General Properties

We establish the existence and uniqueness properties of the decentralized markets equilibrium $\{(q_i(\cdot, \Lambda_i), \Lambda_i)\}_i$. For centralized markets, the (double auction) game is supermodular. This follows from the proportionality of matrices $\{\{\Lambda_i\}_i, \mathcal{V}\}$, which implies that the set of price impact tuples $\{\Lambda_i\}_i$ is a lattice. A weaker condition of commutativity is both necessary and sufficient for the lattice structure. Recall that two matrices A and B commute if $AB = BA$. In our decentralized markets model, price impacts do not commute in general (see Proposition 3.3), and the model does not define a supermodular game; absent commutativity, the set of price impacts (i.e., the set $\mathcal{S}^I$ of tuples of positive semidefinite matrices) is not a lattice.[26] Consequently, Tarski's fixed point theorem cannot be applied to prove the existence of equilibrium and comparative statics. Nevertheless, we show that the equilibrium demand slopes of each agent is monotone increasing in demand slopes of all other agents in the market (Lemma 3.2) and that the symmetric positive definite order emerges as the relevant order for monotonicity (Theorem 3.1) and demand slopes (Lemma 3.1). Let $\mathcal{S}^I$ be the set of I-tuples $\{\Lambda_i\}_i$ of positive semidefinite matrices with $\Lambda_i \in \mathbb{R}^{N(i)\times N(i)}$. On this set, we introduce a partial order: $\{\Lambda_i\}_i \leq \{\Lambda_i'\}_i$ for a pair of tuples $\{\Lambda_i\}_i$, $\{\Lambda_i'\}_i$ if $\Lambda_i \leq \Lambda_i'$ for all $i \in I$. Recalling that the negative X_i of the slope of i's demand and i's price impact are linked through $X_i \equiv (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$, we can rewrite the fixed point condition (4) as a fixed point condition for

[25] This is the equilibrium in Rostek and Weretka (2011). To the best of our knowledge, the uniqueness of the linear equilibrium for many assets (divisible goods) in Proposition 2.1 is new. Its proof is based on the following argument. Multiplying Equations (4) from left and right by $\mathcal{V}^{-1/2}$ and denoting $\tilde{\Lambda}_i = \mathcal{V}^{-1/2}\Lambda\,\mathcal{V}^{-1/2}$, we get

$$\tilde{\Lambda}_i = \left(\left((M_i-1)(\alpha_i \mathrm{Id} + \tilde{\Lambda}_i)^{-1} + \sum_{j\neq i} M_j(\alpha_j \mathrm{Id} + \tilde{\Lambda}_j)^{-1}\right)^{-1}\right), \; i \in I. \tag{5}$$

One expects that any solution to (5) is of the form $\tilde{\Lambda}_i = \beta_i \mathrm{Id}$ for some $\beta_i > 0$, $i \in I$, and consequently $\Lambda_i = \beta_i \mathcal{V}$. In the Appendix, we show that this is indeed the case and equilibrium is unique.

[26] It is generally not possible to define the greatest lower bound and the least upper bound for a bounded set of positive semidefinite matrices. For example, consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Note that both $A \not\geq B$ and $A \not\leq B$ hold, because the positive semidefinite order is incomplete. By definition, matrix $C = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is the least upper bound of A and B if $C \geq A$, $C \geq B$, and any other matrix C' satisfying $C' \geq A$, $C' \geq B$ also satisfies $C \geq C'$. However, $C \geq A$, $C \geq B$ is equivalent to $a > 1, b > 2, (a-1)(b-2) \geq \max\{c^2, (c-1)^2\}$. Clearly, one can decrease a and increase b without violating these inequalities, which implies that C cannot be the least upper bound.

demand slopes, as follows.[27] Define map $G = \{G_i\}_i : \mathcal{S}^I \to \mathcal{S}^I$ via

$$G_i(\{X_i\}_i) = \left(\left(\left((M_i - 1)\bar{X}_i + \sum_{j \neq i} M_j \bar{X}_j \right)^{-1} \right)_{N(i)} + \alpha_i \mathcal{V}_{N(i)} \right)^{-1}, \; i \in I. \tag{6}$$

Essentially, the decentralized market model can be seen as a game in which agents choose their demand slopes (equivalently, demand reduction relative to the competitive schedule, $(\alpha_i \mathcal{V}_{N(i)})^{-1} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$). The following result then follows directly from Theorem 2.1.

Lemma 3.1 *A tuple of linear demand schedules with slopes $\{X_i\}_i$ is an equilibrium if, and only if, it is a fixed point of the best response map. That is, $\{X_i\}_i = G(\{X_i\}_i)$.*

It is interesting to compare the (nonlinear) best response map G to the linear best responses familiar from network models (e.g., Bramoullé, Kranton, and D'Amours (2013); the survey by Jackson and Zenou (2013)). In these games, the best response map of each player takes values in $\mathbb{R}$; is linear in the actions of other agents; and depends only on the actions of direct neighbours on the network. None of these properties holds in our model. First, the best response map takes values in the set of positive semi-definite matrices; this set is not even a lattice. Second, the best response map is non-linear and aggregates non-commuting matrices that live on different parts of the hypergraph. Finally, by (6), the best response of agent i depends on the strategies (the slopes of demand schedules) of all agents in the market who are connected indirectly (i.e., through a sequence of trading partners) with i.

We denote by $G^n(\{\Lambda_i\}_i)$ the n^{th} iteration of the best response map. Standard properties of the positive semidefinite order imply that map G is monotone increasing in $\{X_i\}_i$.

Lemma 3.2 *Map G is monotone increasing on $\mathcal{S}^I$.*

We will also need the following result.

Proposition 3.1 (Monotone Convergence) *Pick an arbitrary starting tuple $\{X_i^0\}_i$ such that $\{X_i^0\}_i \leq G(\{X_i^0\}_i)$ ($\{X_i^0\}_i \geq G(\{X_i^0\}_i)$). Then, iterations $G^n(\{X_i^0\}_i)$ are monotone increasing (decreasing) in n and converge to an equilibrium price impact tuple.*

By Proposition 3.1, equilibrium exists if we can find a tuple $\{X_i^0\}_i$ satisfying the inequality conditions. In the Appendix, we explicitly construct such a tuple. Standard arguments then imply that equilibrium exists and is locally unique. Theorem 3.1 summarizes these equilibrium properties.

Theorem 3.1 (Equilibrium Existence and Determinacy) *Equilibrium exists and it is generically determinate. Equilibrium price impacts are positive semidefinite. If the covariance matrix $\mathcal{V}_{N(i)}$ is invertible for any i, the price impacts are positive definite.*

[27] Note that we are assuming that equilibrium is symmetric within a class: all agents from the same class i submit identical schedules. Lemma C.2 in the Appendix implies that this is indeed the case.

Induced by the covariance matrix Σ and the hypergraph $((I,K),\{(I(n),K(n))\}_n)$, matrix $\mathcal{V}_{N(i)}$ can be degenerate only if agents from class i can trade the same asset at different exchanges.[28] While the lattice structure required for Tarski's fixed point theorem is absent, the maximal and minimal fixed points exist. The following is true.

Proposition 3.2 *For any decentralized market $\mathbb{M}$, the set of equilibrium price impact tuples has unique maximal and minimal elements that are given by $\{\Lambda_{i,\max}\}_i$ and $\{\Lambda_{i,\min}\}_i$, respectively.*

Thus, by Proposition 3.2, the set of equilibria $\{(q_i(\cdot,\Lambda_i),\Lambda_i)\}_i$ has a unique maximal and a unique minimal element.[29] In Appendix A, we provide sufficient conditions for the minimal and maximal equilibria to coincide.[30]

Our analysis demonstrates that the non-commutativity of equilibrium price impact Λ_i and covariance $\mathcal{V}_{N(i)}$ is central to economic mechanisms that do not have centralized market counterparts. The commutativity of centralized market equilibria, which implies the lattice structure, has further implications: The equilibrium (slopes of) demand schedules $(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}$ are proportional to Σ^{-1} and, in equilibrium, each agent's utility is monotone decreasing in price impact Λ_i. Neither of these holds in decentralized markets, in general. In particular, the strategic complementarity of demand schedules is no longer determined by Σ, but it is endogenous, and equilibrium utility is generally not monotone in price impact (Sections 6.4 and 7). Proposition 3.3 shows that commutativity is nongeneric in decentralized markets.

Proposition 3.3 (Genericity of Non-Commutativity) *In any decentralized market $\mathbb{M}$, let i and j be two classes such that $N(i)\neq N(j)$ and suppose that the cardinality of $N(i)\cap N(j)$ is*

[28] If matrix $\mathcal{V}_{N(i)}$ is degenerate, then the inverse demand slope $(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)$ is also degenerate if the rows of Λ_i corresponding to the replica assets are identical. This is the only type of degeneracy that may occur (Corollary $B.1$ in the Appendix). In Section 4, we show that, in this case, agents are indifferent about which exchange to trade in and (replicas of) assets then trade at the same prices. Equilibrium, defined as $\{\Lambda_i, q_i(\cdot,\Lambda_i)\}_i$, is locally unique.

[29] In all of the examples included in this paper, the linear equilibrium is globally unique. While, as in centralized markets, the players' price impacts $\{\Lambda_i\}_i$ and demand schedules $\{q_i(\cdot,\Lambda_i)\}_i$ are strategic complements, the non-commutativity in decentralized markets breaks the equilibrium symmetry present in centralized markets, where agents' strategies are the same up to "scaling" by risk aversion.

With multiple assets (divisible goods), the uniqueness of a *linear* equilibrium in the set of all (potentially nonlinear) equilibria remains an open problem even for centralized markets with quadratic utilities. The linear equilibrium in our decentralized market model is globally unique under the conditions we provide in the Appendix. The comparative-statics results we develop involve monotone comparative statics rather than the Implicit Function Theorem – in particular, monotonicity conditions are separate from second-order conditions.

[30] In the literature on games on networks (and, more generally, linear social interactions; see, e.g., Blume et al. (2012)), the problem is typically linear – conditions on the invertibility of a linear map (matrix) verify nondegeneracy. Unless lifted demand slopes $\{(\alpha_i\bar{\mathcal{V}}_{N(i)}+\bar{\Lambda}_i)^{-1}\}_i$ commute, the equilibrium uniqueness in our model concerns the invertibility of a nonlinear map on tuples of positive definite matrices. The modelof Bramoullé, Kranton, and D'Amours (2013) exhibits a nonlinearity introduced by a constraint $x\geq 0$ on the actions; the best response is $\max\{0,f(x)\}$, where f is linear. This introduces a potential multiplicity of equilibria: If the constraint does not bind, the linear equilibrium is unique; when it does bind, nonlinearity kicks in and multiplicity arises, depending on whose constraint is binding.

Note that the conditions that are derived from Gershgorin disc theorem, such as diagonal dominance, cannot be used in our model even to prove local injectivity (i.e., invertibility of the Jacobian defined in the proof of Lemma $A.3$), as the theorem applies to matrices that are maps from $\mathbb{R}^N\to\mathbb{R}^N$, whereas we deal with maps of matrix tuples into matrix tuples. The isomorphism of $\mathcal{S}^I$ with a cone in $\times_{i=1}^{I}\mathbb{R}^{N(i)\times N(i)}$ is not useful here, as it would ignore the positive definite structure.

strictly greater than 1*. Then, for generic covariance* Σ *and generic risk aversions* $\{\alpha_i\}_i$*, price impacts* $(\Lambda_i)_{N(i)\cap N(j)}$ *and* $(\Lambda_j)_{N(i)\cap N(j)}$ *do not commute.*

4 Equivalence Loops and Regularization

We show that, for any decentralized market, there is an equivalent and unique representation without singularities in the equilibrium price impact that arise when identical assets are traded in different exchanges. Theorem 4.1 explicitly characterizes the set of pairs of exchanges for which this degeneracy occurs, using the conditions on the market structure we define next. The result will allow us to completely characterize price behavior in decentralized markets (Theorem 4.2) and properties of equilibrium hypergraph (Corollary 6.3).

Definition 4.1 *Given two exchanges* n *and* n' *and asset* k*, an* equivalence path *connecting these exchanges with respect to asset* k *is a sequence of exchanges* $\{n_l\}_l$ *and a sequence of agents* $\{a_l\}_{l=1}^{L-1}$ *such that* $n_1 = n$*,* $n_L = n'$*, and for all* l*,* $a_l \in I(n_l) \cap I(n_{l+1})$ *and* $k \in K(n_l)$*. Two equivalence paths are* disjoint *if the corresponding sets of agents are disjoint. Two disjoint equivalence paths form an* equivalence loop.

We now describe a procedure that can be applied to any decentralized market, which we refer to as *regularization*. Fix a decentralized market $\mathbb{M}$. For any pair of exchanges n and n' and any asset k, if there is an equivalence loop with respect to asset k between exchanges n and n', remove k from all of the exchanges in the equivalence loop $\{n_l\}_l$ between $n_1 = n$ and $n_L = n'$, and create a new exchange $(k, \cup_l I(n_l))$ in which only asset k is traded; the initial endowment of any class $i \in \cup_l I(n_l)$ for asset k in the new exchange is the sum of initial endowments $q_{i,k}(n_l)$ in $\{n_l\}_l$. Iterating this procedure, we arrive at a decentralized market with no equivalence loops. Denote by $\mathbb{M}^*$ the so constructed *regularized* market associated with $\mathbb{M}$. For any class i, let $N^*(i)$ be the set of exchanges in $\mathbb{M}^*$ in which class i participates. For any exchange-asset pair $(n,k) \in N^*(i)$, denote by $N(i,n,k)$ the set of exchanges in which class i participates that are connected with n via an equivalence loop with respect to asset k. Then, for any $\Lambda_i^* \in \mathbb{R}^{N^*(i)\times N^*(i)}$, define $\Lambda_i^* \otimes \mathbf{1} \in \mathbb{R}^{N(i)\times N(i)}$ to be the matrix obtained from Λ_i^* by multiplying every matrix element $(\Lambda_i^*)_{(n_1,k_1),(n_2,k_2)}$ by $\mathbf{1}_{N(i,n_1,k_1)\times N(i,n_2,k_2)}$.

Theorem 4.1 (Regularized Market: Equilibrium Equivalence) *For any market* $\mathbb{M}$*, equilibrium prices and allocations in the associated regularized market* $\mathbb{M}^*$ *coincide with those in* $\mathbb{M}$*. The map* $\{\Lambda_i^*\}_i \to \{\Lambda_i^* \otimes \mathbf{1}\}_i$ *defines a one-to-one correspondence between equilibria in* $\mathbb{M}^*$ *and* $\mathbb{M}$*. Furthermore, equilibrium price impacts in* $\mathbb{M}^*$ *are nondegenerate.*

Building on Theorem 4.1, Theorem 4.2 gives the condition on the market structure for replicas of the same asset k to trade at the same prices in different exchanges. With decentralized trading, the diversification opportunities of agents differ, and the value of the same asset can be different across exchanges.

Theorem 4.2 (Price Equalization) *In equilibrium, generically in the initial asset holdings, asset k is traded at the same price in exchanges n and n' if, and only if, there exists an equivalence loop connecting these two exchanges with respect to asset k.*

Suppose that there is no equivalence loop connecting n and n'. Consider a single-agent node (a "network monopolist") connecting exchanges n and n'. If removed from the market, exchanges n and n' become isolated. Let $N = N_1 \cup N_2$ be the disjoint partition of exchanges that is obtained if we remove agent i from the market. Then, in the decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$, price impact of agent i is block diagonal ($M_i = 1$ in Equation (4)). Therefore, since no other agent trades simultaneously in exchanges from N_1 and N_2, it further follows from Equation (4) that price impacts of all other agents are also block diagonal. Thus, no agent is able to equalize prices in these exchanges. By Theorem 4.1, a single-agent node is also the only instance of an absence of an equivalence loop and, hence, is the necessary and sufficient condition for price discrimination in decentralized markets.[31] More subtly, the value of an asset depends on market characteristics other than the span itself (see (Equation (11)). Yet, no arbitrage incentives exist – the schedules are *ex post* optimal.

In the remainder of the paper, unless stated otherwise, we work with regularized markets denoted by $\mathbb{M}$.

5 Equilibrium Liquidity, Allocations and Prices

In this section, we characterize equilibrium price impacts (Section 5.1) and allocations and prices (Section 5.2).

5.1 Price Impact

In a decentralized market with a finite number of risk averse traders, equilibrium price impact of every trader is strictly positive. These conditions are sufficient, but unlike centralized markets (cf. Proposition 2.1), they are not necessary. Consider a (non-regularized) decentralized market $\mathbb{M}$. For any exchange n and an asset k traded on n, let $\mathcal{L}(n,k)$ be the set of all exchanges, including n, connected with n via an equivalence loop with respect to asset k.[32] Let $M(\mathcal{L}(n,k))$ and $I(\mathcal{L}(n,k))$ denote the total number of agents trading in all exchanges from $\mathcal{L}(n,k)$ and the set of corresponding classes, respectively.

Proposition 5.1 (Equilibrium Noncompetitiveness) *Consider a sequence of markets $\mathbb{M}_l$, $l \geq 1$, with a fixed market structure $\{(I(n), K(n))\}_n$ and changing class characteristics $\{\alpha_{i,l}, M_{i,l}\}_i$. Then, in the limit as $l \to \infty$, price impact vanishes in exchange n for asset k if, and only if, either $M(\mathcal{L}(n,k)) \to \infty$, or $\alpha_{i,l} \to 0$ for at least two agents from $I(\mathcal{L}(n,k))$.*

[31] Furthermore, by Theorem 4.1, removing all equivalence loops makes price impacts, and hence demand slopes, nondegenerate. With nondegenerate demand slopes, it follows from Theorem 5.1 in Section 5.2 that there are no linear restrictions on equilibrium prices. In this case vector $\mathbf{Q}$ can take any values in $\mathbb{R}^N$. In fact, prices equalize if, and only if, price impacts have identical rows.

[32] Define $\mathcal{L}(n) \equiv \{n\}$ in case n does not belong to an equivalence loop.

When some agents in exchange n are almost risk neutral, or the number of agents in some class $i \in I(n)$ becomes large, $M_i \to \infty$, or the asset traded in exchange n is riskless, the exchange is essentially perfectly liquid: Equilibrium schedules in exchange n correspond to the marginal utilities. Having access to a perfectly liquid exchange n, agents will, in general, have strictly positive price impact in the other exchanges $N \setminus \{n\}$ in which they participate, such as liquidity pools or a dealer network, unless (by Theorem 4.1) there is an equivalence loop linking these exchanges. Thus, even if the total number of traders in the market is large, traders do not act as price takers.

By Theorem 2.1, in contrast to centralized markets, in general, the equilibrium price impact of trader i in exchanges $N(i)$ depends not only on the price impacts of other traders in exchanges $N(i)$, but also on price impacts of traders in all other exchanges, even if the sets of agents are disjoint from those participating in exchanges from $N(i)$. Lemma 5.1 enables us to characterize further the equilibrium price impact in decentralized markets by permitting the formulation of our model in terms of a single statistic, the market-wide measure of *liquidity,*

$$B \;=\; \left(\sum_j M_j (\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1} \right), \tag{7}$$

the slope of the aggregate market supply. In the literature, liquidity is broadly defined as 'the ease of trading an asset' and, for an individual trader, is measured as the inverse of his price impact. Analogously, letting $B_{-i} \;\equiv\; B - (\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i)^{-1}$, in the equilibrium condition (4), price impact is inversely related to a trader's liquidity in exchanges $N(i)$,

$$\Lambda_i = \left(B_{-i}^{-1}\right)_{N(i)} = \left(\left((B^{-1})_{N(i)}\right)^{-1} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}\right)^{-1} \tag{8}$$

(Lemma *C*.4 in the Appendix). Partition liquidity as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{pmatrix},$$

where $B_{11} \in \mathbb{R}^{N(i)}$ and $B_{22} \in \mathbb{R}^{N \setminus N(i)}$. Define $\mathfrak{S}(B, N(i))$ to be the largest symmetric matrix (in the positive semidefinite order) in $\mathbb{R}^{N(i) \times N(i)}$ that is below B. That is,

$$\mathfrak{S}(B, N(i)) = \max\{A \in \mathbb{R}^{N(i) \times N(i)} : 0 \leq \bar{A} \leq B\}. \tag{9}$$

The maximum exists, is unique, and is given by the *shorted operator* $\mathfrak{S}(B, N(i)) = B_{11} - B_{12} B_{22}^{-1} B_{12}^T$ (see, e.g., Anderson and Trapp (1975)). By exploiting the equilibrium structure of price impact, in the analysis to follow, we aim to identify economic mechanisms that are present in decentralized but not centralized markets. The implications of the following lemma for decentralized markets will be crucial.[33]

[33] A proof follows directly from (4) and Lemma C.4 in the Apendix.

Lemma 5.1 (Equilibrium Price Impact) *In equilibrium, price impact Λ_i of class i is*

$$\begin{aligned}\Lambda_i &= (\mathfrak{S}(B_{-i}, N(i)))^{-1} \\ &= (B_{11} - B_{12}B_{22}^{-1}B_{12}^T - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1})^{-1}.\end{aligned} \tag{10}$$

First, a trader's price impact in exchanges $N(i)$ depends positively (in the sense of positive semidefinite order) both on other trader's price impacts in exchanges $N(i)$ and $N \setminus N(i)$ (cf. Lemma 3.2), as long as the assets in $N(i)$ and $N \setminus N(i)$ are not independent and whenever the exchanges are not in disconnected components of hypergraph $((I, K), \{(I(n), K(n))\}_n)$. The cross-exchange interdependence carries over to equilibrium prices and allocations (Section 5.2).[34] Section 6 characterizes the comparative statics of equilibrium price impact with respect to primitive preferences and assets as well as changes in market structure.

Second, as in a centralized market, the slope of a trader's residual supply B_{-i} determines the quantities he would sell in exchanges in which he participates by lowering price at a margin, subject to market clearing and other traders' optimization. In decentralized markets, given the presence of exchanges $N \setminus N(i)$, absorption of trade by agents in $N(i)$ is greater and price impact smaller than it would be if these agents were trading in isolation; Lemma 5.1 makes precise the sense in which the impact of exchanges $N \setminus N(i)$ is *maximal*. Section 7 explores how the decentralization of markets affects the markets' role in facilitating risk sharing among traders.

Finally, in a centralized market, equilibrium cross-asset price impact is proportional to the primitive covariance Σ of assets (Proposition 2.1). In a decentralized market, the equilibrium substitutability or complementarity of assets is endogenous instead and, in general, differs from Σ (Section 6.4).

5.2 Allocations and Prices

Theorem 5.1 characterizes equilibrium trades and prices in the general model of decentralized markets. For any vector $\mathbf{Q} \in \mathbb{R}^N$, let $\mathbf{Q}_{N(i)} \in \mathbb{R}^{N(i)}$ denote the restriction of this vector to its elements in $\mathbb{R}^{N(i)}$. Let Q_i^0 be the total initial holdings of class i.

[34] The interdependence distinguishes ours from static models of decentralized trading with indivisible assets. With indivisible assets, just as with Nash bargaining (efficient surplus sharing), when the asset swaps hands between a buyer and a seller, the impact of trading links on their optimization arises only through dynamic trading, whereas in our model, the cross-exchange interdependence impacts optimization also in one-period markets. In particular, assuming by each agent that his trade is independent of the choices of traders in exchanges $N \setminus N(i)$ does not satisfy equilibrium conditions but would be without loss of generality in a static game with indivisible assets. Accordingly, the best responses in static network models depend only on the actions of direct neighbors.

In game theoretic terms, the equilibrium price impact of a trader – in a centralized or decentralized market – corresponds to the counterfactual price adjustment needed for other traders to be willing to absorb his marginal increase from the equilibrium trade so that the market clears. In a decentralized market, a trader's counterfactual about off-equilibrium response in exchanges $N(i)$ depends on the counterfactuals of his trading partners in $N(i)$ about the off-equilibrium response to their trades in $N(i)$ as well as other exchanges in which they participate and, hence, the interdependence (cf. the fixed-point condition on price impacts (4)). Tackling counterfactuals that are interdependent not only within exchanges (as is true also in centralized markets) but also across exchanges, allows us to uncover economic mechanisms unique to decentralized trading. In certain decentralized-trade settings, a trader's counterfactual about $N(i)$ is independent of exchanges $N \setminus N(i)$. For example, this is the case in segmented markets whose traders are in disjoint connected components of a hypergraph (see Example 2).

Theorem 5.1 (Equilibrium Prices and Trades) *Let*

$$\mathbf{Q} \equiv B^{-1} \sum_j \left(\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j(B)\right)^{-1} \alpha_j \bar{\mathcal{V}}_{N(j)} Q_j^0.$$

Then, in equilibrium, the vector of market clearing prices is given by

$$p = d - \mathbf{Q}, \tag{11}$$

and the trade of agent i is $q_i = (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}(\mathbf{Q}_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0)$.

Given the strictly positive equilibrium price impact (Proposition 5.1), a trader demands (or sells) less by $(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1} - (\alpha_i \mathcal{V}_{N(i)})^{-1}$, relative to the marginal utilities (i.e., the competitive schedule in exchanges $N(i)$, given the quasilinear utility function (1)). By the noncommutativity of equilibrium price impact and assets' covariance (Proposition 3.3), the extent to which a trader reduces his (net) demand in any given exchange depends on the preferences and assets' riskiness in all exchanges and the market structure. The equilibrium holdings of an agent from class i are $q_i + q_i^0 = (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}(\mathbf{Q}_{N(i)} + \Lambda_i q_i^0)$. Portfolio $\{(\alpha_i \mathcal{V}_{N(i)})^{-1} \mathbf{Q}_{N(i)}\}_i$ evaluated assuming $\Lambda_i = 0$ for all i is the decentralized market counterpart of the (unique) efficient portfolio in a centralized competitive market. Defined by the equalization of a trader's marginal utility with prices, portfolio $\{(\alpha_i \mathcal{V}_{N(i)})^{-1} \mathbf{Q}_{N(i)}\}_i$ depends on the market structure of all exchanges, as does order reduction $((\alpha_i \mathcal{V}_{N(i)})^{-1} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$, explicitly characterized in Lemma *C.2* in the Appendix.)

The linearity of the equilibrium prices leads us to interpret Equation (11) in CAPM terms. Denote by Q_n^k the coordinate of $Q \in R^{\bar{K}}$ corresponding to asset k traded in exchange n. Let $\Sigma_{\kappa(n)}$ be the covariance matrix of the assets that are traded in exchange n and let $\Gamma_n \equiv \Sigma_{\kappa(n)}^{-1} \{\mathbf{Q}_n^k\}_{k \in \kappa(n)} = \{\Gamma_n^k\}_{k \in \kappa(n)} \in \mathbb{R}^{\kappa(n)}$. By the linearity, there exists a vector $\mu_{n,k} = \{\mu_{n,k}^{i,l}\}_{i,l} \in R^{I \times K}$ such that

$$\Gamma_n^k = \langle \mu_{n,k}, Q^0 \rangle = \sum_{i,l} \mu_{n,k}^{i,l} Q_i^{0,l},$$

where $Q^0 = \{Q_i^{0,k}\}_{i,k}$ is the vector of initial holdings. Let $\mathbf{R}_n = \sum_{k \in \kappa(n)} \Gamma_n^k R_k$ denote the random payoff of the portfolio that contains Γ_n^k units of asset k, $k \in K$.

Proposition 5.2 (Decentralized CAPM) *For any exchange n, the price at which asset $k \in \kappa(n)$ is traded in exchange n is equal to the expected payoff net the risk premium, given by the covariance of the asset payoff with the payoff of the exchange-specific market portfolio, $\mathbf{R}_n$,*

$$p_n(k) = \delta_k - \text{Cov}(R_k, \mathbf{R}_n).$$

Proposition 5.2 implies that exchange-specific market portfolios depend in a nontrivial way on the market structure. Even in a public exchange, the market-capitalization-weighted portfolio may not command the highest risk premium if some agents have access to other exchanges.

6 Comparative Statics of Liquidity

In Sections 6 and 7, we examine the comparative statics of equilibrium liquidity and welfare. Two questions are of central interest here: How does liquidity change when a market becomes more decentralized, in the sense that there are more market clearing mechanisms? And what is the impact of market decentralization on welfare? The impact of "local" changes in variables in an equilibrium neighborhood can be examined with the standard Implicit Function Theorem techniques and Lemma 5.1. However, analyzing how changes in market structure ($\{N(i)\}_i$) affect liquidity and welfare in decentralized markets involves "nonlocal" changes. The latter type of comparative statics requires establishing monotonicity with respect to systems of endogenous matrices which are defined by nonlinear maps and live in different subspaces. For centralized markets, lattice-theoretic arguments apply. Our monotone comparative statics for decentralized markets is based on three results from Section 3: the monotone convergence of demand schedules in arbitrary decentralized markets (Proposition 3.1); the existence of the maximal and minimal equilibria (Proposition 3.2), which allow us to prove comparative statics using the arguments analogous to Milgrom and Roberts (1990), despite the inapplicability of Tarski's fixed point theorem; and the positive semidefinite property of price impacts (Theorem 3.1).[35] Let $\mathbb{M}$ and $\mathbb{M}'$ be two different decentralized markets. We say that liquidity is higher in market $\mathbb{M}$ than in market $\mathbb{M}'$, and write

$$(\Lambda_i)_{N(i)\cap N'(i)} \leq (\Lambda'_i)_{N(i)\cap N'(i)} \tag{12}$$

if (12) holds for all i for both the minimal and maximal equilibria $\{\Lambda_{i,\min}\}_i$, $\{\Lambda'_{i,\min}\}$ and $\{\Lambda_{i,\max}\}_i$, $\{\Lambda'_{i,\max}\}$, respectively. The positive semidefinite order possesses properties that are well suited for our analysis. Recall that if $A \geq B$ then the diagonal elements satisfy $A_{ii} \geq B_{ii}$ for all i (however, no implication for the ordering of the off-diagonal elements follows); $A_{N(i)} \geq B_{N(i)}$ for any i; $Z^T AZ \geq Z^T BZ$ for any matrix Z; and $A^{-1} \leq B^{-1}$.

6.1 Liquidity: Equilibrium Properties

Proposition 6.1 characterizes how equilibrium price impact depends on the primitive characteristics of traders and assets – the risk aversion and the number of traders in various exchanges, and the covariance of assets traded.

Proposition 6.1 (Price Impact Monotonicity) *Equilibrium price impact tuple $\{\Lambda_j\}_j$ is*

(i) increasing in risk aversion α_i, for any i;

(ii) increasing and concave in the covariance matrix Σ;

(iii) decreasing in the number M_i of agents in class i, for any i;

(iv) jointly concave in $\{\alpha_i\}_i$ and $\{M_i^{-1}\}_i$; and

[35] Our main results (Proposition 6.1, Theorem 6.1, Proposition 6.2) are based on the monotonicity and concavity properties of the matrix harmonic mean that are derived in Anderson and Trapp (1975) using the theory of shorted operators. The proofs of Theorems 6.1 and Theorem 4.2 use Lemma *A*.1, whose proof is based on the theory of shorted operators. For Propositions 6.3 and 7.4, 7.5 and 7.3, we use the characterization of monotone matrix functions due to Löwner (1934) to show a non-monotone relationships involving price impact, welfare and the hypergraph.

(v) increasing in the number of agent classes I if the new classes do not change the participation $\{N(i)\}_i$ of incumbent agents.

Anticipated by Lemma 5.1, unlike a centralized market, the riskiness of assets traded in a given exchange n, the decreasing marginal utility (risk aversion) and the number of agents who trade there are no longer the sole determinants of market power. Derived from aggregation of traders' schedules and market clearing, an agent's price impact from increasing his trade in exchanges $N(i)$ represents the price concessions required for other agents in exchanges $N(i)$ to be willing to optimally absorb the trade so that exchanges clear. With decreasing marginal utility ($\alpha_j > 0$), more risk averse counterparties j in exchanges $N(i)$ demand larger price concessions to compensate for the trade's impact on their own marginal utilities (cf. the first-order condition (2)), thus making the residual supply less elastic and, hence, price impact larger for all other agents in $N(i)$. In addition, when trading is decentralized, the fewer and more risk averse agent i's counterparties' trading partners in exchanges $N \backslash N(i)$, the larger the counterparties' price impacts in the exchanges $N(j)$ and the larger the price concessions they require in exchanges $N(i)$. The decentralized market effect is strict, even for disjoint classes of agents in $N(i)$ and $N \setminus N(i)$, so long as the agents are indirectly connected through a sequence of counterparties. Thus, improving liquidity in one exchange (e.g., when the number of traders increases) improves liquidity in other exchanges. Notably, the *concavity* result implies that the primitives in exchanges $N \backslash N(i)$ have less than proportional impact on the liquidity in exchanges $N(i)$, compared to the primitives in $N(i)$. Example 2 illustrates the interdependence in price impact Λ_i between exchanges with disjoint sets of traders.

Example 2 (Indirectly Connected Markets) *Consider partition $\{N_1, N_2\}$ of the set of exchanges N and assume that there exist nonempty sets I_1 and I_2, such that $I_1 = \{i \in I; N(i) \in N_1\}$ and $I_2 = \{i \in I; N(i) \in N_2\}$. If we define another set of agents, $I_3 = I - (I_1 \cup I_2)$, then $\{I_1, I_2, I_3\}$ is a partition of the set of agents I. For instance, agents I_3 are dealers who trade assets in a centralized exchange with one another and two sets of clients, I_1 and I_2, who trade a disjoint set of assets. Let M_l be the number of agents in I_l, $l = 1, 2, 3$, $M_1, M_2 \geq 2$. We assume that all agents in I_l, $l = 1, 2, 3$, have the same risk aversion α_l and trade assets in exchanges N_l, where $N_3 = N$. Equation (4) implies that price impact Λ_l is identical for all agents in I_l. Thus,*

$$B = \left(\sum_{l=1}^{3} M_l \left(\alpha_l \bar{\mathcal{V}}_{P_l} + \bar{\Lambda}_l \right)^{-1} \right).$$

When $I_3 = \emptyset$ (i.e., the network is disconnected), the inter-partition block components of B are zero-matrices,

$$B \;=\; \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} = \begin{pmatrix} M_1 \left(\alpha_1 \mathcal{V}_{P_1} + \Lambda_1 \right)^{-1} & 0 \\ 0 & M_2 \left(\alpha_l \mathcal{V}_{P_2} + \Lambda_2 \right)^{-1} \end{pmatrix}.$$

Therefore, $B_{11} = \mathfrak{S}(B, N(1))$ and, hence,

$$\Lambda_l \;=\; (B_{11} - (\alpha_l \mathcal{V}_{P_l} + \Lambda_l)^{-1})^{-1} = (M_1 - 1)^{-1} (\alpha_l \mathcal{V}_{P_l} + \Lambda_l).$$

The price impact of $i \in I_l$ does not depend on the price impacts, risk aversion, or number of traders in $N \setminus N_l$, as expected.

Assume $I_3 \neq \emptyset$, $M_3 \geq 1$, so that exchanges N_1 and N_2 are connected via I_3. The market-wide liquidity is

$$B = \begin{pmatrix} M_1\Sigma_1 + M_3\Sigma_{3,11} & M_3\Sigma_{3,12} \\ M_3\Sigma_{3,12}^T & M_2\Sigma_2 + M_3\Sigma_{3,22} \end{pmatrix},$$

where $\Sigma_l \equiv (\alpha_l \mathcal{V}_l + \Lambda_l)^{-1}$, $l = 1, 2, 3$ and

$$\Lambda_3 \equiv \begin{pmatrix} \Lambda_{3,11} & \Lambda_{3,12} \\ \Lambda_{3,12}^T & \Lambda_{3,22} \end{pmatrix}, \Sigma_3^{-1} = \begin{pmatrix} \Sigma_{3,11} & \Sigma_{3,12} \\ \Sigma_{3,12}^T & \Sigma_{3,22} \end{pmatrix}^{-1} = \begin{pmatrix} \alpha_3\mathcal{V}_1 + \Lambda_{3,11} & \alpha_3\mathcal{V}_{12} + \Lambda_{3,12} \\ \alpha_3\mathcal{V}_{12}^T + \Lambda_{3,12}^T & \alpha_3\mathcal{V}_2 + \Lambda_{3,22} \end{pmatrix}.$$

Then, in particular, the price impact of agents in I_1 is

$$\Lambda_1 = ((M_1 - 1)\Sigma_1 + M_3\Sigma_{3,11} - (M_3\Sigma_{3,12})(M_2\Sigma_2 + M_3\Sigma_{3,22})^{-1}(M_3\Sigma_{3,12})^T)^{-1}.$$

Assume that Λ_l is a positive definite matrix for all $l = 1, 2, 3$. We consider three comparative statics for Λ_1 and one for Λ_3.

(1) Price impact Λ_2: Consider a matrix $\widetilde{\Lambda}_2 \geq \Lambda_2$, such that $(\widetilde{\Lambda}_2 - \Lambda_2)$ is a positive semidefinite matrix. By the properties of positive definite matrices, we have $\widetilde{\Sigma}_2^{-1} \equiv \alpha_2\mathcal{V}_2 + \widetilde{\Lambda}_2 \geq \Sigma_2^{-1}$ and, thus, $\widetilde{\Sigma}_2 \leq \Sigma_2$ and

$$\widetilde{\Lambda}_1 \equiv ((M_1 - 1)\Sigma_1 + M_3\Sigma_{3,11} - (M_3\Sigma_{3,12})(M_2\widetilde{\Sigma}_2 + M_3\Sigma_{3,22})^{-1}(M_3\Sigma_{3,12})^T)^{-1} \geq \Lambda_1.$$

Thus, the price impact in exchanges N_1 depends positively on the price impact in the indirectly connected exchanges N_2.

(2) Risk aversion α_2: For any $\hat{\alpha}_2 \geq \alpha_2 > 0$, $\widehat{\Sigma}_2^{-1} \equiv \hat{\alpha}_2\mathcal{V}_2 + \Lambda_2 \geq \Sigma_2^{-1}$ and $\widehat{\Sigma}_2 \leq \Sigma_2$. By the same argument, $\widehat{\Lambda}_1 \geq \Lambda_1$. Therefore, the more risk averse the traders in exchanges N_2, the greater the price impacts in exchanges N_1.

(3) The number of traders: The price impacts of traders in exchanges N_1 increase when the number of traders decreases in N_1 or in the indirectly connected exchanges N_2. Consider an increase in the number of traders M_3 in the linking classes on the price impact,

$$\Lambda_1 \equiv (M_1 - 1)\Sigma_1 + M_3(\Sigma_{3,11} - \Sigma_{3,12}(\frac{M_2}{M_3}\Sigma_2 + \Sigma_{3,22})^{-1}\Sigma_{3,12}^T).$$

When M_3 increases, $(\Sigma_{3,11} - \Sigma_{3,12}(\frac{M_2}{M_3}\Sigma_2 + \Sigma_{3,22})^{-1}\Sigma_{3,12}^T)$ decreases. However, by Corollary 6.2 and Proposition 6.3 below, if $\alpha_3 < \min\{\alpha_1, \alpha_2\}$, the more connected agents of class I_3 have larger price impact. This also illustrates how the equilibrium effects of market segmentation (i.e., traders in exchanges N_1 and N_2 are disjoint) differ from those of the absence of lack of payoff correlation (between the assets traded in N_1 and N_2).

(4) Cross-exchange effects: The dealers' price impacts in exchanges N_1 and N_2 are not inde-

pendent,

$$\Lambda_3 = \left((M_3 - 1) \begin{pmatrix} \Sigma_{3,11} & \Sigma_{3,12} \\ \Sigma_{3,12}^T & \Sigma_{3,22} \end{pmatrix} + \begin{pmatrix} M_1\Sigma_1 & 0 \\ 0 & M_2\Sigma_2 \end{pmatrix} \right)^{-1},$$

as long as the assets traded in N_1 and N_2 are not independent. Depending on the substitutability or complementarity of assets N_1 and N_2, the dealer's liquidity is higher or lower, relative to the case of independent assets in N_1 and N_2 ($\Lambda_{3,12}^T \sim \alpha_3 \mathcal{V}_{12}^T$).

6.2 Liquidity and Decentralized Market Structure

In this section, we consider two sets of comparative statics. First, we examine how equilibrium liquidity changes when a market becomes more decentralized; that is, the number of the market clearing mechanisms increases for the fixed sets of traders I and assets K. Then, we analyze how an agent's participation in the market determines his liquidity relative to other agents in the exchanges $N(i)$ in which he trades.

Given the set of agents, suppose a new exchange is created for a subset of agents which operates along with the existing exchanges. This change in market structure corresponds to an increase in the subset of exchanges in which class i from the subset participates, $N'(i) \supseteq N(i)$. Since, unlike in the experiment of Proposition 6.1, the dimensions of price impacts change, we compare the price impacts for exchanges $N(i)$ only (i.e., in the sub-hypergraphs induced by $N(i) \cap N'(i)$ in markets $\mathbb{M}$ and $\mathbb{M}'$ in definition (12)).

Theorem 6.1 (Price Impact and Participation) *For any i, an increase in participation of class i lowers equilibrium price impact of all agents in all exchanges.*

Thus, creating a new exchange always (weakly) improves liquidity in all existing exchanges.

Suppose instead that the number of exchanges in a market increases by breaking up an existing exchange. Theorem 6.1 implies that breaking up exchanges always (weakly) reduces liquidity. Furthermore, this holds for any asset structure in the new market.

Corollary 6.1 *Suppose exchange $(I(n), K(n))$ is split into two exchanges $(I_1(n), K_1(n))$ and $(I_2(n), K_2(n))$ with arbitrary (not necessarily disjoint) subsets of assets $K_1(n) \cup K_2(n) = K(n)$ and agent classes $I_1(n) \cup I_2(n) = I(n)$. Equilibrium price impact increases in all exchanges.*

More generally, liquidity is monotone in both the set inclusion of traders and assets. Qualitatively, the liquidity effects of changing the market structure characterized by Theorem 6.1 and Corollary 6.1 are independent of the number and risk aversion of traders, and the covariance matrix of assets in the exchanges before and after the change.

Next, we consider how the extent to which a trader is connected with the market and his more or less central position in the market, measured by participation in different exchanges, influence his equilibrium price impact relative to other traders in a given exchange. By Equation (8), the equilibrium price impacts of different market participants are linked through the aggregate liquidity measure B. Namely, let

$$\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}) \equiv (\Lambda_i^{-1} + (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1})^{-1} \tag{13}$$

be the harmonic mean of two matrices Λ_i and $\Lambda_i + \alpha_i \mathcal{V}_{N(i)}$. By Equation (8), $\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}) = (B^{-1})_{N(i)}$, for any class i. In particular, the price impacts of two classes i and j that are connected (i.e., $N(i) \cap N(j) \neq \emptyset$) are related as follows

$$\big(\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)})\big)_{N(i)\cap N(j)} = \big(\Phi(\Lambda_j, \alpha_j \mathcal{V}_{N(j)})\big)_{N(i)\cap N(j)} = (B^{-1})_{N(i)\cap N(j)}. \tag{14}$$

Suppose that $N(i) \supset N(j)$; for instance, class i is better connected than class j. A concavity property of the harmonic mean (13) implies the following relationship among the price impacts in the exchanges in which both classes i and j participate, $(\Lambda_i)_{N(j)}$ and Λ_j.

Proposition 6.2 *Suppose that class i has greater market participation than class j, $N(i) \supset N(j)$. Then,*

$$\Phi\big((\Lambda_i)_{N(j)}, \alpha_i \mathcal{V}_{N(j)}\big) \geq \Phi\big(\Lambda_j\,,\, \alpha_j \mathcal{V}_{N(j)}\big)\,. \tag{15}$$

Function $\Phi(\Lambda, \alpha\mathcal{V})$ is monotone increasing in Λ and, therefore, for the case of scalar Λ_j, inequality (15) immediately yields the following result.

Corollary 6.2 (Relative Price Impact: One Asset) *Suppose that $N(i) \supset N(j)$ and that class j participates in a single exchange for one asset. Then, if $\alpha_i \leq \alpha_j$, the equilibrium price impact of class i in exchanges $N(j)$ is larger than that of class j,*

$$(\Lambda_i)_{N(j)} \geq \Lambda_j\,. \tag{16}$$

That is, more connected agents have larger price impact.

Nevertheless, with many assets, one cannot extrapolate Corollary 6.2 by using (15) to conclude that $(\Lambda_i)_{N(j)} \geq \Lambda_j$. The non-commutativity is, again, the key. Let $A_1 \equiv \Phi\big((\Lambda_i)_{N(j)}\,,\, \alpha_i \mathcal{V}_{N(j)}\big)$ and $A_2 \equiv \Phi\big(\Lambda_j\,,\, \alpha_j \mathcal{V}_{N(j)}\big)$. Then (using Lemma *C.2* in the Appendix),

$$(\Lambda_i)_{N(j)} = \alpha_i \mathcal{V}_{N(j)}^{1/2} f(\alpha_i \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2}) \mathcal{V}_{N(j)}^{1/2}\ ,\ \Lambda_j = \alpha_j \mathcal{V}_{N(j)}^{1/2} f(\alpha_j \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2}) \mathcal{V}_{N(j)}^{1/2}\,,$$

where

$$f(a) = \frac{2 - a + \sqrt{a^2+4}}{2a} \tag{17}$$

is monotone decreasing in a. Inequality $A_1 \geq A_2$ (Proposition 6.2) implies $X_1 \equiv \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2} \leq \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2} \equiv X_2$. However, given two non-commuting symmetric matrices X_1 and X_2 and a monotone decreasing function $f(x)$, inequality $X_1 \leq X_2$ does *not* generally imply $f(X_1) \geq f(X_2)$. A function f that satisfies $f(X_1) \geq f(X_2)$ for any $X_1 \leq X_2$ is called *matrix monotone.* In particular, to conclude that $(\Lambda_i)_{N(j)} \geq \Lambda_j$, function f in (17) must be matrix-monotone, which is not the case.[36] In Section 7, we show that this non-monotonicity has important welfare implications.

One can still compare price impacts through an eigenvalue order instead of the (weaker) positive semidefinite order, using that with positive semidefinite matrices, there is a min-max interpretation of eigenvalues. For the eigenvalues of a symmetric $m \times m$ matrix A ordered to be decreasing, $\text{eig}(A) = \{\mu_1(A) \geq \cdots \geq \mu_m(A)\}$, we write $\text{eig}(A) \geq \text{eig}(B)$ if $\mu_i(A) \geq \mu_i(B)$ for all $i = 1, \cdots, m$.[37]

Proposition 6.3 (Relative Price Impact: Many Assets) *Suppose that class i has greater market participation than class j, $N(i) \supset N(j)$. Then, if $\alpha_i \leq \alpha_j$, equilibrium price impact of class i in exchanges $N(j)$ is larger than that of class j in the following sense:*

$$\text{eig}\big(\alpha_i^{-1}\mathcal{V}_{N(j)}^{-1/2}(\Lambda_i)_{N(j)}\mathcal{V}_{N(j)}^{-1/2}\big) \geq \text{eig}\big(\alpha_j^{-1}\mathcal{V}_{N(j)}^{-1/2}\Lambda_j\mathcal{V}_{N(j)}^{-1/2}\big).$$

If the matrices $\mathcal{V}_{N(j)}^{-1/2}(\Lambda_i)_{N(j)}\mathcal{V}_{N(j)}^{-1/2}$ and $\mathcal{V}_{N(j)}^{-1/2}\Lambda_j\mathcal{V}_{N(j)}^{-1/2}$ commute, then the stronger inequality (16) holds.

In particular, if agents are equally risk averse, $\alpha_i = \alpha_j$, liquidity in exchanges $N(j)$ is lower for the more connected agents i (e.g., dealers or intermediaries) than their less connected counterparties. (In contrast, in a centralized market with assets from exchanges $N(j)$ and classes i and j, the agents would have the same price impact.) This can be understood through Lemma 5.1: Since the price impact of each agent in a given exchange is determined by the risk exposure of other agents in that exchange, class i's ability to diversify risk in exchanges $N(i) \setminus N(j)$ lowers their risk exposure in exchanges $N(j)$ relative to class j's exposure and, thus, the price impact of class j in exchanges $N(j)$.

The eigenvalues of matrix $\mathcal{V}_{N(j)}^{-1/2}\Lambda_j\mathcal{V}_{N(j)}^{-1/2}$ have a clear economic interpretation based on the min-max representation of eigenvalues.

Lemma 6.1 (Courant-Fisher Theorem) *Let $U \subset \mathbb{R}^{N(j)}$ be a subspace. For any symmetric matrix $A \in \mathbb{R}^{N(j)\times N(j)}$, the eigenvalues of $\mathcal{V}_{N(j)}^{-1/2}A\mathcal{V}_{N(j)}^{-1/2}$, $\mu_1 \leq \ldots \leq \mu_{N(j)}$ satisfy*

$$\begin{aligned}\mu_k &= \min_{\dim U = N(j)-k+1}\left\{\max\left\{y^T A y \mid y \in U,\ y^T\mathcal{V}_{N(j)}y \leq 1\right\}\right\} \\ &= \max_{\dim U = k}\left\{\min\left\{y^T A y | y \in U,\ y^T\mathcal{V}_{N(j)}y \leq 1\right\}\right\}.\end{aligned} \tag{18}$$

[36] In fact, f is not matrix monotone on any interval. This noteworthy property does not have any scalar analogues. This implies that, with sufficiently many assets, for any $A \geq 0$ there exists B, $B \leq A$, such that B is sufficiently close to A and the monotonicity fails (by the Löwner's Theorem). A function $f(z)$ is matrix monotone on some (even an arbitrarily small) interval if, and only if, it can be approximated by convex combinations of simple hyperbolic functions $\frac{\alpha}{z+\beta}$, $\alpha \in R_+$, $\beta \in R$. For the general theory of monotone matrix functions, see Löwner (1934) and Donoghue (1974).

Note that non-commutativity is essential here. If A and B commute, they can be diagonalized in the same basis and, clearly, the implication $A \geq B \Rightarrow f(A) \leq f(B)$ holds for diagonal matrices.

[37] The eigenvalue functional is not a linear functional (except in dimension one), but it is a minimax expression of linear functionals. Eigenvalue order and the positive semidefinite order are equivalent for commuting matrices. We also use techniques from the theory of monotone matrix functions to prove Proposition 7.2 in the welfare section.

Recall that $y^T \Lambda_i y$ is the execution cost from buying portfolio y for a class-i agent. In analogy with the mean-variance efficient portfolios, call a portfolio y *liquidity-efficient* if it minimizes the execution cost $y^T \Lambda_i y$ among all portfolios with the same riskiness $y^T \mathcal{V}_{N(j)} y = 1$. By Equation (18), $\mu_1(\mathcal{V}_{N(j)}^{-1/2} \Lambda_i \mathcal{V}_{N(j)}^{-1/2})$ (respectively, $\mu_{N(j)}(\mathcal{V}_{N(j)}^{-1/2} \Lambda_i \mathcal{V}_{N(j)}^{-1/2})$) correspond to the maximal (minimal) liquidity-efficiencies that can be achieved by class i. A subspace $U \subset R^{N(j)}$ can be interpreted as a set of investable funds (i.e., portfolios of traded assets) that an agent is allowed to hold. Then, for a k-dimensional subspace U of investable funds, each eigenvalue μ_k, $k = 1, ..., N(i)$ characterizes the maximal and minimal liquidity efficiencies that can be achieved by a class-i agent who chooses a portfolio of these funds. Thus, Proposition 6.3 captures that the maximal and minimal liquidity efficiency is lower for agents with greater participation. The minimal and maximal liquidity efficiencies are attained when y is an eigenvector of the appropriate eigenvalue. Since matrices $\mathcal{V}_{N(j)}$ and Λ_i do not commute in general, the maximal-efficiency portfolios differ for agents' whose participation in the market and endowments differ. The heterogeneous liquidity efficiencies generate additional welfare gains in decentralized markets that that have no counterparts in centralized markets (see Section 7).

6.3 Liquidity and Standardization of Assets

By Theorems 4.1 and 4.2, a change in the market structure has no impact on equilibrium liquidity if it does not affect the hypergraph of the associated regularized market – prices and liquidity in all exchanges along the equivalence loop with respect to an asset coincide and correspond to the one liquidity and one price in the associated regularized market. The converse is also true for generic covariance matrices: A local improvement in liquidity implies strict global improvement in liquidity for all indirectly connected exchanges and correlated assets.[38] Hence, the liquidity improvement in Theorem 6.1 is generically strict.

With regularization (equivalently) defined with respect to agents, and not assets, by Theorem 4.1, a change in participation affects liquidity if, and only if, it changes, for some trader i, the subsets of agent classes I in some exchanges $N(i)$ or the subsets of assets K he trades.[39] Respectively, these changes in participation correspond to a change in cross-exchange price impact, for a fixed set of exchanges, and the creation of a new exchange (the dimensions of price impacts changes) with some assets that are not part of an equivalence loop.

It follows from Corollary 6.1 that liquidity is the highest when all agents participate in all potential exchanges – a market structure equivalent to a centralized market.[40] Given Theorem 4.1, trader participation in all potential exchanges is not necessary – by Theorem 6.1, markets equivalent to

[38] With uncorrelated assets, price impacts are diagonal, and there is no relationship between liquidity for different assets in indirectly connected exchanges. Hence, strict liquidity improvement may fail for nongeneric covariance matrices.

[39] Price impact is neutral when participation increases from $((K_1(n), I(n)), (K_2(n), I(n)))$ to $((K(n), I(n)), (K(n), I(n)))$, but breaking up an exchange $(I(n), K(n))$ into $(I_1(n), K(n))$ and $(I_2(n), K(n))$ increases price impact (see the proof of Corollary 6.1). Given the set of assets traded by each agent, equilibrium depends only on the set of agents' trading counterparties.

[40] Equilibrium liquidity of class i in a decentralized market is always between i's liquidity in a centralized market ($M = \sum_i M_i$ agents, K assets) and the liquidity of class i who trades assets $N(i)$ only among themselves. That is,

a centralized market in the sense of Corollary 6.3 attain the maximal liquidity among all market structures for the same agents and assets.

Corollary 6.3 *Equilibrium schedules and price impacts* $\{(\Lambda_i, q_i(\cdot, \Lambda_i))\}_i$ *in a market with assets* K *and agent classes* I *coincide with those in a centralized market with the same assets and agents if, and only if, (i) for any agent* i *and any asset* k*, we have* $k \in K(N(i))$*; and (ii) for any asset* k *and any two exchanges* n, n' *with* $k \in K(n) \cap K(n')$*, there exists an equivalence loop with respect to* k *connecting* n *and* n'*.*

Consider the following sufficient condition for the equivalence: For any two exchanges n and n' with assets $K(n)$ and $K(n')$, there exists a path of exchanges $\{(n_l, I(n_l))\}_{l=1}^{L_{n,n'}}$ with respect to assets $K(n) \cup K(n')$ such that $n_1 = n$, $n_L = n'$, and $K(n_l) \cap (K(n_{l-1}) \cup K(n_{l+1})) = K(n_l)$, for all $l \neq \{1, L\}$ and $\sum_{i \in I(n_l)} M_i > 2$, for all l. In particular, disjoint subsets of assets K traded by disjoint subsets of agents I may trade at the liquidity as high as in a centralized market with a single exchange $n = (K, I)$.

While our analysis takes the market structure as given, the model has strong implications for the topological structure of the regularized market. Denote by $\mathbb{M}_k^*$ the sub-hypergraph of the regularized market that consists only of exchanges in which asset k is traded. By construction, the hypergraph of $\mathbb{M}_k^*$ does not contain loops and is, thus, a forest.[41]

Definition 6.1 *Agent* i *is a* monopolistic bridge *with respect to asset* k *if removing* i *from the market strictly increases the number of connected components in the hypergraph of* $\mathbb{M}_k^*$*.*

Corollary 6.4 (Equilibrium Hypergraph) *A market is regularized if, and only if, the hypergraph is such that if* $I(n) \cap I(n') \neq \varnothing$*, then* $K(n) \cap K(n') = \varnothing$ *or* $I(n) \cap I(n')$ *is a monopolistic bridge, for any* $n' \neq n$*.*

Corollary 6.3 implies that, to the extent that assets K are *standardized* in that they can hedge risks of sufficiently many market participants, provided there is no "local" monopoly power (in the sense of Definition 6.1), decentralized markets behave essentially as centralized markets.[42]

for $b \in \mathbb{R}_+$ defined in Proposition 2.1, in equilibrium, for all i,

$$\frac{2 - \alpha_i b + \sqrt{(\alpha_i b)^2 + 4}}{2b} \mathcal{V}_{N(i)} \leq \Lambda_i \leq \frac{\alpha_i}{M_i - 2} \mathcal{V}_{N(i)} . \tag{19}$$

Note that the lower bound does not require $M_i > 2$. Indeed, Λ_i / α_i can become arbitrarily large when $M_i = 2$.

41 A *forest* is an (undirected) graph, all of whose connected components are trees; its graph consists of a disjoint union of trees. Equivalently, a forest is a cycle-free graph.

42 Our results indicate that, insofar as the intermediaries who create dark pools have no monopoly power, the value created by liquidity pools, where publicly listed securities are traded, does not derive from pure liquidity motives. Institutional investors choose to participate in liquidity (dark) pools seeking privacy in execution of trades – to avoid exposure of their orders and front running (Knight Capital Group (2010); Angel, Harris, and Spatt (2011)).

Consistent with our results, liquidity pools (e.g., internal crossing networks in large banks), which trade publicly listed stocks, typically have no internal price setting mechanism but execute orders at the best price currently available at the public exchange ("price matching"). While there is no hard requirement for broker-dealers to match prices this way and the term "best execution" has not been defined clearly by U.S. regulation, this is the general business practice. See, e.g., `http://www.wallstreetandtech.com/internalization-is-it-really-that-bad/60404324.`

Likewise, so do markets with traders whose hedging needs are "less idiosyncratic."[43] Nevertheless, many products traded over the counter are notoriously hard to standardize. Types of financial products traded over the counter typically include either bespoke products or standardized but relatively illiquid assets.[44] Our results suggest that differences in the standardization of assets (relative to traders' hedging risks) endogenously give rise to a dichotomy in the types of traded assets and the associated market structures which, for homogeneous assets, are forests with respect to traders and, for multiple assets or bespoke products – with respect to assets and traders.[45]

6.4 Complementarity and Substitutability of Asset Payoffs

In centralized markets, equilibrium price impact is determined solely by the primitive covariance matrix Σ (cf. Proposition 2.1). With decentralized trading, agents' participation in different exchanges alters the riskiness of assets relative to Σ, and payoff substitutability or complementarity is no longer determined by Σ. Furthermore, both the riskiness and the covariance of assets traded in decentralized markets are typically heterogeneous across traders.

We illustrate the endogeneity in the riskiness of asset payoffs in a family of markets which include (at least) one perfectly liquid exchange and are otherwise arbitrary. The family allows us to isolate the endogenous cross-asset effects in a fully explicit fashion, while accommodating a variety of markets (e.g., any model from Example 1). The effects discussed carry over to general decentralized markets.

Example 3 Markets with a liquid exchange: *I classes of agents all trade assets $\{1, \cdots, K_1\}$ in exchange n. In addition, there are exchanges $\{n_l\}_l$, $l \in L$, $n_l \neq n$ in which assets $\{K_1 + 1, \cdots, K\}$*

[43] There is widespread evidence (cited in the Introduction) for equity markets in various countries, that the creation of new exchanges has increased competition and resulted in a single virtual market with multiple points of entry.

[44] Market participants looking to hedge specific risks may not find a standardized product that would effectively match their exposure and instead may prefer to use a bespoke product. Bespoke products often do not have the level of standardization required for trading on organized exchanges. For many, there may be no secondary market pricing sources. Demand for bespoke products comes from a variety of market participants, including end-users of the non-financial sector (e.g., airlines), end-users of the financial sector (e.g., banks and insurance companies), and institutional investors (e.g., mutual funds, pension funds, university endowments, and sovereign wealth funds). Furthermore, a variety of financial instruments that may not have an official market are traded by multilateral trading facilities (MTFs), which have fewer restrictions on the admittance of financial instruments for trading. Standardization of assets is considered as the central challenge in the ongoing changes in the regulation of OTC markets (Financial Stability Board (2010); Duffie (2013)).

[45] The literature on the microstructure of decentralized markets finds that a core-periphery and hub-and-spoke architectures are robust features in a number of markets for homogeneous assets. 1,000 banks participate in the U.S. Federal Funds market of overnight unsecured loans; however, each bank provides loans on average to only 3.3 other banks, and most banks have few counterparties while few have many (Bech and Atalay (2010); Afonso, Kovner, and Schoar (2012)). Craig and Peter (2010) document a core-periphery structure in the German banking system and Li and Schürhoff (2012) in the U.S. municipal bonds market. See also Cocco, Gomes and Martins (2009) for the Portugese interbank market. Moreover, in these markets, gross notional outstanding is highly asymmetric between smaller and larger institutions (e.g., Atkeson, Eisfeldt, and Weill (2013); Schachar (2013)). This is consistent with a (nontrivial) forest topology being associated with "local" monopoly power and underlying it asymmetric distribution of risks in the market (see Section 7).

In turn, bespoke financial products are typically traded by large financial institutions that design complex products for other institutional investors – a form of intermediation (Financial Stability Board (2010)). OTC credit default swaps markets are segmented; most small banks do not participate at all, and large banks serve as intermediaries between medium-size banks (e.g., Bech and Atalay (2010); Atkeson, Eisfeldt, and Weill (2013)).

are traded, and each class i can trade in a subset of these exchanges together with class $j \notin I$. Assume α_j/M_j is sufficiently small and $M_j \geq 2$. Let, for any i,

$$\mathcal{V}_{N(i)} = \begin{pmatrix} \mathcal{V}_i^{11} & \mathcal{V}_i^{12} \\ \mathcal{V}_i^{21} & \mathcal{V}_i^{22} \end{pmatrix}$$

be the block decomposition of $\mathcal{V}$ in $\mathbb{R}^{N(i)} = \mathbb{R}^{N(i)\cap N(j)} \oplus \mathbb{R}^{N(i)\setminus N(j)}$. Define, for any $i \neq j$,[46] *$\mathcal{V}_{i\setminus j} \equiv \mathfrak{S}(\mathcal{V}_i, N(i) \setminus N(j)) \in \mathbb{R}^{(N(i)\setminus N(j))\times(N(i)\setminus N(j))}$.*

Proposition 6.4 *Assume $M_j \geq 2$. Then, in the limit as $\alpha_j/M_j \to 0$, equilibrium price impacts in exchanges $N(j)$ vanish, $\Lambda_j \to 0$, whereas equilibrium price impacts in exchanges $N(i) \setminus N(j)$, $\Lambda_{i\setminus j} \equiv \Lambda_{i,N(i)\setminus N(j)}$, solve the system*

$$\Lambda_{i\setminus j} = \Pi_{N(i)\setminus N(j)} \left((M_i - 1)(\alpha_i \bar{\mathcal{V}}_{i\setminus j} + \bar{\Lambda}_{i\setminus j})^{-1} + \sum_{k\neq i,j} M_k \left(\alpha_k \bar{\mathcal{V}}_{k\setminus j} + \bar{\Lambda}_{k\setminus j}\right)^{-1} \right)^{-1} \lceil_{\mathbb{R}^{N(i)\setminus N(j)}}.$$

Furthermore, the demand slope for class-j agents in exchanges $N(i)$ coincides with $(\alpha_i \bar{\mathcal{V}}_{i\setminus j} + \bar{\Lambda}_{i\setminus j})^{-1}$ and $\mathbf{Q}_{N(j)} = 0$.

Proposition 6.4 can be understood through the standard results on Gaussian conditioning: Consider an agent from class i who chooses portfolio $q = \binom{y}{x} \in \mathbb{R}^{N(i)\cap N(j)} \oplus \mathbb{R}^{N(i)\setminus N(j)}$. The variance of this portfolio is given by $\mathrm{Var}(R_{N(i)}^T q) = q^T \mathcal{V} q$, and the minimal risk that the agent can achieve by trading in exchanges $N(j)$ is given by

$$\min_{y\in\mathbb{R}^{N(i)\cap N(j)}} \mathrm{Var}\left(R_{N(i)}^T \begin{pmatrix} y \\ x \end{pmatrix}\right) = x^T \mathfrak{S}(\mathcal{V}_i, N(i) \setminus N(j)) x\,. \tag{20}$$

Thus, $\mathcal{V}_{i\setminus j}$ is the agent's covariance matrix for the residual risks in $N(i) \setminus N(j)$, which cannot be hedged in the liquid exchanges $N(j)$. The greater the extent to which the market participation of class i overlaps with $N(j)$, the less the residual risk that i must bear. Thus, $\mathcal{V}_{i\setminus j}$ can be interpreted as *effective riskiness* of assets traded by class i, given participation $\{N(i)\}_i$.

Observe that matrices $\mathcal{V}_{i\setminus j}$ are no longer sub-matrices of the covariance matrix $\mathcal{V}$; the conditional covariances $\mathcal{V}_{i\setminus j}$ are linked to $\mathcal{V}$ is a nonlinear way. In general, there is no direct relationship between the residual covariance matrices $\mathcal{V}_{i_1\setminus j}$ and $\mathcal{V}_{i_2\setminus j}$ for different classes i_1 and i_2, unless they are equally connected with the risk neutral (or large) class j. If $N(j) \cap N(i_1) = N(j) \cap N(i_2)$, it follows directly from (20) that the effective riskiness for classes i_1 and i_2 is the same in the exchanges in which they both trade. Decentralized trading changes the riskiness of assets for different agents, even if it is common knowledge that asset payoffs are distributed $N(d, \Sigma)$. Example 4 illustrates.

Example 4 (Endogenous Payoff Riskiness) *Consider a market from Example 3.*

(1) Class j with a sufficiently small α_j/M_j trades J assets κ_i, $i = 1, \cdots, J$, in I_1 exchanges $N(j) = \{\kappa_1, \cdots, \kappa_J\}$; that is, we identify exchange κ_i with asset κ_i traded in this exchange.

[46] Recall that $\mathfrak{S}(\mathcal{V}_i, N(i) \setminus N(j)) = \mathcal{V}_i^{22} - \mathcal{V}_i^{21}(\mathcal{V}_i^{11})^{-1}\mathcal{V}_i^{12}$.

Some of the assets κ_i can be identical. In addition, I_1 classes $i \neq j$ can all participate in a common exchange n for $K(n)$ assets with the covariance matrix $V_{\{n\}} = (V_{k_1,k_2})_{k_1,k_2=1}^{K(n)}$, and class i can also trade asset κ_i in exchange κ_i; that is, $N(i) = \{n \cup \kappa_i\},\ i = 1, \cdots, I_1$. Let $V_{\kappa_i,k} = Cov(R_{\kappa_i}, R_k) = V_{\kappa_i,k},\ k = 1, \cdots, K(n),\ i = 1, \cdots, I_1$. Then, $N(i) \setminus N(j) = \{n\}$, and Proposition 6.4 implies that, in element-by-element notation, $V_{i\setminus j} \in R^{K(n)\times K(n)}$ of agent i is given by

$$\mathcal{V}_{i\setminus j} = \mathcal{V}_{\{n\}} - \mathcal{V}_{\kappa_i,\kappa_i}^{-1} \left(\mathcal{V}_{\kappa_i,k_1}\mathcal{V}_{\kappa_i,k_2}\right)_{k_1,k_2=1}^{|N(J)|}, \tag{21}$$

where $\mathcal{V}_{\kappa_i,\kappa_i}$ is the variance of asset κ_i. Thus, effective riskiness for the assets in exchange n differs across agents; in particular, the implied covariance between assets k_1 and k_2 for agent i is $\mathcal{V}_{k_1,k_2} - \mathcal{V}_{\kappa_i,\kappa_i}^{-1}\mathcal{V}_{\kappa_i,k_1}\mathcal{V}_{\kappa_i,k_2}$.

(2) *Suppose that the covariances $\mathcal{V}_{\kappa_i,k}$ and the variances $\mathcal{V}_{\kappa_i,\kappa_i},\ i = 1, \cdots, I_1,\ k = 1, \cdots, K(n)$ are multiplied by a common factor β, $|\beta| > 1$. Then, by (21), the implied covariance matrices decrease and a slight modification of Proposition 6.1 implies that equilibrium price impacts of all classes in exchange n decrease. That is, the more the assets in $K(n)$ covary with those in the liquid exchanges, the more risk can be hedged; this improves liquidity in exchange n. In particular, when the set $K(n)$ contains a single asset, the price impacts of all agents in n are monotone decreasing in the absolute value of any covariance $|\mathcal{V}_{\kappa_i,k}|$.*

As Example 4 shows, with decentralized trading, diagonal price impact $(\Lambda)_{nn}$ and cross-exchange price impact $(\Lambda)_{nm}$, $n \neq m$, are affected by asset covariances traded in different exchanges. Note that these results contrast sharply with centralized markets, where the equilibrium price impact matrix of every agent is proportional to matrix Σ; thus, introducing new assets (i.e., market completion) does not influence liquidity in trading the existing assets. The following corollary of Proposition 6.4 demonstrates that, in a decentralized market, depending on the market structure, an introduction of an asset (e.g., through an increase in participation) may give rise to essentially arbitrary[47] heterogeneity in effective riskiness, and consequently, in the cross-exchange price impact.

Corollary 6.5 *Consider a market with K assets with the covariance matrix Σ and I classes of agents with risk aversion $\{\alpha_i\}_i$. Let $Z_i,\ i \in I$ be an arbitrary collection of positive semidefinite matrices satisfying $Z_i \leq \Sigma$ for all i and such that either $\sum_i(\Sigma - Z_i) > \Sigma$, or $Z_1 \leq Z_2 \leq \cdots \leq Z_I$. Then, one can increase participation $\{N(i)\}_i$ that introduces additional assets available for trading such that class i's effective riskiness for the K assets is given by Z_i.*

Being linked to the non-commutativity of price impacts, by Proposition 3.3, the heterogeneity in the effective riskiness is a generic phenomenon in decentralized markets.

[47] Specifically, the diagonal elements $(\mathcal{V}_{i\setminus j})_{nn}$ of the effective riskiness matrix $\mathcal{V}_{i\setminus j}$ are lower than those for $\mathcal{V}$, whereas the off-diagonal elements $(\mathcal{V}_{i\setminus j})_{nm},\ n \neq m$, (i.e., effective cross-exchange covariances) can take any values satisfying $|(\mathcal{V}_{i\setminus j})_{nm}| < (\mathcal{V}_{nn}\mathcal{V}_{mm})^{1/2}$ (the Cauchy-Schwartz inequality).

7 Welfare in Decentralized Markets

In centralized markets, any improvement in liquidity also increases equilibrium utility and, hence, welfare. In this section, we show that the link between liquidity and equilibrium utility does not hold in decentralized markets. Thus, changes in market structure that lower liquidity may increase an agent's utility and total welfare.[48]

In hindsight, the relation between liquidity and welfare in a decentralized market depends on how exposure to aggregate and idiosyncratic risk contributes to an agent's equilibrium utility. Proposition 7.1 separates these contributions. Define

$$\Gamma_i(\Lambda_i) \equiv (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1} \left(\frac{1}{2} \alpha_i \mathcal{V}_{N(i)} + \Lambda_i \right) (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$$

and

$$\Delta_i(\Lambda_i) \equiv \frac{1}{2} \Lambda_i (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1} \alpha_i \mathcal{V}_{N(i)} (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1} \Lambda_i \,.$$

Proposition 7.1 (Indirect Utility) *The equilibrium utility of an agent from class i with initial holdings q_i^0 can be decomposed as*

$$U_i(\Lambda_i; q_i^0) = \underbrace{\mathbf{Q}_{N(i)}^T \Gamma_i(\Lambda_i)\, \mathbf{Q}_{N(i)}}_{\text{Utility from Aggregate Risk Exposure}} \tag{22}$$

$$\underbrace{-(q_i^0)^T \Delta_i(\Lambda_i)\, q_i^0}_{\text{Utility Loss from Idiosyncratic Risk Exposure}} \tag{23}$$

$$\underbrace{-2(\alpha_i \mathcal{V}_{N(i)} q_i^0)^T \Gamma_i(\Lambda_i) \mathbf{Q}_{N(i)}.}_{\text{Utility Loss from Covariance of Risks}} \tag{24}$$

Since matrices $\Gamma_i(\Lambda_i)$ and $\Delta_i(\Lambda_i)$ are positive definite, it is immediate that the utility compensation (22) for aggregate risk exposure and the utility loss (23) due to nondiversifiable idiosyncratic risk are positive and increasing in the aggregate (per capita) risk $\mathbf{Q}_{N(i)}$ and the idiosyncratic risk in the initial portfolio q_i^0. The covariance term (24) accounts for diversification benefits or losses that idiosyncratic risk exposure q_i^0 provides against the nondiversifiable aggregate risk portfolio $\mathbf{Q}_{N(i)}$. Using the identity[49] $\Delta_i(\Lambda_i) = 0.5\, \alpha_i \mathcal{V}_{N(i)} - \alpha_i \mathcal{V}_{N(i)}^T \Gamma_i(\Lambda_i) \alpha_i \mathcal{V}_{N(i)}$, it is useful to write the indirect utility net of the autarky disutility associated with idiosyncratic risk exposure from the initial holdings,

$$U_i(\Lambda_i; q_i^0) = \big(\mathbf{Q}_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0\big)^T \Gamma_i(\Lambda_i) \big(\mathbf{Q}_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0\big) - \frac{1}{2} (q_i^0)^T \alpha_i \mathcal{V}_{N(i)} q_i^0. \tag{25}$$

The gains from trade (and, hence, the individual utility) of a class-i agent are determined by two factors: the extent to which his initial holdings q_i^0 differ from the 'market portfolio'

[48] Again, the positive semidefinite order is incomplete; that is, for two matrices A and B, neither $A \geq B$ nor $A \leq B$ necessarily holds. Moreover, if $A \geq B \geq 0$ and $AB = BA$ then $A^2 \geq B^2$; if $A \geq B$, but $AB \neq BA$ then, generally, $A^2 \ngeq B^2$.

[49] See (42).

$(\alpha_i \mathcal{V}_{N(i)})^{-1}\mathbf{Q}_{N(i)}$ and the diversification opportunities relative to autarky, $\Gamma_i(\Lambda_i)$ versus $\alpha_i \mathcal{V}_{N(i)}$.

Suppose that, for some i, Λ_i and $\mathcal{V}_{N(i)}$ are proportional; $\Lambda_i = \lambda\, \mathcal{V}_{N(i)}$. Then the centralized market predictions follow: Larger price impact unambiguously lowers equilibrium utility; the utility compensation for bearing aggregate risk and the negative of the loss from residual idiosyncratic risk exposure both decrease.[50] Proposition 7.1 establishes that the proportionality is also necessary for each of the monotone relations.

Proposition 7.2 *Fix the risk premium vector* $\mathbf{Q}$. *Then, an increase in price impact decreases the utility of an agent from class* i *if, and only if,* $\mathbf{Q}_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0$ *is an eigenvector of* $(\alpha_i \mathcal{V}_{N(i)} + \Lambda)^{-1}(\alpha_i \mathcal{V}_{N(i)})^{-1}$.

Two matrices are proportional only if every vector is an eigenvector for both of them. Proposition 7.2 yields the following result.

Corollary 7.1 *In any decentralized market with equilibrium price impact* $\{\Lambda_i\}_i$, *an increase in price impact always decreases equilibrium compensation for aggregate risk exposure and increases equilibrium utility loss from idiosyncratic risk exposure if, and only if,* Λ_i *is proportional to* $\mathcal{V}_{N(i)}$.

In equilibrium, a change in the market structure impacts equilibrium utility (25) through the equilibrium price impact Λ_i and the equilibrium risk premium vector $\mathbf{Q}$. While the latter effect depends on the distribution of endowments in the economy, we can characterize the former explicitly. Proposition 7.2 thus concerns the utility effects of the changes in the market structure when the impact of $\mathbf{Q}$ is sufficiently small.[51] Note that, in the absence of aggregate risk, if traders had zero price impact, all risk could be diversified within each class and the classes would have no incentive to trade with one another. With positive equilibrium price impact, changes in participation affect the diversificaion opportunities within and across classes.

It follows from Proposition 7.2 that, in centralized markets, proportional order reduction per unit of marginal utility (hence the scaling by $(\alpha_i \mathcal{V}_{N(i)})^{-1}$) is Pareto efficient in the sense that it maximizes the equilibrium utility, given $\mathbf{Q}_{N(i)}$. By Proposition 3.3, the optimality of proportional order reduction is not generic in decentralized markets. The non-proportionality (and non-commutativity) of Λ_i and $\mathcal{V}_{N(i)}$ changes the welfare properties of decentralized markets in three central ways. First, breaking the link between the equilibrium demand slopes (i.e., $-(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$) and assets' covariance (Section 6.4) severs the centralized market proportionality between price

[50] When Λ_i and $\mathcal{V}_{N(i)}$ are scalar (i.e., one-dimensional), by direct calculation, $\Gamma_i(\Lambda_i)$ and $-\Delta_i(\Lambda_i)$ are monotone decreasing in Λ_i. For proportional price impact (i.e., if $\Lambda_i = \lambda \mathcal{V}_{N(i)}$),

$$\Gamma(\Lambda_i) = \frac{0.5\alpha_i + \lambda}{(\alpha_i + \lambda)^2}\, \mathcal{V}_{N(i)}^{-1} \text{ and } \Delta_i(\Lambda_i) = \frac{1}{2}\frac{\lambda^2 \alpha_i}{(\alpha_i + \lambda)^2}\, \mathcal{V}_{N(i)},$$

and, hence, the monotonicity in λ follows.

[51] For an example of a class of distributions for which the risk premium vector $\mathbf{Q}$ is independent of the equilibrium price impacts, take an arbitrary vector $Q' \in \mathbb{R}^N$ and suppose that the total initial holding of class i is $Q_i^0 = M_i\,(\alpha_i \mathcal{V}_{N(i)})^{-1}\, Q'_{N(i)}$, for all i. Then, $\mathbf{Q} = Q'$, there is no inter-class trade, and only agents within the same class trade with one other – by direct calculation using Theorem 5.1. Thus, the risk premium vector $\mathbf{Q}$ is independent of the equilibrium price impacts. In particular, if participation of some agents decreases from $N(i)$ to $N'(i) \subset N(i)$, $\alpha_i \mathcal{V}_{N'(i)}\, Q_i^0 = M_i\, Q'_{N'(i)}$ continues to hold and, hence, $\mathbf{Q} = Q'$ does too.

impact Λ_i and equilibrium order reduction $(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1} - \big(\alpha_i \mathcal{V}_{N(i)}\big)^{-1}$ as well. This affects the extent to which gains to trade $\mathbf{Q}_{N(i)} - \alpha_i \mathcal{V}_{N(i)} q_i^0$ are exhausted; for example, agents who trade riskier assets do not necessarily reduce their orders more. Second, non-proportional Λ_i and $\mathcal{V}_{N(i)}$ may act differently on different subspaces of portfolios. Consequently, while price impact is independent of endowments, through non-commutativity, decentralized trading changes the way aggregate and idiosyncratic endowment risks affect agents' utility. In particular, both an agent's compensation for aggregate risk exposure may increase and his utility loss from idiosyncratic risk exposure may decrease if his price impact increases (Propositions 7.3 and 7.4, Example 8).[52] Finally, given the noncompetitive trading, the equilibrium risk premium $\mathbf{Q}$ is not monotone in price impact. In fact, equilibrium response of $\mathbf{Q}$ to changes in the market structure may overturn the effects due to a change in Λ_i (Example 5). The remainder of this section characterizes and illustrates the new decentralized markets effects.

Proposition 7.3 relaxes the centralized markets eigenvector condition.

Proposition 7.3 (Breaking up an Exchange May Increase Welfare) *For any market from Example 3 such that, for some i, $\tilde{\Lambda}_i$ is not proportional to $\tilde{\mathcal{V}}_i$, there exists a participation and initial endowments such that the utility of every agent increases if some exchanges are split or some agents are excluded from participating in some exchanges.*

Thus, restricting participation of traders in some exchanges may improve equilibrium utility.

When the vector of risk premia $\mathbf{Q}$ is not independent of price impact, any change in the market structure also changes the aggregate risk premium vector $\mathbf{Q}$. This may lead to nonlinear changes in the allocation of risk across agents, as Example 5 illustrates.

Example 5 *Consider markets from Example 3 with only one asset traded in exchange n and normalize $M_i = 1$, for simplicity. Let $v_i \equiv \mathcal{V}_{i\setminus j} = \mathcal{V}^{11} - \mathcal{V}_i^{12}(\mathcal{V}_i^{22})^{-1}(\mathcal{V}_i^{12})^T$ be the effective riskiness of asset 1 for agents of class i. Then, Propositions 2.1 and 6.4 imply that the price impact $\lambda_i \equiv \Lambda_{i\setminus j}$ satisfies*

$$\Lambda_i = (2 - \alpha_i v_i b + \sqrt{(\alpha_i v_i b)^2 + 4})/(2b)$$

and b is the unique positive solution to $\sum_i (2 + \alpha_i v_i b + \sqrt{(\alpha_i v_i b)^2 + 4})^{-1} = 1/2$. Furthermore, agent i's equilibrium risky holdings are

$$q_i = \frac{1}{\alpha_i v_i + \Lambda_i} q^{Av} + \frac{\Lambda_i}{\alpha_i v_i + \Lambda_i} Q_i^0,$$

where the average portfolio is given by $q^{Av} \equiv b^{-1} \sum_i \gamma_i Q_i^0$, where $\gamma_i \equiv (\alpha_i v_i)/(\alpha_i v_i + \Lambda_i)$ is the order reduction of agent i. Equilibrium total welfare is given by

$$d_1 \sum_i Q_i^0 - \sum_i \frac{1}{2} \alpha_i v_i \left((\alpha_i v_i + \Lambda_i)^{-1} q^{Av} + \left(1 - \frac{\alpha_i v_i}{\alpha_i v_i + \Lambda_i}\right) Q_i^0 \right)^2$$

[52] In the proof of Proposition 7.1, we explicitly characterize the price impact matrices $\hat{\Lambda}_i$ that maximize $\mathbf{Q}_{N(i)}^T \Gamma_i(\hat{\Lambda}_i) \mathbf{Q}_{N(i)}$ over all $\hat{\Lambda}_i \geq \Lambda_i$ and the price impact matrices $\hat{\Lambda}_i'$ that maximize the loss $(q_i^0)^T \Delta_i(\hat{\Lambda}_i') q_i^0$ over all $\hat{\Lambda}_i' \leq \Lambda_i$.

and is monotone decreasing in q^{Av} when q^{Av} is sufficiently large.

Suppose that either the participation of some classes increases or a new, liquid public exchange for a nonredundant asset is introduced. Then, for all i, v_i decreases and, hence, Λ_i does too; however, $\{v_i, \Lambda_i\}_i$ are otherwise unrestricted. It follows that, for generic changes in participation and generic new assets introduced, vector $\{\gamma_i\}_i$ of equilibrium order reduction changes non-proportionally. Suppose that $\{\gamma_i\}_i$ changes to $\{\hat{\gamma}_i\}_i$. Thus, one can pick $\{Q^0_i\}_i$ that is orthogonal to $\{\gamma_i\}_i$ but not orthogonal to $\{\hat{\gamma}_i\}_i$. Then, a change in market structure gives rise to aggregate risk (i.e., q^{Av} becomes non-zero), which may reduce total welfare.

Two implications of Example 5 are worth noting. First, if the number of traders is not large and liquidity needs measured by endowment size are heterogeneous (cf. the determinants of the risk premium $\mathbf{Q}$ in Theorem 5.1), moving an asset traded OTC to a single centralized exchange can create systemic risk and lower welfare.[53] Also, markets in which specialists intermediate trading of particular assets can be associated with higher welfare than markets in which these assets are traded in a single exchange. More generally, our analysis sheds light on the instances in which different forms of intermediation improve efficiency; for instance, when dealers or brokers (who as opposed to dealers need not trade on their own account) or specialists (in trading particular assets) would be efficient. While adding a trader with a zero net endowment always increases liquidity, it may decrease utility. The model points to a relation between distribution of endowment risk and the market structure (the hypergraph; e.g., core versus periphery traders) and, thus, types of intermediation.[54]

The analysis thus far demonstrates that changes in market structure that increase price impact may also increase equilibrium utility. Example 6 shows that a decentralized market structure may Pareto-dominate a centralized market with the same set of agents and assets. In light of Proposition 7.2, a decentralized market may permit a better alignment of traders' gains to trade with trading opportunities in exchanges $\{N(i)\}_i$ in a way that is not feasible with centralized trading.[55]

[53] Indeed, OTC trading is considered superior for more customized products, but also for products that are standardized but less frequently traded, such as many single-name credit default swaps (e.g., Duffie (2012, 2013)). For the pros and cons of introducing centralized clearing houses for standardized assets traded OTC, see the discussion between the Director of Research at Bloomberg (Litan (2013)) and Duffie (2013).

[54] Although the term "dealer" carries some legal distinctions in certain markets, the main difference between a dealer and other market participants is that, by convention, an OTC dealer is usually expected to make "two-way markets"; a dealer usually has less difficulty in adjusting inventories. Central dealers hold larger and more volatile (risky) inventories on average and keep bonds longer than peripheral dealers (Li and Schürhoff (2012)). In the market for CDSs, Atkeson, Eisfeld and Weil (2013) document a relation between order imbalances and trader position in the network; Shachar (2012) finds that order imbalances of end-users in the CDSs market are associated with significant price impact, which depends on their direction relative to the direction of dealers' inventory.

[55] The proportionality constant in the eigenvector condition represents a 'single degree of freedom' in exhausting gains to trade $(\alpha_i \mathcal{V}_{N(i)})^{-1}\mathbf{Q}_{N(i)} - q^0_i$ in a centralized market,

$$(\alpha_i \mathcal{V}_{N(i)} + \Lambda)^{-1}(\alpha_i \mathcal{V}_{N(i)})^{-1}[(\alpha_i \mathcal{V}_{N(i)})^{-1}\mathbf{Q}_{N(i)} - q^0_i] = \mu[(\alpha_i \mathcal{V}_{N(i)})^{-1}\mathbf{Q}_{N(i)} - q^0_i].$$

In a decentralized market, Γ_i does not commute with $\mathcal{V}$, generically, and agents can benefit from different effective riskiness of the assets. Additionally, the impact of the initial holdings on the risk premium vector $\mathbf{Q}$ differs from that in a centralized market, where $\mathcal{V}^{-1}\mathbf{Q}$ is always a *linear* combination of the initial asset holdings. The decentralized market coefficients with which initial asset holdings enter $\mathbf{Q}$ are (non-commuting) matrices. Depending on the endogenous effective covariances, the coordinates of the risk premium $\mathbf{Q}$ may get amplified, thus improving the

Example 6 *Consider the market from Example 2 with two assets (e.g., complete market with two states), two classes of agents of size M_1 and M_2 with risk aversions α_1, α_2 and a third class with a single agent (an intermediary; $M_3 = 1$) whose risk aversion is normalized to one. Suppose first that there are two exchanges, A in which class-1 agents trade asset 1 with the intermediary and B in which class-2 agents trade asset 2 with the intermediary ($N_1 = \{A\}$ and $N_2 = \{B\}$). Thus, the two classes cannot trade directly. The initial holdings of asset j of the agents in class i are q_i^j, $j = 1, 2$. We show that the equilibrium utility can be higher for every agent in the dealer-intermediated market, compared to the centralized market, if the agents' risk exposures in the covariance between the initial portfolios of the two classes is negative. Let us parametrize the covariance matrix as follows,*

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_{11} & \varepsilon \mathcal{V}_{12} \\ \varepsilon \mathcal{V}_{12}^T & \mathcal{V}_{22} \end{pmatrix},$$

where the parameter ε measures the correlation between the assets. Let ε be sufficiently small. To examine when the two agent classes $i = 1, 2$ can be better off in a dealer-intermediated market, assume for simplicity that the initial endowments are such that, for the class-2 agents, there are no gains from trade in the centralized market. We show that this happens if, and only if, class 1 holds the same risk (in proportions) as class two, but with a larger exposure; $\alpha_1 q_1 = \kappa_2 \alpha_2 q_2 \equiv q = (q^1, q^2)$, $\kappa_2 > 1$, where $q_i = (q_i^1, q_i^2)$, $i = 1, 2$. In this case, class 2 is always better off in the decentralized market, whereas class 1 is better off if, and only if, two conditions hold: the risk, and hence the hedging benefit, in the second exchange, $\langle \mathcal{V}_{22} q^2, q^2 \rangle$, is sufficiently small, and q_1 differs sufficiently from $\mathbf{Q}$ in the decentralized market, so that the gains to trade are large enough for class 1. (Note that, while the compensation for the aggregate risk exposure is smaller in the decentralized market, the 'size' of aggregate risk differs in centralized and decentralized markets.) As we show, the gains from trade are the largest when α_2 is sufficiently larger than α_1 (e.g., $\alpha_1 < 0.35$ and $\alpha_2 > 5$).

Turning to the intermediary, his utility is affected differently by the covariance term in the centralized and decentralized markets: In the centralized market, the intermediary's equilibrium utility is increasing in the covariance between the portfolios of the two classes, whereas in the decentralized market, it is decreasing. Namely, in a centralized market, the standard CAPM holds and the risk premium that the intermediary receives for taking a fraction of other agents' risks is proportional to the covariance between the agents' risk exposures. In a decentralized market, being the only agent who trades in both exchanges, the intermediary has no cross-exchange price impact. Since, in equilibrium, he has to absorb significant fractions of the agents' risk exposures, he can only profit from these exposures provided that they provide good hedges against each other. When these initial exposures are sufficiently negatively correlated, the intermediary can be better off in the decentralized market.

In Section 6.2, we showed that agents who are better connected with the market, in the sense of having more links or being more centrally located, may have larger price impact. Here, we show that the agents who face larger price impact may also have greater equilibrium utility. We illustrate

agents' compensation for the risks taken.

this in the class of markets with liquid exchanges (Example 3, Section 6.4). By Proposition 6.4, the problem reduces to studying the price impact $\tilde{\Lambda}_i = \Lambda_{i\setminus j}$ of class i in the (illiquid) exchange n. Let $\Pi_{K(n)}$ be the orthogonal projection onto the subspace of assets traded in exchange n and let $\tilde{\mathbf{Q}} \equiv \mathbf{Q}_{K(n)}$ and $\tilde{q}_i^0 \equiv (q_i^0)_{K(n)}$ With $\tilde{\mathcal{V}}_i = \mathcal{V}_{i\setminus j}$ defined as in Proposition 6.4, and $\tilde{\Gamma}_i \equiv \tilde{\Gamma}_i(\tilde{\mathcal{V}}_i, \tilde{\Lambda}_i)$, $\tilde{\Delta}_i \equiv \tilde{\Delta}_i(\tilde{\mathcal{V}}_i, \tilde{\Lambda}_i)$, and by Equation (14), $\tilde{\Lambda}_i^{-1} + (\tilde{\Lambda}_i + \alpha_i \tilde{V}_i)^{-1} = \tilde{B}$ for all i, which implies a global upper bound on the price impact of all agents,

$$\tilde{\Lambda}_i < 2\tilde{B}^{-1}. \tag{26}$$

A direct calculation gives the following result.

Lemma 7.1 *In the markets from Example 3,* $\tilde{\Gamma}_j = 0.5\left(\tilde{B}\tilde{\Lambda}_j\tilde{B} - \tilde{B}\right)$ *and* $2\tilde{\Delta}_j = 3\tilde{\Lambda}_j\tilde{B}\tilde{\Lambda}_j - \tilde{\Lambda}_j\tilde{B}\tilde{\Lambda}_j\tilde{B}\tilde{\Lambda}_j$ *.*

Formulae of Lemma 7.1 enable us to identify the effects of the heterogeneous price impact on individual welfare. For example, Lemma 7.1 immediately yields that $\tilde{\Gamma}_{j_1} \geq \tilde{\Gamma}_{j_2}$ if, and only if, $\tilde{\Lambda}_{j_1} \geq \tilde{\Lambda}_{j_2}$. That is, agents with higher price impact receive a higher compensation for aggregate risk exposure. An analogous result for $\tilde{\Delta}_j$ only holds if all matrices commute. Since, in general, with multiple assets, there is no link between participation and price impact (see the discussion following Corollary 6.2), the welfare effects can go either way. We conclude by illustrating these phenomena.

Example 7 *In the markets from Example 3, suppose that the matrices* $\tilde{\Lambda}_{j_1}, \tilde{\Lambda}_{j_2}, \tilde{B}$ *commute. The utility of agents with larger price impact is larger (resp. smaller) than that of agents with smaller price impact if the aggregate risk is sufficiently large (resp. small) relative to the idiosyncratic risk. Consider exchange* n *with one asset traded, agents* j_1 *and* j_2 *with identical risk aversion and initial endowments* $\tilde{q}_{j_1}^0 = \tilde{q}_{j_2}^0 > 0$*, and assume that agent* j_1 *is better connected than agent* j_2*,* $N(j_1) \supset N(j_2)$*. Then, by Corollary 6.2, the price impact of agent* j_1 *is larger and, consequently, there exist thresholds* $r_1, r_2 > 0$ *such that agent* j_1 *attains higher (resp. lower) utility than* j_2 *if, and only if,* $\tilde{\mathbf{Q}}/\tilde{q}_{j_1}^0 \notin (-r_1, r_2)$*.*

Proposition 7.4 considers a market with a non-commutative equilibrium.

Proposition 7.4 (Non-commutativity and Idiosyncratic Risk) *In the markets from Example 3, suppose that the matrices* $\tilde{\Lambda}_{j_1}$ *and* $\tilde{B}$ *do not commute. Then, the loss due to idiosyncratic risk exposure can be smaller for the class with larger price impact.*

Based on Proposition 7.4, Example 8 provides a non-commutative counterpart of Example 7.

Example 8 *In the markets from Example 3, consider exchange* n *with at least two assets traded, and agents* j_1 *and* j_2 *with identical risk aversions, non-zero initial endowments* $\tilde{q}_{j_1}^0 = \tilde{q}_{j_2}^0 > 0$ *and no aggregate risk (*$\tilde{\mathbf{Q}} = 0$*), so that the agents only receive negative utility from the residual idiosyncratic risk exposure. Then, there exists participation such that* $N(j_1) \supset N(j_2)$ *and agent* j_1 *has a larger price impact than* j_1*, and yet agent* j_1 *attains a higher equilibrium utility.*

Finally, we would like to understand the link between connectedness and price impact. The next result demonstrates the role of non-commutativity in Proposition 6.3.

Proposition 7.5 (Commutativity, Connectedness and Price impact) *If $\tilde{B}^{1/2}\tilde{\mathcal{V}}_{j_1}\tilde{B}^{1/2}$ and $\tilde{B}^{1/2}\tilde{\mathcal{V}}_{j_2}\tilde{B}^{1/2}$ commute and $\tilde{\mathcal{V}}_{j_1} \leq \tilde{\mathcal{V}}_{j_2}$, then $\tilde{\Lambda}_{j_1} \geq \tilde{\Lambda}_{j_2}$. However, for any $\tilde{B}$ and $\tilde{\mathcal{V}}_{j_1}$ that do not commute and satisfy $\tilde{B} > 2\tilde{\mathcal{V}}_{j_1}^{-1}$, there exists $\mathcal{V}_{j_2} \geq \mathcal{V}_{j_1}$ such that $\tilde{\Lambda}_{j_1} \not\geq \tilde{\Lambda}_{j_2}$.*

Example 9 builds on Proposition 7.5.

Example 9 *Within Example 3, consider exchange n with at least two assets traded and agents j_1 and j_2 with identical risk aversions and* zero *initial endowments $\tilde{q}^0_{j_1} = \tilde{q}^0_{j_2} = 0$. Then, there exists participation such that agent j_1 is better connected than agent j_2, that is, $N(j_1) \supset N(j_2)$, and agent j_1 attains lower equilibrium utility.*

8 Discussion

This paper takes the exchanges and the assets available to trade in decentralized markets as exogenous. Our analysis suggests that the study of endogenous formation of exchanges in decentralized markets, with respect to welfare or other objectives, should not be separated from security design. Furthermore, the endogeneity and heterogeneity of residual riskiness of assets traded in decentralized markets imply that efficiency and profit opportunities from security design as well as specialization in trading certain assets exist that are not available in centralized markets. Endogenizing exchange creation and asset structure in decentralized markets is studied in Malamud and Rostek (2013).

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A Equilibrium Existence and Uniqueness

A profile of demand functions $\{q_i(\cdot)\}_i$ is a *robust Nash Equilibrium* if, for each trader i, $q_i(p_{N(i)})$ is a best response given $\{q_j(p_{N(j)})\}_{j\neq i}$ for an arbitrary additive noise.

Proof of Theorem 2.1. Consider market $\mathbb{M} = \{\{\alpha_i, M_i, N(i)\}_i, (K, d, \mathcal{V})\}$ and let N be the total number of exchanges in $\mathbb{M}$. Let $q_i(p_{N(i)}, \Lambda_i)$ be trader i's optimal demand given his assumed price impact Λ_i, defined in Equation (2), given the quasilinear utility function (1): for all prices $p_{N(i)}$, i equalizes marginal utility and marginal payment,

$$d_{N(i)} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i) q_i(p_{N(i)}, \Lambda_i) \ = p_{N(i)} + \alpha_i \mathcal{V}_{N(i)} q_i^0 \,. \tag{27}$$

For the "if" part, fix demand schedules $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ submitted by all traders given their assumed price impact $\hat{\Lambda}_i$. Assume that the Jacobian of each trader's residual supply, defined by $\{q_j(p_{N(j)}, \hat{\Lambda}_j)\}_{j\neq i}$, is $\hat{\Lambda}_i = -(\sum_{j\neq i} D_p q_j(\cdot, \hat{\Lambda}_j))^{-1}_{N(i)}$, for each i. The price $\hat{p}_n$ that clears exchange n is determined by $\sum_{i\in I(n)} q_i(p_n, p_{N(i)\setminus\{n\}}, \hat{\Lambda}_i) = 0$. Since for each i and n, demand $q_i(\cdot, \hat{\Lambda}_i)$ satisfies condition (27) for all prices p_n, it does so for the exchange-clearing price $\hat{p}_n$. By the global concavity of the maximization problem of each trader, demand functions $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ are best responses at $\hat{p}_n$. With (nondegenerate support) uncertainty about the traders' initial holdings, trader i's residual supply with slope $\hat{\Lambda}_i$ has a stochastic intercept. Since, for each i and n, condition (27) holds for all prices p_n, it holds for each realization of the residual supply. Hence, given the global concavity, $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ is a Nash equilibrium with an arbitrary additive noise and, thus, a robust Nash Equilibrium.

For the "only if" part, suppose that traders submit demand functions $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ such that price impact $\hat{\Lambda}_i \neq -(\sum_{j\neq i} D_{p_{N(j)}} q_j(\cdot, \hat{\Lambda}_j))^{-1}$ for some i. Then, schedule $q_i(p_{N(i)}, \hat{\Lambda}_i)$ is not a best response to $\{q_j(p_{N(j)}, \hat{\Lambda}_j)\}_{j\neq i}$ at the exchange-clearing price $\hat{p}_n$, defined by $\sum_{i\in I(n)} q_i(\hat{p}_n, \hat{p}_{N(i)\setminus\{n\}}, \hat{\Lambda}_i) = 0$. With an additive perturbation, by the linearity of demands, for each trader i, for almost all prices p_n, $d_{N(i)} - (\alpha_i \mathcal{V}_{N(i)} + \Lambda_i) q_i(p_{N(i)}, \Lambda_i) \neq p_{N(i)} + \alpha_i \mathcal{V}_{N(i)} q_i^0$, where Λ_i is the Jacobian of the residual supply of trader i. The prices for which the equality is violated have measure one and the schedule that equalizes marginal utility with marginal payment for all prices p_n (i.e., satisfies condition (27)) gives a strictly higher utility for measure one of noise realizations (and, hence, a strictly higher expected utility). It follows that $q_i(p_{N(i)}, \hat{\Lambda}_i)$ is not a robust best response and noise exists for which $\{q_i(p_{N(i)}, \hat{\Lambda}_i)\}_i$ is not a robust Nash Equilibrium. ■

Lemma A.1 *Let A be a positive definite matrix and $A_{11} = \Pi_{N(i)} A\lceil_{\mathbb{R}^{N(i)}}$. Then, $A^{-1} \geq \bar{A}_{11}^{-1}$.*

Proof of Lemma A.1. Let $B = A^{-1}$, $x \in \mathbb{R}^{N(i)}$ and $y \in Y^{N\setminus N(i)}$. Then,

$$\min_{y\in\mathbb{R}^{N\setminus N(i)}} \langle B(x,y),(x,y)\rangle = \langle (B_{11} - B_{12}B_{22}^{-1}B_{21})x, x\rangle. \tag{28}$$

By the Frobenius formula (Lemma C.4), $B_{11} - B_{12}B_{22}^{-1}B_{21} = ((B^{-1})_{11})^{-1} = A_{11}^{-1}$. Therefore,

$$\langle A^{-1}(x,y),(x,y)\rangle = \langle B(x,y),(x,y)\rangle \geq \langle A_{11}^{-1}x, x\rangle = \langle \bar{A}_{11}^{-1}(x,y),(x,y)\rangle,$$

for any $(x,y) \in \mathbb{R}^N$, and the claim follows. ■

Let $D = \text{diag}(z),\ z \in \mathbb{R}^N$, be a diagonal matrix. Multiplication of (4) by $D_{N(i)}$ from the left and from the right gives the following scale invariance property of price impacts.

Lemma A.2 *Let $D = \text{diag}(z),\ z \in \mathbb{R}^N$, be a diagonal matrix and $\mathcal{V}' = D\mathcal{V}D$. Then, the map $\{\Lambda_i\}_i \to \{D_{N(i)}\Lambda_i D_{N(i)}\}$ defines a one-to-one correspondence between equilibria in markets defined by $\mathcal{V}$ and $\mathcal{V}'$, respectively.*

Proof of Proposition 2.1. The fact that this is indeed an equilibrium follows by direct calculation. The uniqueness follows from Lemma *A.5* below after one diagonalizes $\mathcal{V}$. ■

Let $F = \{F_i\}_i : \mathcal{S}^I \to \mathcal{S}^I$ be the map defined by the right-hand side of (4). By construction, the maps F and G are simple transformation of each other. However, analytically, it is more convenient to work directly with the map F. For this reason, all the proofs in the sequel are performed using this map. Passing from F to G is then straightforward.

Proof of Lemmas 3.1 and 3.2. By Theorem 2.1, $\{\Lambda_i\}_i$ is an equilibrium if and only if $\{\Lambda_i\}_i = F(\{\Lambda_i\}_i)$. By direct calculation, defining $X_i = (\Lambda_i + \alpha_i \mathcal{V}_{N(i)})^{-1}$, we get that $\{\Lambda_i\}_i = F(\{\Lambda_i\}_i)$ if and only if $\{X_i\}_i = G(\{X_i\}_i)$. The fact that both F and G are monotone increasing follows because the map $Y \to Y^{-1}$ is monotone decreasing, whereas $Y \to Y_{N(i)}$ is monotone increasing in the positive semi-definite order. ∎

Proof of Proposition 3.1. Pick an arbitrary starting tuple $\{X_i^0\}_i$ such that $\{X_i^0\}_i \le G(\{X_i^0\}_i)$. By direct calculation, the corresponding price impacts $\Lambda_i^0 = (X_i^0)^{-1} - \alpha_i \mathcal{V}_{N(i)}$ satisfy $\{\Lambda_i^0\}_i \ge F(\{\Lambda_i^0\}_i)$. Since map F is continuous and monotonic with respect to the defined partial order, recursively applying F to the inequality $\{\Lambda_i^0\}_i \ge F(\{\Lambda_i^0\}_i)$, we get that $F^n(\{\Lambda_i^0\}_i)$ is monotone decreasing and, therefore, converges to a fixed point of F. For the second case, we get a price impact tuple satisfying $\{\Lambda_i^0\}_i \le F(\{\Lambda_i^0\}_i)$, so that the sequence $F^n(\{\Lambda_i^0\}_i)$ is monotone increasing. Therefore, to prove convergence to a fixed point, we need to show that it is bounded from above. To this end, pick $\alpha > 0$ sufficiently large so that $\{\tilde{\Lambda}_i\}_i$ defined by

$$\tilde{\Lambda}_i = \alpha \mathrm{diag}((M(n) - 2)^{-1})_{k \in N(i)} \tag{29}$$

satisfies $\{\Lambda_i^0\}_i \le \{\tilde{\Lambda}_i\}_i$. An analogous argument implies that

$$F(\{\tilde{\Lambda}_i\}_i) \le \{\tilde{\Lambda}_i\}_i.$$

Let $\Omega = \{\{\Lambda_j\}_j \in \mathcal{S}^{\mathbb{M}} : \Lambda_j \le \tilde{\Lambda}_j,\ \forall j\}$. Then, for any $\{\Lambda_j\}_j \in \Omega$,

$$F(\{\Lambda_j\}_j) \le F(\{\tilde{\Lambda}_j\}_j) \le \{\tilde{\Lambda}_i\}_i \tag{30}$$

and, hence, F maps Ω into itself. Therefore, the sequence $F^n(\{\Lambda_i^0\}_i)$ is monotone increasing, bounded from above by $\{\tilde{\Lambda}_i\}_i$, and hence converges to a fixed point of F. ∎

Proof of Theorem 3.1. Existence of equilibria for the case when $\mathcal{V}_{N(i)}$ is nondegenerate for any i follows from Proposition 3.1. For the general case, let $\mathcal{V}^\varepsilon \equiv \mathcal{V} + \varepsilon \mathrm{Id}$ and let F^ε be the corresponding map. By Proposition 3.1, for any $\varepsilon > 0$, there exists an equilibrium $\{\Lambda_i^\varepsilon\}$ corresponding to $\mathcal{V}^\varepsilon$. Pick a sequence $\varepsilon_k = 1/k$ and an equilibrium $\{\Lambda_i^{\varepsilon_k}\}_i$. Since F^ε is monotone increasing in ε, we have

$$\{\Lambda_i^{\varepsilon_k}\}_i = F^{\varepsilon_k}(\{\Lambda_i^{\varepsilon_k}\}_i) \ge F^{\varepsilon_{k+1}}(\{\Lambda_i^{\varepsilon_k}\}_i)$$

and, hence, by Proposition 3.1, there exists an equilibrium $\{\Lambda_i^{\varepsilon_{k+1}}\}_i \le \{\Lambda_i^{\varepsilon_k}\}_i$. Thus, we can construct a monotone decreasing sequence $\{\Lambda_i^{\varepsilon_k}\}_i$ which converges to an equilibrium corresponding to $\varepsilon = 0$.

To prove generic determinacy, let, for each i, $X_i \equiv \Lambda_i + \alpha_i \mathcal{V}_{N(i)}$, and define a map $\Phi : \mathcal{S}^I \to \mathcal{S}^I$ via

$$\Phi_i(\{X_j\}) = X_i - \left(\left((M_i - 1)\, X_i^{-1} + \sum_{j \neq i} M_j\, X_j^{-1} \right)^{-1} \right)_{N(i)} .$$

The equilibrium equation can be written as $\Phi(\{X_j\}_j) = \{\alpha_j \mathcal{V}_{N(j)}\}_j$. Let Ψ^I be the image of the map $\mathcal{V} \to \{\alpha_i \mathcal{V}_{N(i)}\}_i$ defined on the set of positive semidefinite matrices. Let Θ^I be the subset of $\mathcal{S}^I$ such that, for any $\{X_j\} \in \Theta^I$, we have that $\Phi(\{X_j\}) = \{\alpha_j \mathcal{V}_{N(j)}\}_j$ for some positive definite matrix $\mathcal{V}$. Θ^I is an algebraic variety and, therefore, can be represented as a finite union of irreducible algebraic sets that are smooth manifolds. The same is true for Ψ^I. By Sard's theorem, almost every $\{\alpha_i \mathcal{V}_{N(i)}\}_i \in \Psi^I$ has a regular pre-image under Φ, that is, equilibria are determinate for generic covariance matrices. The finiteness of the set of equilibria follows by the standard compactness arguments and the fact that, by Proposition 3.2, all equilibria belong to a compact set. ■

Since equilibrium uniqueness is equivalent to the uniqueness of the fixed point of map F (Theorem 2.1), we can identify the strategy of an agent in the game (i.e., his demand schedule) with its slope $(\alpha_i \mathcal{V}_{N(i)} + \Lambda_i)^{-1}$, and find conditions on the demand slopes. Equilibrium uniqueness is equivalent to the uniqueness of the fixed point of map F It therefore suffices to show that F is a contraction on a suitably defined normed space.

Lemma A.3 *For any i, suppose that $0 \leq \{B_i\}_i \leq \{A_i\}_i$ are such that any equilibrium tuple $\{\Lambda_i\}_i$ satisfies $\{B_i\}_i \leq \{(\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i)^{-1}\}_i \leq \{A_i\}_i$. Suppose that, for any $\{X_i\}_i$, $\{B_i\}_i \leq \{X_i\}_i \leq \{A_i\}_i$,*

$$(M_j - 1) X_j^2 + \sum_{i \neq j} M_i X_i^2 < \left((M_j - 1) X_j + \sum_{i \neq j} M_i X_i \right)^2 , \; j = 1, \cdots, I . \tag{31}$$

Then, map F is a contraction on the set $\{B_i\} \leq \{X_i\} \leq \{A_i\}$ and, hence, there exists a unique equilibrium.

Proof of Lemma A.3. Let us calculate the derivative of map F. That is, consider an infinitesimal change $\{\Lambda_i\} \to \{\Lambda_i + \varepsilon\, Y_i\}$. Then, a direct calculation based on the identity

$$(U + \varepsilon V)^{-1} \approx U^{-1} - \varepsilon\, U^{-1} V U^{-1}$$

implies that the Frechet derivate of F is given by

$$\begin{aligned} \frac{\partial F_j}{\partial(\{\Lambda_i\}_i)}(\{Y_i\}_i) \;&=\; \Pi_{N(j)} \left((M_j - 1) X_j + \sum_{i \neq j} M_i X_i \right)^{-1} \left((M_j - 1) X_j Y_j X_j + \sum_{i \neq j} M_i X_i Y_i X_i \right) \\ &\quad \times \left((M_j - 1) X_j + \sum_{i \neq j} M_i X_i \right)^{-1} . \end{aligned}$$

Introduce a norm of the set of I-tuples of positive semidefinite matrices via $|||\{Y_i\}||| = \max_i \|Y_i\|_{N(i)}$, where $\|\cdot\|_{N(i)}$ is the standard norm on matrices in $\mathbb{R}^{N(i)}$ defined by

$$\|Y\| = \max_{x\in\mathbb{R}^N, x\neq 0} \frac{\|Yx\|}{\|x\|}.$$

For simplicity, in the sequel we omit the index $N(i)$ for the norms. For a symmetric matrix, $\|Y\| = \max|\mathrm{eig}(Y)|$ and, therefore, $Y_i \in [-\|Y_i\|\mathrm{Id}_{N(i)}\,,\, \|Y_i\|\mathrm{Id}_{N(i)}]$. Suppose now that condition (31) holds. Then,

$$\left\|\left((M_j-1)X_j+\sum_{i\neq j}M_iX_i\right)^{-1}\left((M_j-1)X_j^2+\sum_{i\neq j}M_iX_i^2\right)\left((M_j-1)X_j+\sum_{i\neq j}M_iX_i\right)^{-1}\right\|\leq 1\,,$$

and hence

$$\begin{aligned}
&\frac{\partial F_j}{\partial\{\Lambda_i\}}(\{Y_i\})\\
=\;& \Pi_{N(j)}\left((M_j-1)X_j+\sum_{i\neq j}M_iX_i\right)^{-1}\left((M_j-1)X_jY_jX_j+\sum_{i\neq j}M_iX_i|||\{Y_i\}|||\mathrm{Id}_{N(i)}X_i\right)\\
=\;& \left((M_j-1)X_j+\sum_{i\neq j}M_iX_i\right)^{-1} \;<\; |||\{Y_i\}|||\mathrm{Id}_{N(j)}.
\end{aligned}$$

The same argument implies

$$\frac{\partial F_j}{\partial\{\Lambda_i\}}(\{Y_i\}) \;>\; -|||\{Y_i\}|||\mathrm{Id}_{N(j)}\,.$$

That is,

$$\left\||\frac{\partial F_j}{\partial\{\Lambda_i\}}(\{Y_i\})\right\|| \;<\; |||\{Y_i\}|||\,.$$

That is, map F is a contraction on this set and, consequently, cannot have more that one fixed point. ■

Note that, when X_i are positive numbers (or commuting matrices, in which case they can be simultaneously diagonalized), a direct calculation implies that condition (31) holds. However, absent commutativity, this is generally not true. The usefulness of Lemma A.3 depends on a good choice of the bounds $\{B_i\}_i$ and $\{A_i\}_i$. The next result provides a simple and easily verifiable condition that guarantees the applicability of Lemma A.3, based on the choice $\{B_i\}_i = \{(\bar\Lambda^0_{i,\max} + \alpha_i\bar{\mathcal{V}}_{N(i)})^{-1}\}_i$ and $\{A_i\}_i = \{(\bar\Lambda^0_{i,\min} + \alpha_i\bar{\mathcal{V}}_{N(i)})^{-1}\}_i$.[56]

[56] As an example, consider the case when all pairs of assets are equally correlated with correlation ρ, all agents have same risk aversion α, and $\max_i |N(i)| \leq \hat N$ Then, $\max(\mathrm{eig}(C(\mathcal{V}_{N(i)}))) \leq \max\{1+\rho(\hat N-1), 1-\rho\}$ and $\min(\mathrm{eig}(C(\mathcal{V}_{N(i)}))) \geq \min\{1+\rho(\hat N-1), 1-\rho\}$ and (32) thus imposes upper and lower bounds on the correlation ρ. For example, in the symmetric case when $M(n) = \hat M$ is independent of n and $\rho > 0$, we obtain the simple condition $\rho < \frac{\hat M-2}{\hat M+\hat N-2}$. For Corollary A.1, one could also pick $\{B_i\}_i = \{(\bar\Lambda^k_{i,\max} + \alpha_i\bar{\mathcal{V}}_{N(i)})^{-1}\}_i$

Corollary A.1 *Suppose that*

$$\min_n \frac{M(n)-2}{\lambda^*(n)} \geq \max_n \frac{M(n)-2}{(M(n)-1)\lambda^*(n)}. \tag{32}$$

Then, equilibrium is unique.

Roughly, the left-hand side of (32) measures how competitive an exchange is, whereas the right-hand side reflects the dispersion of payoff riskiness across exchanges. If this dispersion is high, there is a lot of 'room' for non-commutativity and uniqueness can only be guaranteed when strategic effects are small; that is, when $M(n)$ is sufficiently large.

and $\{A_i\}_i = \{(\bar{\Lambda}^k_{i,\min} + \alpha_i \bar{\mathcal{V}}_{N(i)})^{-1}\}_i$, for any $k \geq 1$.

Supplementary Appendix

To proceed further, we first establish auxiliary results.

Lemma A.4 *If there is only one asset, $K = 1$, then equilibrium is unique.*

Proof. The proof follows directly from Lemma A.3. Indeed, in this case, the conditions of Lemma A.3 hold and, therefore, map F is a contraction and has a unique fixed point. ■

Lemma A.5 *Let $\{A_i\}_i \in \mathcal{S}^I$ be a tuple of diagonal matrices. Consider the map $F_A : \mathcal{S}^I \to \mathcal{S}^I$ defined via*

$$F_i(\{\Lambda_j\}_j) = \left(\left((M_i - 1)(\bar{A}_i + \alpha_i \bar{\Lambda}_i)^{-1} + \sum_{j \neq i} M_j (\bar{A}_j + \alpha_j \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)} .$$

Then, this map has a unique fixed point in the class of diagonal matrices.

Proof. The proof follows directly from Lemma A.3 because, on the set of diagonal matrices, F is a contraction. ■

Lemma A.6 *Let $\{A_i\}_i \in \mathcal{S}^I$ be a tuple of diagonal matrices. Then, map F_A from Lemma A.5 has a unique fixed point.*

Proof. Let $\{\Lambda_i^A\}_i$ be an arbitrary fixed point of F_A, and let $\{\Lambda_i^*\}_i$ be the diagonal fixed point, which is unique by Lemma A.5. Pick $\beta_1 \in \mathbb{R}_+$ so that β_1 satisfies $\beta_1 \mathrm{Id}_{N(i)} \leq \Lambda_i^A$ for all i and $\beta_1 \leq \min_i \min(\mathrm{eig}(A_i))$. Similarly, pick $\beta_2 \in \mathbb{R}_+$ so that β_2 satisfies $\beta_2 \mathrm{Id}_{N(i)} \geq \Lambda_i^A$ for all i and $\beta_2 \geq \max_i \max(\mathrm{eig}(A_i))$. Define $\{B_{1k}\}_i \equiv \{\beta_k \mathrm{Id}_{N(i)}\}_i$, $k = 1, 2$, and let F_{B_k}, $k = 1, 2$ be the corresponding maps. Then, define $\{\Lambda_i^{B_k}\}_i = \beta_k \{\mathrm{diag}((M(n) - 2)^{-1})_{N(i)}\}_i$. We have,

$$\{\Lambda_i^{B_k}\}_i = F^{B_k}(\{\Lambda_i^{B_k}\}_i).$$

Therefore, iterating the inequality

$$\{\Lambda_i^{B_1}\}_i \ = \ F^{B_1}(\{\Lambda_i^{B_1}\}_i) \ \leq \ F_A(\{\Lambda_i^{B_1}\}_i),$$

we obtain that $F_A^n(\{\Lambda_i^{B_1}\}_i)$ converges to a diagonal fixed point of F_A, and hence, by Lemma A.5, converges to $\{\Lambda_i^*\}_i$. A similar argument implies that $F_A^n(\{\Lambda_i^{B_2}\}_i)$ also converges to $\{\Lambda_i^*\}_i$.

Now, by the definition of β_k, $k = 1, 2$, we also have

$$F^{B_1}(\{\Lambda_i\}_i) \ \leq \ F_A(\{\Lambda_i\}_i) \ \leq \ F^{B_2}(\{\Lambda_i\}_i),$$

for any $\{\Lambda_i\}_i \in \mathcal{S}^I$. Therefore, by the monotonicity of map F_A,

$$F_A^n(\{\Lambda_i^{B_1}\}_i) \ \leq \ F_A^n(\{\Lambda_i^A\}_i) \ \leq \ F_A^n(\{\Lambda_i^{B_2}\}_i) \, .$$

Taking $n \to \infty$ and using the fact that $F_A^n(\{\Lambda_i^A\}_i) = \{\Lambda_i^A\}_i$, we get $\{\Lambda_i^*\}_i \le \{\Lambda_i^A\}_i \le \{\Lambda_i^*\}_i$, and the claim follows. ■

Let $C(\mathcal{V}) = \operatorname{diag}(\{\mathcal{V}_{nn}^{-1/2}\})\mathcal{V}\operatorname{diag}(\{\mathcal{V}_{nn}^{-1/2}\})$ be the correlation matrix of the assets. For any exchange n and agent i, define $\alpha_{i,*} \equiv \alpha_i \min(\operatorname{eig}(C(\mathcal{V})_{N(i)}))$ and $\alpha_i^* \equiv \alpha_i \max(\operatorname{eig}(C(\mathcal{V})_{N(i)}))$; $\alpha_{i,*}$ and α_i^* can be interpreted as the bounds on the effective riskiness (see Section 6.4). For any exchange n, define two constants $\lambda^*(n) \equiv \min_{i\in I(n)} \alpha_{i,*}$ and $\lambda^*(n) \equiv \max_{i\in I(n)} \alpha_i^*$ and let $M(n) = \sum_{i\in I(n)} M_i$ be the number of agents trading in n. Let further $\{\Lambda_{i,\min}^0\} = \left\{\operatorname{diag}(\frac{\lambda_*(n)\mathcal{V}_{nn}}{M(n)-2})_{N(i)}\right\}_i$ and $\{\Lambda_{i,\max}^0\} = \left\{\operatorname{diag}(\frac{\lambda^*(n)\mathcal{V}_{nn}}{M(n)-2})_{N(i)}\right\}_i$ and $\{X_{i,\min}^0\}_i = \{(\alpha_i\mathcal{V}_{N(i)} + \Lambda_{i,\max}^0)^{-1}\}_i$, $\{X_{i,\max}^0\}_i = \{(\alpha_i\mathcal{V}_{N(i)} + \Lambda_{i,\min}^0)^{-1}\}_i$. A direct calculation implies that

$$\{X_{i,\min}^0\} \le G(\{X_{i,\min}^0\}) \text{ and } \{X_{i,\max}^0\} \ge G(\{X_{i,\max}^0\})\,.$$

and, similarly,

$$\{\Lambda_{i,\min}^0\} \le F(\{\Lambda_{i,\min}^0\}) \text{ and } \{\Lambda_{i,\max}^0\} \ge F(\{\Lambda_{i,\max}^0\})\,.$$

We now construct the minimal and maximal equilibria by the explicit iterative procedure described in Proposition 3.1. To this end, define recursively two sequences $\{\Lambda_{i,\min}^k\}_i \in \mathcal{S}^I$ and $\{\Lambda_{i,\max}^k\}_i \in \mathcal{S}^I,\ \ k \ge 1$ via $\{\Lambda_{i,\min}^k\}_i \equiv F(\{\Lambda_{i,\min}^{k-1}\}_i)$ and $\{\Lambda_{i,\max}^k\}_i \equiv F(\{\Lambda_{i,\max}^{k-1}\}_i)$. By Proposition 3.1, the sequence $\{\Lambda_{i,\min}^k\}_i,\ k \ge 0$, is monotone increasing, whereas $\{\Lambda_{i,\max}^k\}_i,\ k \ge 0$, is monotone decreasing and they converge to equilibria (the fixed points of map F) that we denote by $\{\Lambda_{i,\min}\}_i$ and $\{\Lambda_{i,\max}\}_i$, respectively. The corresponding demand slopes are determined via $\{X_{i,\min}\}_i = \{(\alpha_i\mathcal{V}_{N(i)} + \Lambda_{i,\max})^{-1}\}_i,\ \{X_{i,\max}\}_i = \{(\alpha_i\mathcal{V}_{N(i)} + \Lambda_{i,\min})^{-1}\}_i$.

Proof of Proposition 3.2. Pick an arbitrary equilibrium $\{\Lambda_i\}_i$. Then, for all $i \in I$,

$$\begin{aligned}\Lambda_i &= \left(\left((M_i-1)(\alpha_i\mathcal{V}+\bar{\Lambda}_i)^{-1} + \sum_{j\ne i} M_j(\alpha_j\mathcal{V}+\bar{\Lambda}_j)^{-1}\right)^{-1}\right)_{N(i)} \\ &\le \left(\left((M_i-1)(\alpha_i^*\mathrm{Id}_{N(i)}+\bar{\Lambda}_i)^{-1} + \sum_{j\ne i} M_j(\alpha_j^*\mathrm{Id}_{N(j)}+\bar{\Lambda}_j)^{-1}\right)^{-1}\right)_{N(i)}.\end{aligned} \tag{33}$$

Let F_A be the map corresponding to the right-hand side of(33). Then, iterating F_A and using Proposition 3.1, we get that F_A has a fixed point $\{\Lambda_i^*\}_i$ satisfying $\{\Lambda_i\}_i \le \{\Lambda_i^*\}_i$. By Lemma A.6, this is the unique diagonal fixed point. Then, Λ_i^* is diagonal, and, for any exchange n, the scalar price impacts $\{(\Lambda_i^*)_{nn}\}_i$ coincide with price impacts in a centralized exchange for a single asset with variance 1 and risk aversions α_i^*. The same iteration argument as above implies that these price impacts are monotone increasing in α_i^* and therefore satisfy

$$\Lambda_i^* \le \frac{\lambda^*(n)}{M(n)-2}\,\mathrm{Id}_{N(i)},\ \ i \in I.$$

Therefore, by the monotonicity of map F,

$$\{\Lambda_i\}_i = F^n(\{\Lambda_i\}_i) \leq F^n(\{\Lambda^0_{i,\max}\}) \to \{\Lambda_{i,\max}\}.$$

Similarly,

$$\begin{aligned}\Lambda_i &= \left(\left((M_i-1)(\alpha_i\mathcal{V}+\bar{\Lambda}_i)^{-1}+\sum_{j\neq i}M_j(\alpha_j\mathcal{V}+\bar{\Lambda}_j)^{-1}\right)^{-1}\right)_{N(i)} \\ &\leq \left(\left((M_i-1)(\alpha_{i,*}\mathrm{Id}_{N(i)}+\bar{\Lambda}_i)^{-1}+\sum_{j\neq i}M_j(\alpha_{j,*}\mathrm{Id}_{N(j)}+\bar{\Lambda}_j)^{-1}\right)^{-1}\right)_{N(i)}.\end{aligned} \tag{34}$$

and the same argument as above implies that

$$\Lambda_i^* \geq \frac{\lambda^*(n)}{M(n)-2}\mathrm{Id}_{N(i)},\ i\in I,$$

and the same iterative procedure implies that

$$\{\Lambda_i\}_i = F^n(\{\Lambda_i\}_i) \geq F^n(\{\Lambda^0_{i,\min}\}) \to \{\Lambda_{i,\min}\}.$$

This completes the proof. ■

Proof of Proposition 3.3. By standard arguments based on the Sard Theorem and the analytic implicit function theorem, it suffices to show that, locally, we can always find a small perturbation of the covariance matrix such that price impacts do not commute. This follows by direct calculation, using the expressions in Lemma *A.3*. ■

Everywhere in the sequel, we use the following convenient notation:
Notation. For any $x, y \in \mathbb{R}^N$, we write $y^T x = \langle x, y\rangle$.

Given a triplet X, A, B of symmetric matrices of same dimension, we use the notation $X \in [B, A]$ when $B \leq X \leq A$. We need the following auxiliary result.

Lemma A.7 *If $X \in [B, A]$ and $Xq = z$ then*

$$\langle q, z\rangle \geq \max\{\langle Bq, q\rangle, \langle A^{-1}z, z\rangle\}.$$

Proof. Since $X \geq B$, we have $\langle Bq, q\rangle \leq \langle Xq, q\rangle = \langle q, z\rangle$ and the first claim follows. To prove the second claim, pick an $\varepsilon > 0$. Then, $X \leq A$ implies $(X+\varepsilon\mathrm{Id})^{-1} \geq (A+\varepsilon\mathrm{Id})^{-1}$ and, therefore,

$$XA^{-1}X \leq X(X+\varepsilon\mathrm{Id})^{-1}X.$$

Since $x^2(x+\varepsilon)^{-1}x \le x$ of any $x \ge 0$, the functional calculus implies that $X(X+\varepsilon \mathrm{Id})^{-1}X \le X$. Taking the limit as $\varepsilon \downarrow 0$, we get $XA^{-1}X \le X$. Therefore,

$$\langle A^{-1}z, z\rangle = \langle A^{-1}Xq, Xq\rangle = \langle XA^{-1}Xq, q\rangle \le \langle Xq, q\rangle = \langle z, q\rangle\,,$$

and the proof is complete. ■

Define $M_i' = M_i - \mathbf{1}_{i=j},\ i = 1, \cdots, I$.

Lemma A.8 *Consider the function*

$$\Psi_j(z_1, \cdots, z_I) \equiv \sum_i M_i' \|z_i\|^2 - \|\sum_{i \ne j} M_i' z_i\|^2$$

and let $\mu(q) = \max\{\Psi_j(z_1, \cdots, z_I) : z_i \in \mathbb{R}^{N(i)}, \langle q, z_i\rangle \ge \max\{\langle \bar{B}_i q, q\rangle, \langle A_i^{-1} z_i, z_i\rangle\},\ i \in I\}$. *If*

$$\max_{q \in \mathbb{R}^N} \mu(q) < 0,$$

then the conditions of Lemma A.3 are satisfied.

Proof. The claim follows directly from Lemma A.7 if we define $\bar{X}_i q = z_i$. ■

Lemma A.9 *Let* $a_i = \|A_i\|$ *and* $a = \max_{i \in I} a_i$. *Suppose that*

$$a\mathrm{Id} \ \le\ \sum_i M_i' \bar{B}_i\,.$$

Then, the hypothesis of Lemma A.8 is satisfied.

Proof. Pick a tuple $z_i \in \mathbb{R}^{N(i)}, i \in I$, satisfying $\langle q, z_i\rangle \ \ge\ \max\{\langle \bar{B}_i q, q\rangle,\ \langle A_i^{-1} z_i, z_i\rangle\},\ i \in I$. Then,

$$a_i^{-1}\|z_i\|^2 \le \langle A_i^{-1} z_i, z_i\rangle \le \langle q, z_i\rangle,\ i \in I.$$

Normalize q so that $\|q\| = 1$. Then, we can decompose $z_i = \langle q, z_i\rangle q + z_i^{\perp}$ with $z_i^{\perp} \in \mathbb{R}^N$, $\langle z_i^{\perp}, q\rangle = 0$. Let $\beta_i \equiv M_i' \langle q, z_i\rangle$. Then,

$$\|\sum_{i \ne j} M_i' z_i\|^2 = \left(\sum_i \beta_i\right)^2 + \|\sum_i M_i' z_i^{\perp}\|^2 \ge \left(\sum_i \beta_i\right)^2$$

and, therefore,

$$\begin{aligned}\Psi_j(z_1, \cdots, z_I) &\equiv \sum_i M_i'\|z_i\|^2 - \|\sum_{i \ne j} M_i' z_i\|^2 \le \sum_i a_i \beta_i - \|\sum_{i \ne j} M_i' z_i\|^2 \le \sum_i a_i\beta_i - \left(\sum_i \beta_i\right)^2 \\ &\le \left(\sum_i \beta_i\right)\left(a - \sum_i \beta_i\right).\end{aligned}$$

and the claim follows because, by assumption,

$$\sum_i \beta_i \geq \sum_i M_i' \langle q, z_i \rangle \geq \sum_i M_i' \langle q, \bar{B}_i q \rangle \geq a.$$

■

Proof of Corollary A.1. For simplicity, we work directly with the correlation matrix and assume that $\mathcal{V} = C(\mathcal{V})$. By Proposition 3.2, any equilibrium $\{\Lambda_i\}_i$ satisfies

$$\Lambda^0_{i,\min} \leq \Lambda_{i,\min} \leq \Lambda_i \leq \Lambda_{i,\max} \leq \Lambda^0_{i,\max},\ i \in I.$$

Therefore,

$$\operatorname{diag}\left(\frac{M(n)-2}{(M(n)-1)\lambda^*(n)}\right)_{N(i)} \leq (\Lambda_i + \alpha_i \mathcal{V}_{N(i)})^{-1} \leq \operatorname{diag}\left(\frac{M(n)-2}{(M(n)-1)\lambda_*(n)}\right)_{N(i)},\ i \in I\,,$$

and the claim follows from Lemma *A*.9. ■

B Equivalence Loops: Proof of Theorems 4.1 and 4.2

Proof of Theorems 4.1 and 4.2. For notational convenience, in this proof, we relabel exchanges in which multiple assets are traded by creating a separate exchange for each asset without changing participation. That is, in the new notation, a single asset is traded in every exchange.

The proof of the "if" part (liquidity in two exchanges coincides *if* there is an equivalence loop connecting these two exchanges) consists of several steps. Our goal is to show that, for any price impact Λ, the rows corresponding to n and n' coincide. This is done in Lemmas *B*.1 and *B*.3. The proof of Lemma *B*.1 is established through a sequence of auxiliary results. The goal is to show that price impacts are degenerate. To this end, we use Lemma *A*.1 to conclude that price impacts in a market are lower than those in a smaller market in which only trading along the loop is possible. To proceed further, we need to get hold of equilibria in this simpler "single loop" market. Since we cannot exclude existence of asymmetric equilibria, we use the concavity of map F to reduce the problem to studying symmetric equilibria. The latter can be characterized explicitly as solutions to a simple algebraic system and Lemma *B*.2 completes the proof.

Lemma B.1 *Suppose that an asset k is traded along an equivalence loop $n = n_1, n_2, \cdots, n_L = n'$. Then, for any class $i \in I(n_l) \cap I(n_{l+1})$, price impact $(\Lambda_i)_{n_1 \cup n_2}$ is proportional to* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Proof. For simplicity, we normalize variance $\mathcal{V}_{kk}$ to 1. As in the proof of Theorem 3.1, we add an ε to the diagonal of $\mathcal{V}$ and then study the limit as $\varepsilon \downarrow 0$. Let a_l, $l \in L$, be the classes corresponding to the equivalence loop and let $\Lambda^c_l \equiv (\Lambda_l)_{n_l \cup n_{l+1}} \in \mathbb{R}^{2\times 2}$ their price impacts in the corresponding exchanges. We only consider the (harder) case when each class l is a monopolistic node.[57] By

[57] If there are at least two agents trading in two linked exchanges, a direct calculations based on arguments below implies that price impacts are always equalized across these exchanges.

assumption, there are at least three agents trading in every exchange. Denote by A_l the projection of their demand slope on the exchange n_l. Since all $\{\Lambda_i\}$ are uniformly bounded from above, the demand slopes A_l are uniformly bounded away from zero by a constant $A > 0$, independent of ε. Furthermore, let also $\alpha = \max_l \alpha_l$ be the maximal risk aversion for the agents in the sequence that defines the equivalence loop. Denote by $\hat{\Lambda}_l^c$ the corresponding price impacts, lifted only to the collection of exchanges in the equivalence loop. Then, by Lemma *A.1*,

$$\Lambda_l^c \le \left(\left(\sum_{k \ne l} (\hat{\Lambda}_k^c + \alpha(\mathbf{1} + \varepsilon \mathrm{Id}))^{-1} + A\,\mathrm{Id} \right)^{-1} \right)_{n_l \cup n_{l+1}} . \tag{35}$$

Denote by $\hat{F}$ the map on the right-hand side of this inequality. Then, we can rewrite (35) as

$$\{\Lambda_l^c\} \le \hat{F}(\{\Lambda_l^c\}).$$

Let

$$Z \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, it follows from Theorem 5 in Anderson (1971) that

$$\{Z\Lambda_l^c Z\}_l \le \hat{F}(\{Z\Lambda_l^c Z\}_l).$$

Let $\Lambda_l^a \equiv 0.5(\Lambda_l^c + Z\Lambda_l^c Z),\ l = 1, \cdots, L$. Then, Theorem 24 in Anderson and Duffin (1969) implies that

$$\{\Lambda_l^a\}_l \le \hat{F}(\{\Lambda_l^a\}_l).$$

Note that, for each matrix Λ_l^a, the two diagonal elements are identical. Note further that inequalities (35) are symmetric with respect to Λ_l.[58] Therefore, adding up the inequalities $\Lambda_l^a \le \hat{F}_l(\{\Lambda_k^a\}_k)$ and using the concavity of map F (based on Theorem 24 in Anderson and Duffin (1969)), we get that

$$\{\Lambda_l^{av}\}_l \le \hat{F}(\{\Lambda_l^{av}\}_l),$$

where $\Lambda_l^{av} = \frac{1}{L}\sum_{l=1}^{L} \Lambda_l^a$ is independent of l.[59]

Proposition 3.1 implies that iterations $\hat{F}^n(\{\Lambda_l^{av}\}_l),\ n \ge 1$, converge to a fixed point $\{\Lambda_l^*\}_l$ satisfying $\{\Lambda_l^{av}\}_l \le \{\Lambda_l^*\}_l$. Furthermore, by symmetry, $\Lambda_l^* = Z\Lambda_l^* Z$ for all l and Λ_l^* is independent of l. That is, these matrices are all identical and have identical diagonal elements. Let

$$(\hat{\Lambda}_l^c + \alpha(\mathbf{1} + \varepsilon \mathrm{Id}))^{-1} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix},\ l = 1, \cdots, L.$$

[58] In fact, the equivalence loop is a circle, and so a rotation of the loop moves the classes without affecting their price impacts.

[59] I.e., tuple $\{\Lambda_l^{av}\}$ consists of identical elements, all equal to the average of Λ_l^a.

A direct (but tedious) calculation, based on Lemma C.2, implies that

$$x_1 - x_2 = \varepsilon^{-1} g(\varepsilon/w), \tag{36}$$

where $w = y_1 - y_2$ and

$$Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix} = \left((\hat{B} + A\mathrm{Id})^{-1} \right)_{n_1 \cup n_2},$$

where

$$\hat{B} = \sum_{l=1}^{L} (\hat{\Lambda}_l^c + \alpha(\mathbf{1} + \varepsilon \mathrm{Id}))^{-1}.$$

We now show that, in the limit as $\varepsilon \to 0$, we have $w \to 0$. Suppose the contrary. Then, by the definition of the function $g(a)$, we have $\varepsilon^{-1} g(\varepsilon/w) \to w^{-1}$ as $\varepsilon \to 0$, and hence,

$$x_1 - x_2 \to (y_1 - y_2)^{-1}. \tag{37}$$

We have

$$\hat{B} = \begin{pmatrix} 2x_1 + A & x_2 & 0 & \cdots & 0 & x_2 \\ x_2 & 2x_1 + A & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2 & 0 & \cdots & 0 & x_2 & 2x_1 + A \end{pmatrix}.$$

A direct calculation implies that the discrete Fourier transform

$$F = \frac{1}{\sqrt{N}} (e^{-i2\pi kn/N})_{k,n=0}^{N-1}$$

diagonalizes $\hat{B}$ and the eigenvalues are $2x_1 + A + 2x_2 \cos(2\pi n/N)$. That is,

$$B = F \mathrm{diag}(2x_1 + A + 2x_2 \cos(2\pi n/N)) \bar{F}.$$

Therefore,

$$B^{-1} = F \mathrm{diag}((2x_1 + A + 2x_2 \cos(2\pi n/N))^{-1}) \bar{F},$$

and hence,

$$y_1 = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{2x_1 + A + 2x_2 \cos(2\pi n/N)} \tag{38}$$

$$y_2 = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\cos(2\pi n/N)}{2x_1 + A + 2x_2 \cos(2\pi n/N)}. \tag{39}$$

We use the following lemma.

Lemma B.2 *We have*

$$y_1 - y_2 < \frac{1}{x_1 - x_2}. \tag{40}$$

Proof. Since $A > 0$, we have

$$(y_1 - y_2)(x_1 - x_2) < \frac{1}{N} \sum_{n=0}^{N-1} \frac{(1 - \cos(2\pi n/N))(x_1 - x_2)}{2x_1 + 2x_2 \cos(2\pi n/N)} \equiv \zeta(x_1)$$

whenever $x_1 > x_2$. By direct calculation,

$$\frac{d\zeta(x_1)}{dx_1} = \frac{1}{N} \sum_{n=0}^{N-1} \frac{(1 - \cos^2(2\pi n/N))x_2}{2(x_1 + x_2 \cos(2\pi n/N))^2}\,.$$

Thus, $\zeta(x_1)$ is monotone increasing (decreasing) in x_1 if x_2 is positive (negative). In the first case, $\zeta(x_1) < \zeta(+\infty)$. But, when $x_1 \to \infty$, $\hat{B}^{-1} \approx (2x_1)^{-1}\mathrm{Id}$ and, hence,

$$\frac{1}{2}\frac{1}{N} \sum_{n=0}^{N-1} (1 - \cos(2\pi n/N)) = \zeta(+\infty) = \frac{1}{2}$$

and the claim follows. If $x_2 < 0$, then

$$\zeta(x_1) < \zeta(-x_2) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{(1 - \cos(2\pi n/N))}{1 - \cos(2\pi n/N)} = 1,$$

and the claim follows. Inequality (40) contradicts (37), and hence we conclude that $w \to 0$ when $\varepsilon \to 0$. But this cannot happen if x_1, x_2 stay bounded in this limit. Thus, x_1 must blow up, that is, the matrix $\hat{\Lambda}_l^c + \alpha \mathbf{1}$ becomes degenerate when $\varepsilon \to 0$. But this can only happen if $\hat{\Lambda}_l^c$ is proportional to $\mathbf{1}$. Indeed, $\mathbf{1}$ is a projection onto $\binom{1}{1}$. If $\hat{\Lambda}_l^c$ is not proportional to $\mathbf{1}$, it has an eigenvector not proportional to $\binom{1}{1}$, implying that the image of $\hat{\Lambda}_l^c + \alpha \mathbf{1}$ spans $\mathbb{R}^2$, and hence the invertibility. We conclude that $\hat{\Lambda}_l^c = \kappa \mathbf{1}$ for some $\kappa > 0$. The inequality $\Lambda_l^m \le \hat{\Lambda}_l^c$ implies that Λ_l^m cannot have any eigenvectors except for $\binom{1}{1}$ and hence $\Lambda_l^m = \kappa_1 \mathbf{1}$ for some $\kappa_1 > 0$. The inequality

$$\Lambda_l^c \le \frac{1}{2L} \Lambda_l^a \le \frac{1}{2L} \sum_l \Lambda_l^a = \Lambda_l^m = \kappa_1 \mathbf{1}$$

implies by the same argument that Λ_l^c is proportional to $\mathbf{1}$. ■ ■

Lemma B.3 *Suppose that $\Lambda \in \mathbb{R}^{N \times N}$ is a symmetric, positive semidefinite matrix. If $\Lambda_{11} = \Lambda_{22} = \Lambda_{12} = \lambda$ then the first two rows of Λ coincide.*

Proof. Indeed, for any $i > 2$, the sub-matrix

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{1i} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{2i} \\ \Lambda_{1i} & \Lambda_{2i} & \Lambda_{ii} \end{pmatrix}$$

must be positive semidefinite. This implies that

$$\begin{pmatrix} \lambda & \lambda \\ \lambda & \lambda \end{pmatrix} - \begin{pmatrix} \Lambda_{1i} \\ \Lambda_{2i} \end{pmatrix} \Lambda_{ii}^{-1} (\Lambda_{1i}, \Lambda_{2i})$$

is positive semidefinite, which is only possible if $\Lambda_{1i} = \Lambda_{2i}$. ■

It follows that the rows of Λ_l corresponding to exchanges n_l and n_{l+1} coincide. Therefore, for any vector in the image of $\Lambda_l + \alpha_l \mathcal{V}_{N(l)}$, the corresponding coordinates are equal; that is, prices in the two exchanges coincide and the equivalence result of Theorem 4.2 also follows.

Suppose now that there is no equivalence loop connecting exchanges n and n'. Treating the set of exchanges in which asset k is traded as a graph,where two vertices (exchanges) are connected if there is an agent who trades in both exchanges, consider its associated regularized graph obtained using the procedure described in the main text. On this tree, there is a unique path connecting n and n'. Since, by assumption, n and n' are not connected by a loop, an agent i exists on this path who, if removed from n and n', the exchanges become isolated. Let $N = N_1 \cup N_2$ be the disjoint partition of exchanges that is obtained if we remove agent i from the market. It then follows from the equilibrium Equation (4) that in the decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$, price impact of agent i is block diagonal. Furthermore, since no other agent can trade simultaneously in exchanges from both N_1 and N_2, it follows from (4) that price impacts of all other agents are also block diagonal. Thus, no agent is able to equalize prices in these exchanges, and the claim follows. ■

Corollary B.1 *The only degeneracy of $\Lambda_i + \alpha_i \mathcal{V}_{N(i)}$ that may occur in equilibrium is when some rows and columns of this matrix are identical.*

Proof. The claim follows directly from the proof of Theorem 4.1 above: After the regularization procedure (i.e., the procedure of removing identical rows and columns from $\Lambda_i + \alpha_i \mathcal{V}_{N(i)}$), the resulting matrices are nondegenerate. ■

C Price Impact Characterization

We now use Lemma C.4 and the following result.

Lemma C.1 (Functional Calculus for Symmetric Matrices) *For any continuous function $f(x)$ and any symmetric matrix A, we can define $f(A)$ as follows. By the eigen-decomposition theorem, there exists an orthogonal matrix U and a diagonal matrix D such that $A = U^T D U$ where $D = \mathrm{diag}(d_i)$ where d_i are the eigenvalues of A. Then,*

$$f(A) = U^T \mathrm{diag}(f(d_i)) U.$$

In general, the matrix U is not unique. The uniqueness holds only if eigenvalues of A are all distinct. However, even if U is not unique, $f(A)$ is uniquely determined, and so it is well-defined. The following lemma explicitly links price impact Λ_i with the aggregate liquidity measure B. Let $f_1(a) = (2 - a + \sqrt{a^2 + 4})/2$ and $f(a) = f_1(a)/a$.

Lemma C.2 *Let* $Y_i = (B^{-1})_{N(i)}$. *Then,*

$$\Lambda = Y^{1/2} f_1(Y_i^{-1/2} \alpha_i \mathcal{V}_{N(i)} Y_i^{-1/2}) Y_i^{1/2}.$$

If $\mathcal{V}_{N(i)}$ *is invertible then*

$$\Lambda_i = \alpha_i \mathcal{V}_{N(i)}^{1/2} f(\alpha_i \mathcal{V}_{N(i)}^{1/2} Y_i^{-1} \mathcal{V}_{N(i)}^{1/2}) \mathcal{V}_{N(i)}^{1/2}.$$

Lemma is a direct consequence of Lemma 5.1 and the following auxiliary result.

Lemma C.3 *Let* Y, Z *be nonnegative definite, with* Y *positive definite. The unique positive definite symmetric matrix* Λ *solving*

$$\Lambda = (Y^{-1} - (Z + \Lambda)^{-1})^{-1}$$

is given by

$$\Lambda = Y^{1/2} f_1(Y^{-1/2} Z Y^{-1/2}) Y^{1/2},$$

where $f_1(a) = (2 - a + \sqrt{a^2 + 4})/2$. *If* Z *is invertible, then we can also write*

$$\Lambda = Z^{1/2} f(Z^{1/2} Y^{-1} Z^{1/2}) Z^{1/2}$$

with $f(a) = f_1(a)/a$. *Furthermore,*

$$(Z + \Lambda)^{-1} = Z^{-1/2} g(Z^{1/2} Y^{-1} Z^{1/2}) Z^{-1/2}$$

with $g(a) = (f(a) + 1)^{-1} = 2a/(2 + a + \sqrt{a^2 + 4})$.

Proof. Multiplying by $(Y^{-1} - (Z + \Lambda)^{-1})$ from the right gives

$$\Lambda(Y^{-1} - (Z + \Lambda)^{-1}) = \mathrm{Id}.$$

Multiplying by Λ from the left gives

$$Y^{-1} = \Lambda^{-1} + (Z + \Lambda)^{-1}. \tag{41}$$

Multiplying from the left and right by $Y^{1/2}$ (we do this to preserve symmetry), we have

$$\mathrm{Id} = L^{-1} + (Y^{-1/2} Z Y^{-1/2} + L)^{-1},$$

where we defined $L = (Y^{-1/2} \Lambda Y^{-1/2})$. Let $A = Y^{-1/2} Z Y^{-1/2}$. Let us first show that A and L commute. Indeed, multiplying $(A + L)$ from the left and the right, gives

$$(A + L)L^{-1} + \mathrm{Id} = (A + L) = L^{-1}(A + L) + \mathrm{Id}.$$

Subtracting Id from both sides and multiplying by L from the left and the right gives

$$LA + L^2 = L(A + L) = (A + L)L = AL + L^2$$

and the claim follows.

Thus, A and L commute and, therefore, there exists an orthonormal basis such that both A and L are diagonal in this basis. For an orthogonal matrix U, both UAU^T and ULU^T are diagonal, and

$$\mathrm{Id} = U\mathrm{Id}U^T = UL^{-1}U^T + U(A + L)^{-1}U^T = (ULU^T)^{-1} + (UAU^T + ULU^T)^{-1}.$$

Since all matrices on both sides are diagonal, each diagonal element has the same form with the unique positive solution $f(a)$ of

$$1 = \frac{1}{x} + \frac{1}{a + x}.$$

Therefore, we obtain

$$L = U^T f(UAU^T)U = f(A) = f(Y^{-1/2}ZY^{-1/2}).$$

Similarly, assume that Z is invertible (also, symmetric and positive definite). Then, there exists a positive-definite invertible matrix $Z^{1/2}$. Multiplying (41) by $Z^{1/2}$ from the left and right, we get

$$K = B^{-1} + (\mathrm{Id} + B)^{-1},$$

where $K = Z^{1/2}Y^{-1}Z^{1/2}$ and $B = Z^{-1/2}\Lambda Z^{-1/2}$. Multiplying $(\mathrm{Id} + B)$ from the left and right,

$$K + BK = (\mathrm{Id} + B)K = B^{-1} + 2\mathrm{Id} = K(\mathrm{Id} + B) = K + KB,$$

which implies that K and B commute. By the argument analogous to the above, with the unique positive solution $f_1(a)$ to

$$a = \frac{1}{x} + \frac{1}{1 + x},$$

we get that $B = f_1(K)$. ∎

Proof of Propositions 5.1 and 6.4. Suppose that $M_i \geq 2$. Then, since, by assumption, there are at least three agents participating in each exchange, there exists an $\varepsilon > 0$ such that

$$\begin{aligned}
\Lambda_i &= \left(\left((M_i - 1)\,(\alpha_i \mathcal{V}_{N(i)} + \bar{\Lambda}_i)^{-1} + \sum_{j \neq i} M_j\,(\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1}\right)^{-1}\right)_{N(i)} \\
&\leq \left(\left(\varepsilon \mathrm{Id} + (M_i - 1)(\alpha_i \|\mathcal{V}\| \mathrm{Id} + \bar{\Lambda}_i)^{-1}\right)^{-1}\right)_{N(i)} = \left(\varepsilon \mathrm{Id} + (M_i - 1)(\alpha_i \|\mathcal{V}\| \mathrm{Id} + \Lambda_i)^{-1}\right)^{-1}.
\end{aligned}$$

Let $\ell \geq 0$ be the largest eigenvalue of Λ_i. Then, we get that

$$\ell \leq \left(\varepsilon \mathrm{Id} + (M_j - 1)(\alpha_j \|\mathcal{V}\| \mathrm{Id} + \ell)^{-1}\right)^{-1}.$$

By direct calculation, this inequality implies that $\ell \to 0$ when $\alpha_j \to 0$ or $M_j \to 0$.

Now, pick any class $i \neq j$. Then,

$$(\Lambda_i)_{N(j)\cap N(i)} \leq ((M_j(\alpha_j\|\mathcal{V}\| + \bar{\Lambda}_j)^{-1} + \varepsilon \mathrm{Id})^{-1})_{N(j)\cap N(i)} = ((M_j(\alpha_j\|\mathcal{V}\| + \Lambda_j)^{-1} + \varepsilon \mathrm{Id})^{-1})_{N(j)\cap N(i)}.$$

Since $(\alpha_j\|\mathcal{V}\| + \ \Lambda_i)^{-1}$ converges to ∞, we get the required.

Finally, the last claim follows because

$$\lim_{\alpha_j \to 0} \Lambda_i = (\bar{\Lambda}_{i\setminus j})_{N(i)}$$

and hence, by the Frobenius formula (Lemma C.4)

$$((\mathcal{V}_{N(i)} + \Lambda_i)^{-1})_{N(j)\cap N(i)} \to (\Lambda_{i\setminus j} + \mathfrak{S}(\mathcal{V}_{N(i)}, N(i) \setminus N(j)))^{-1}.$$

To prove the result about the limit allocation, we need to study the asymptotic behaviour in greater detail. This is done in the following proposition.

Proposition C.1 *Let $M_j > 2$. Then, for sufficiently small $\alpha = \alpha_j$, there exists an equilibrium price impact tuple $\{\Lambda_i(\alpha)\}$ satisfying $\Lambda_j(\alpha) \approx \frac{\alpha}{M_j - 2}\mathcal{V}_{N(j)}$ and, for all $i \neq j$, to the first order in α,*

$$\Lambda_i(\alpha) \approx \Pi_{N(i)} \begin{pmatrix} \alpha \mathcal{V}_{N(j)} \frac{M_j - 1}{(M_j-2)M_j} & -\alpha \frac{M_j-1}{(M_j-2)M_j} \mathcal{V}_{N(j)}\, W_{12}(i) W_{22}(i)^{-1} \\ -\alpha \frac{M_j-1}{(M_j-2)M_j} W_{22}(i)^{-1}\, W_{12}(i)^T \mathcal{V}_{N(j)} & \Lambda_{i\setminus j} + \alpha \Lambda^{(1)}_{i\setminus j} \end{pmatrix}$$

where

$$W(i) = \begin{pmatrix} W_{11}(i) & W_{12}(i) \\ W_{12}^T(i) & W_{22}(i) \end{pmatrix} \equiv (M_i - 1)(\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i(0))^{-1} + \sum_{k \neq i,j} M_k (\alpha_k \bar{\mathcal{V}}_{N(k)} + \bar{\Lambda}_k(0))^{-1}.$$

The first order equilibrium response $\{\Lambda^{(1)}_{i\setminus j}\}_{j\neq i}$ is the unique solution to the system

$$\begin{aligned} \Lambda^{(1)}_{i\setminus j} \;&=\; \Pi_{N(i)\setminus N(i)} W_{22}(i)^{-1} \Bigg((M_i - 1) Z_i \Lambda^{(1)}_{i\setminus j} Z_i + \sum_{k \neq i,j} M_k Z_k \Lambda^{(1)}_{k\setminus j} Z_k \\ &\quad + W_{12}(i)^T \mathcal{V}_{11} \frac{M_j - 1}{(M_j - 2) M_j} W_{12}(i) \Bigg) W_{22}(i)^{-1} \lceil_{\mathbb{R}^{N(i)}}. \end{aligned}$$

where $Z_i \equiv (\alpha_i \bar{\mathcal{V}}_{N(i)} + \bar{\Lambda}_i(0))^{-1},\ i \neq j$.

Proof. The fixed point equation is

$$\Lambda_i(\alpha) = \Pi_{N(i)}((\alpha \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j)^{-1} + W(i,\alpha))^{-1}$$

and the claim follows by direct calculation from the Frobenius formula (Lemma C.4). ■

Furthermore,

$$B^{-1} \approx \begin{pmatrix} \alpha_j \mathcal{V}_{N(j)} \frac{M_j-1}{(M_j-2)M_j} & -\alpha_j \frac{M_j-1}{(M_j-2)M_j} \mathcal{V}_{N(j)} W_{12} W_{22}^{-1} \\ -\alpha_j \frac{M_j-1}{(M_j-2)M_j} W_{22}^{-1} W_{12}^T \mathcal{V}_{N(j)} & W_{2\setminus 1} + \alpha_j W_{2\setminus 1}^{(1)} \end{pmatrix}$$

where

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{pmatrix} \equiv \sum_{k \neq j} M_k (\alpha_k \bar{\mathcal{V}}_{N(k)} + \bar{\Lambda}_k(0))^{-1}.$$

Thus, the trade of an agent of class j is approximately given by

$$(\alpha_j \mathcal{V}_{N(j)} + \Lambda_j)^{-1} \mathbf{Q}_{N(j)} - \frac{M_j-2}{M_j-1} q_j^0 = \alpha_j^{-1} \frac{M_j-2}{M_j-1} \mathcal{V}_{N(j)}^{-1} \mathbf{Q}_{N(j)} - \frac{M_j-2}{M_j-1} q_j^0.$$

And we have

$$\begin{aligned} & \alpha_j^{-1} \frac{M_j-2}{M_j-1} \mathcal{V}_{N(j)}^{-1} \mathbf{Q}_{N(j)} \\ \approx\ & \alpha_j^{-1} \frac{M_j-2}{M_j-1} \mathcal{V}_{N(j)}^{-1} \alpha_j \mathcal{V}_{N(j)} \frac{M_j-1}{(M_j-2)M_j} X_{N(j)}^{(0)} - \alpha_j^{-1} \frac{M_j-2}{M_j-1} \mathcal{V}_{N(j)}^{-1} \alpha_j \frac{M_j-1}{(M_j-2)M_j} \mathcal{V}_{N(j)} W_{12} W_{22}^{-1} X_{N\setminus N(j)}^{(0)} \\ =\ & M_j^{-1} \left(X_{N(j)}^{(0)} - W_{12} W_{22}^{-1} X_{N\setminus N(j)}^{(0)} \right). \end{aligned}$$

where

$$X^{(0)} = \sum_{j \neq i} \left(\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j(0) \right)^{-1} \alpha_j \bar{\mathcal{V}}_{N(j)} Q_j^0 + \frac{M_i-2}{M_i-1} Q_i^0.$$

In contrast, agents from class $i \neq j$ trade

$$(\alpha_i \mathcal{V}_{N(i)} + \bar{\Lambda}_i(0))^{-1} \left(\mathbf{Q}_{N(i)\setminus N(j)} - \alpha_i \mathcal{V}_{N(i)} q_i^0 \right),$$

because $\mathbf{Q}_{N(j)} = 0$. ■

Proof of Theorem 5.1. The market clearing takes the form

$$\sum_{j=1}^{I} \left(\alpha_j \bar{\mathcal{V}}_{N(j)} + \bar{\Lambda}_j(B) \right)^{-1} (d - p - \alpha_j \bar{\mathcal{V}}_{N(j)} q_j^0) = 0$$

and the required expressions follow by direct calculation. ■

Proof of Proposition 5.2. The claim is a direct consequence of Theorem 5.1.

■

We need the following auxiliary lemma that can be verified by direct calculation.

Lemma C.4 (Frobenius Formula) *We have*

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}B^T)^{-1} & -A^{-1}B(D - B^T A^{-1} B)^{-1} \\ -(D - B^T A^{-1} B)^{-1} B^T A^{-1} & (D - B^T A^{-1} B)^{-1} \end{pmatrix}.$$

D Comparative Statics

Proof of Proposition 6.1. Fix a parameter α and let us rewrite equilibrium equation as $\{\Lambda_j\}_j = F(\{\Lambda_j\}_j, \alpha)$. By definition, both for $\alpha = M_i$ and $\alpha = \alpha_j$, map F is monotone increasing in α in the sense of the positive semidefinite partial order.

Fix $\alpha_1 \leq \alpha_2$ and let $\{\Lambda_j(\alpha_1)\}_j$ be an equilibrium. Then,

$$\{\Lambda_j(\alpha_1)\}_j = F(\{\Lambda_j(\alpha_1)\}_j, \alpha_1) \leq F(\{\bar{\Lambda}_j(\alpha_1)\}_j, \alpha_2).$$

By Proposition 3.1, there exists an equilibrium $\{\Lambda_j(\alpha_2)\}_j$ satisfying $\{\Lambda_j(\alpha_2)\}_j \geq \{\Lambda_j(\alpha_1)\}_j$.

The last claim follows by a similar argument. Namely, let $\hat{F}(\{\Lambda_j\}_{j=1}^I, \Lambda_{I+1}\})$ be the equilibrium map corresponding to the market with $I+1$ classes and let $\{\Lambda_j^*\}_{j=1}^{I+1}$ be an equilibrium in that market. Define the map

$$\hat{F}_1(\{\Lambda_j\}_{j=1}^I) \equiv \hat{F}(\{\Lambda_j\}_{j=1}^I, \Lambda_{I+1}^*\}).$$

Then, using the same monotonicity arguments as above, we get that

$$\hat{F}_1(\{\Lambda_j\}_{j=1}^I) \leq F(\{\Lambda_j\}_{j=1}^I)$$

for any tuple $\{\Lambda_j\}_{j=1}^I$. By construction, $\{\Lambda_j^*\}$ is a fixed point of $\hat{F}_1$ and therefore

$$\{\Lambda_j^*\}_{j=1}^I = \hat{F}_1(\{\Lambda_j^*\}_{j=1}^I) \leq F(\{\Lambda_j^*\}_{j=1}^I),$$

and Proposition 3.1 implies that there exists an equilibrium $\{\Lambda_j\}_{j=1}^I$ with I classes satisfying $\{\Lambda_j\}_{j=1}^I \geq \{\Lambda_j^*\}_{j=1}^I$.

It remains to prove the concavity results. We only prove concavity in the covariance matrix. The other Pick two covariance matrices $\mathcal{V}^m, m = 1, 2,$ and let $\{\Lambda_j^m\}_j$ be two corresponding equilibria. Let $\mathcal{V}^3 = 0.5(\mathcal{V}^1 + \mathcal{V}^2)$ and denote by $F^m,\ m = 1, 2, 3$ the equilibrium maps corresponding to these covariance matrices. A direct application of Theorem 24 in Anderson and Duffin (1969) implies that, for any $\{\Lambda_j^1\}_j,\ \{\Lambda_j^2\}_j \in \mathcal{S}^I$,

$$0.5(F^1(\{\Lambda_j^1\}_j) + F^1(\{\Lambda_j\}_j)) \leq F^3(\{0.5(\Lambda_j^1 + \Lambda_j^2)\}_j)\,.$$

Consequently, for any two fixed points satisfying $\{\Lambda_j^m\}_j = F^1(\{\Lambda_j^m\}_j)$, Proposition 3.1 implies that F^3 has a fixed point $\{\Lambda_j^3\}_j$ satisfying $\{\Lambda_j^3\}_j \geq\ 0.5(\Lambda_j^1 + \Lambda_j^2)\}_j$, and the required concavity follows. ■

Proof of Theorem 6.1. Consider two markets which differ only in participation $\{N(i)\}_i$ and $\{N'(i)\}_i,\ N(i)' \supseteq N(i)$ for all $i \in I$. Pick an equilibrium $\{\Lambda_i\}_i$ corresponding to $\{N'(i)\}_i$. By

Lemma *A*.1,

$$
\begin{aligned}
(\Lambda_i)_{N(i)} &= \left(\left((M_i-1)(\alpha_i\bar{\mathcal{V}}_{N'(i)}+\bar{\Lambda}_i)^{-1}+\sum_{j\neq i}M_j(\alpha_j\bar{\mathcal{V}}_{N'(j)}+\bar{\Lambda}_j)^{-1}\right)^{-1}\right)_{N(i)}\\
&\leq \left(\left((M_i-1)(\alpha_i\bar{\mathcal{V}}_{N(i)}+(\bar{\Lambda}_i)_{N(i)})^{-1}+\sum_{j\neq i}M_j(\alpha_j\bar{\mathcal{V}}_{N(j)}+(\bar{\Lambda}_j)_{N(j)})^{-1}\right)^{-1}\right)_{N(i)}.
\end{aligned}
$$

For all $\in I$. Therefore, by Proposition 3.1, there exists an equilibrium $\{\hat{\Lambda}_i\}_i$ corresponding to participation $\{N(i)\}_i$ and satisfying $\{\hat{\Lambda}_i\}_i \geq \{(\Lambda_i)_{N(i)}\}_i$ and the claim follows. ■

Proof of Corollary 6.1. Let $\mathbb{M}_{-n}$ be market $\mathbb{M}$ with all exchanges but $n = (I(n), K(n))$.

Consider exchanges $(I_1(n), K_1(n))$ and $(I_2(n), K_2(n))$ in market $\mathbb{M}'$. Let all classes $i \in I(n)$ trade in each exchange for assets $K_1(n)$ and $K_2(n)$. Then, by Theorem 6.1, price impact in the resulting market $\widetilde{\mathbb{M}} = (\mathbb{M}_{-n}, (K_1(n), I(n)), (K_2(n), I(n)))$ is lower. Thus, price impact in $\mathbb{M}'$ is higher than that in $\mathbb{M}$.

Instead, in market $\mathbb{M}'$, let the participation of classes $I_1(n)$ in the new exchanges increase from $K_1(n)$ to $K(n)$ and classes $I_2(n)$ from $K_2(n)$ to $K(n)$ such that each group of agent classes trades assets $K(n)$ in a separate exchange. Then, by Theorem 6.1, price impact in the resulting market $\widetilde{\mathbb{M}}' = (\mathbb{M}_{-n}, (K(n), I_1(n)), (K(n), I_2(n)))$ is lower. Define $n^l = (K(n), I_l(n))$, $l = 1, 2$. Next, let the participation of classes $I_1(n)$ increase to include n^2 and the participation of classes $I_2(n)$ increases to include n^1. Denote by $\widetilde{\mathbb{M}}'' = (\mathbb{M}_{-n}, (I(n), K(n)), (I(n), K(n)))$ the corresponding decentralized market. Then, by Theorem 6.1, price impact in $\widetilde{\mathbb{M}}''$ is lower than that in $\mathbb{M}'$. Since market $\widetilde{\mathbb{M}}''$ is equivalent to one with the centralized exchange $n = (I(n), K(n))$, price impact in $\widetilde{\mathbb{M}}''$ and $\mathbb{M}$ coincides.

Thus, price impact in $(\mathbb{M}_{-n}, (I_1(n), K_1(n)), (I_2(n), K_2(n))) = \mathbb{M}'$ is higher than in $(\mathbb{M}_{-n}, (K(n), I_1(n)), (K(n), I_2(n)))$, which is higher than in $(\mathbb{M}_{-n}, (K_1(n), I(n)), (K_2(n), I(n)))$, which equals that in $(\mathbb{M}_{-n}, (I(n), K(n)) = \mathbb{M}$. ■

Proof of Proposition 6.2. By (14),

$$
(\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}))_{N(j)} = \Phi(\Lambda_j, \alpha_j \mathcal{V}_{N(j)}).
$$

By Theorem 5 in Anderson (1971),

$$
(\Phi(\Lambda_i, \alpha_i \mathcal{V}_{N(i)}))_{N(j)} \leq \Phi((\Lambda_i)_{N(j)}, \alpha_i \mathcal{V}_{N(j)}),
$$

and the claim follows. ■

Proof of Proposition 6.3. Let $W_1 = \alpha_i^{-1}\mathcal{V}_{N(j)}^{-1/2}(\Lambda_i)_{N(j)}\mathcal{V}_{N(j)}^{-1/2}$ and $W_2 = \alpha_j^{-1}\mathcal{V}_{N(j)}^{-1/2}\Lambda_j\mathcal{V}_{N(j)}^{-1/2}$. Then, $W_k = f(\alpha_i\mathcal{V}_{N(j)}^{1/2}A_k^{-1}\mathcal{V}_{N(j)}^{1/2})$, $k = 1, 2$. By Lemma 6.1, eigenvalues are increasing in the positive semi-definite order and hence, by Proposition 6.2

$$\mathrm{eig}(\alpha_i \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2}) \le \mathrm{eig}(\alpha_j \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2})\,.$$

Therefore,

$$\mathrm{eig}(W_1) \;=\; f(\mathrm{eig}(\alpha_i \mathcal{V}_{N(j)}^{1/2} A_1^{-1} \mathcal{V}_{N(j)}^{1/2})) \ge f(\mathrm{eig}(\alpha_j \mathcal{V}_{N(j)}^{1/2} A_2^{-1} \mathcal{V}_{N(j)}^{1/2})) = \mathrm{eig}(W_2)\,.$$

If W_1 and W_2 commute, diagonalizing them in the same basis implies that eigenvalue order and the positive semi-definite order are equivalent. ■

E Welfare

We will need the identity

$$\begin{aligned}
&\Delta_i(\Lambda_i) = \frac{1}{2}\Lambda_i(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\alpha_i\mathcal{V}_{N(i)}(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\Lambda_i\\
&= (\mathrm{Id}-\alpha_i\mathcal{V}_{N(i)}(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1})(0.5\alpha_i\mathcal{V}_{N(i)}+\Lambda_i-\Lambda_i)(\mathrm{Id}-(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\alpha_i\mathcal{V}_{N(i)})\\
&= \;0.5\alpha_i\mathcal{V}_{N(i)}+\alpha_i\mathcal{V}_{N(i)}\Gamma_i(\Lambda_i)\alpha_i\mathcal{V}_{N(i)}-(0.5\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\alpha_i\mathcal{V}_{N(i)}\\
&-\alpha_i\mathcal{V}_{N(i)}(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)(0.5\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)\\
&+\Lambda_i(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\alpha_i\mathcal{V}_{N(i)}+\alpha_i\mathcal{V}_{N(i)}(\Lambda_i+\alpha_i\mathcal{V}_{N(i)})^{-1}\Lambda_i\\
&-\alpha_i\mathcal{V}_{N(i)}(\Lambda_i+\alpha_i\mathcal{V}_{N(i)})^{-1}\Lambda_i(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\alpha_i\mathcal{V}_{N(i)}\\
&= \;0.5\alpha_i\mathcal{V}_{N(i)}+\alpha_i\mathcal{V}_{N(i)}\Gamma_i(\Lambda_i)\alpha_i\mathcal{V}_{N(i)}-\alpha_i\mathcal{V}_{N(i)}(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\alpha_i\mathcal{V}_{N(i)}\\
&-\alpha_i\mathcal{V}_{N(i)}(\Lambda_i+\alpha_i\mathcal{V}_{N(i)})^{-1}\Lambda_i(\alpha_i\mathcal{V}_{N(i)}+\Lambda_i)^{-1}\alpha_i\mathcal{V}_{N(i)}\\
&= 0.5\alpha_i\mathcal{V}_{N(i)}-\alpha_i\mathcal{V}_{N(i)}\Gamma_i(\Lambda_i)\alpha_i\mathcal{V}_{N(i)}
\end{aligned} \tag{42}$$

Proof of Proposition 7.1 and Corollary 7.2. We need a couple of lemmas. The first one is a direct consequence of the Frobenius formula (Lemma C.4).

Lemma E.1 *Let $H \subset \mathbb{R}^N$ be a subspace, B a symmetric positive definite matrix on H and A a positive definite matrix on $\mathbb{R}^N$. Then, $A \ge \bar{B}$ if, and only if,*

$$(A^{-1})_H \le B^{-1}.$$

Proof. We have

$$A - \bar{B} \;=\; \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} - B \end{pmatrix}$$

and hence, by (28), $A - \bar{B} \ge 0$ if, and only if, $A_{22} - A_{12}^T A_{11}^{-1} A_{12} - B \ge 0$. By Lemma C.4, this is equivalent to $(A^{-1})_{22} \le B^{-1}$. ■

Lemma E.2 *There exists a matrix $B \le A$ such that $Bq = z$ if, and only if, $\langle A^{-1}z, z\rangle \;\le \langle q, z\rangle$.*

Proof. We normalize z so that $\|z\| = 1$. Suppose first that $B \le A$ satisfies $Bq = z$. Then, $\langle q, z\rangle = \langle B^{-1}z, z\rangle \ge \langle A^{-1}z, z\rangle$. Now, suppose that $\langle A^{-1}z, z\rangle \le \langle q, z\rangle$ and define $B = (\langle q, z\rangle)^{-1}\langle \cdot, z\rangle z$. Let H be the span of the vector z. By Lemma E.1, it suffices to check that $(A^{-1})_H \le B^{-1}$. But $(A^{-1})_H = \langle A^{-1}z, z\rangle$ and B^{-1} acts as $(\langle q, z\rangle)$ on this subspace. The claim follows. ■

Now, we have

$$\langle \Gamma_i(\Lambda)q, q\rangle = \langle X_1 q, q\rangle - 0.5\alpha_i\langle \mathcal{V}_{N(i)}X_1 q, X_1 q\rangle,$$

where $X_1 = (\alpha_i\mathcal{V}_{N(i)} + \Lambda)^{-1}$ and $q = \mathbf{Q}_{N(i)} - \alpha_i\mathcal{V}_{N(i)}q_i^0$. Denote $X_2 = (\alpha_i\mathcal{V}_{N(i)} + \hat{\Lambda})^{-1}$. Suppose first that we do not change the participation of class i. Then, $\hat{\Lambda} \ge \Lambda$ if, and only if, $X_2 \le X_1$ and the problem becomes

$$\max_{X_2 \le X_1}\{\langle X_2 q, q\rangle - 0.5\alpha_i\langle \mathcal{V}_{N(i)}X_2 q, X_2 q\rangle\}.$$

If the participation of class i increases from $N(i)$ to $N'(i) \supset N(i)$, it follows from Theorem 6.1 that $\hat{\Lambda}_i > (\Lambda_i)_{N(i)}$ and, therefore, by Lemma *A*.1,

$$X_1 = (\alpha_i\mathcal{V}_{N'(i)} + \Lambda_i)^{-1} \ge (\alpha_i\bar{\mathcal{V}}_{N(i)} + (\bar{\Lambda}_i)_{N(i)})^{-1} \ge (\alpha_i\bar{\mathcal{V}}_{N(i)} + \bar{\hat{\Lambda}}_i)^{-1} = X_2\,,$$

and so the same argument applies.

Denote $z = X_2 q$. Then, by Lemma *E*.2, the problem is equivalent to

$$\max_{z:\langle X_1^{-1}z, z\rangle \le \langle q, z\rangle}\{\langle z, q\rangle - 0.5\alpha_i\langle \mathcal{V}_{N(i)}z, z\rangle\}.$$

This is a concave maximization problem over a convex domain and, hence, the global maximum is achieved when $z = (\alpha_i\mathcal{V}_{N(i)})^{-1}q$. By direct calculation, $\langle X_1^{-1}z, z\rangle > \langle q, z\rangle$ holds and, hence, the constraint is binding at the optimum. Therefore, the first-order condition is

$$q - \alpha_i\mathcal{V}_{N(i)}z - \lambda(2X_1^{-1}z - q) = 0 \Rightarrow z = (1+\lambda)(\alpha_i\mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1}q,$$

where the Lagrange multiplier λ is determined from the binding constraint

$$0 = (1+\lambda)^2\langle X_1^{-1}(\alpha_i\mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1}q, (\alpha_i\mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1}q\rangle - (1+\lambda)\langle(\alpha_i\mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1}q, q\rangle.$$

Denote the right-hand side of this equation by $\varphi(\lambda)$. Then, $\varphi(0) = \langle X_1^{-1}(\alpha_i\mathcal{V}_{N(i)})^{-1}q, (\alpha_i\mathcal{V}_{N(i)})^{-1}q\rangle - \langle(\alpha_i\mathcal{V}_{N(i)})^{-1}q, q\rangle > 0$, because $X_1 < (\alpha_i\mathcal{V}_{N(i)})^{-1}$ implies $(\alpha_i\mathcal{V}_{N(i)})^{-1}X_1^{-1}(\alpha_i\mathcal{V}_{N(i)})^{-1} > (\alpha_i\mathcal{V}_{N(i)})^{-1}$. Furthermore, $\varphi(\infty) = (1/4)\langle X_1 q, q\rangle - (1/2)\langle X_1 q, q\rangle < 0$. The maximum is achieved at $z = X_1 q$ if, and only if,

$$(1+\lambda)(\alpha_i\mathcal{V}_{N(i)} + 2\lambda X_1^{-1})^{-1}q = X_1 q$$

or, equivalently, $(\alpha_i\mathcal{V}_{N(i)} + 2\lambda X_1^{-1})X_1 q = (1+\lambda)q$. This happens if, and only if, q is an eigenvector of $\alpha_i\mathcal{V}_{N(i)}X_1 = (\mathrm{Id} + \Lambda(\alpha_i\mathcal{V}_{N(i)})^{-1})^{-1}$ and the assertion follows because the latter matrix has the same eigenvectors as $\Lambda\mathcal{V}_{N(i)}^{-1}$. ■

Example 6. In the dealer-intermediated market, from the Frobenius formula (Lemma C.4), up to the terms of order ε^2, we have

$$(\Lambda+\mathcal{V})^{-1} \approx \begin{pmatrix} \gamma_1 \mathcal{V}_{11}^{-1} & -\varepsilon\gamma_1\gamma_2\mathcal{V}_{11}^{-1}\mathcal{V}_{12}\mathcal{V}_{22}^{-1} \\ -\varepsilon\gamma_1\gamma_2\mathcal{V}_{22}^{-1}\mathcal{V}_{12}^T\mathcal{V}_{11}^{-1} & \gamma_2\,\mathcal{V}_{22}^{-1} \end{pmatrix},$$

whereas, up to the order of ε^2, we have $(\Lambda_{ii}+\alpha_i\mathcal{V}_{ii})^{-1} \approx (\alpha_i+\beta_i)^{-1}\mathcal{V}_{ii}^{-1},\ i=1,2$. Using Lemma C.4 once again, we get that, up to the terms of order ε^2,

$$B^{-1} \approx \begin{pmatrix} b_1^{-1}\mathcal{V}_{11} & \varepsilon\,\frac{\gamma_1\gamma_2}{b_1b_2}\mathcal{V}_{12} \\ \varepsilon\frac{\gamma_1\gamma_2}{b_1b_2}\mathcal{V}_{12}^T & b_2^{-1}\mathcal{V}_{22} \end{pmatrix}.$$

Therefore, the market risk premium vector $\mathbf{Q}$ is given by

$$\begin{aligned}
\mathbf{Q} \ &= \ B^{-1}\begin{pmatrix}(\Lambda_{11}+\alpha_1\mathcal{V}_{11})^{-1}\alpha_1\left(\mathcal{V}_{11}q_1^1+\varepsilon\mathcal{V}_{12}q_1^2\right) \\ (\Lambda_{22}+\alpha_2\mathcal{V}_{22})^{-1}\alpha_2\left(\mathcal{V}_{22}q_2^2+\varepsilon\mathcal{V}_{12}^Tq_2^1\right)\end{pmatrix} \approx \begin{pmatrix} b_1^{-1}\mathcal{V}_{11} & \varepsilon\,\frac{\gamma_1\gamma_2}{b_1b_2}\mathcal{V}_{12} \\ \varepsilon\frac{\gamma_1\gamma_2}{b_1b_2}\mathcal{V}_{12}^T & b_2^{-1}\mathcal{V}_{22} \end{pmatrix}\begin{pmatrix}\frac{M_1\alpha_1}{\alpha_1+\beta_1}\mathcal{V}_{11}^{-1}\left(\mathcal{V}_{11}q_1^1+\varepsilon\mathcal{V}_{12}q_1^2\right) \\ \frac{M_2\alpha_2}{\alpha_2+\beta_2}\mathcal{V}_{22}^{-1}\left(\mathcal{V}_{22}q_2^2+\varepsilon\mathcal{V}_{12}^Tq_2^1\right)\end{pmatrix} \\
&\approx \ \begin{pmatrix} b_1^{-1}\frac{M_1\alpha_1}{\alpha_1+\beta_1}\left(\mathcal{V}_{11}q_1^1+\varepsilon\mathcal{V}_{12}q_1^2\right)+\varepsilon\frac{\gamma_1\gamma_2}{b_1b_2}\mathcal{V}_{12}\frac{M_2\alpha_2}{\alpha_2+\beta_2}q_2^2 \\ b_2^{-1}\frac{M_2\alpha_2}{\alpha_2+\beta_2}\left(\mathcal{V}_{22}q_2^2+\varepsilon\mathcal{V}_{12}^Tq_2^1\right)+\varepsilon\frac{\gamma_1\gamma_2}{b_1b_2}\mathcal{V}_{12}^T\frac{M_1\alpha_1}{\alpha_1+\beta_1}q_1^1 \end{pmatrix}.
\end{aligned}$$

To compute the individual indirect utilities, we will need to calculate the matrices Γ for each of the three agent classes. Again, ignoring the terms of order ε^2, we have

$$\Gamma_i \approx \frac{0.5\alpha_i+\beta_i}{(\alpha_i+\beta_i)^2}\mathcal{V}_{ii}^{-1},\ i=1,2,$$

whereas for the intermediary we have

$$\Gamma \approx \begin{pmatrix} \xi_{11}\mathcal{V}_{11}^{-1} & \varepsilon\xi_{12}\mathcal{V}_{11}^{-1}\mathcal{V}_{12}\mathcal{V}_{22}^{-1} \\ \varepsilon\xi_{12}\mathcal{V}_{22}^{-1}\mathcal{V}_{12}^T\mathcal{V}_{11}^{-1} & \xi_{22}\mathcal{V}_{22}^{-1} \end{pmatrix}$$

where we have defined

$$\xi_{ii} \equiv \gamma_i^2\left(0.5+\frac{\alpha_1+\beta_1}{M_1}\right),\ i=1,2$$

and

$$\xi_{12} \equiv \gamma_1\gamma_2\left(1-\gamma_1\left(0.5+\frac{\alpha_1+\beta_1}{M_1}\right)-\gamma_2\left(0.5+\frac{\alpha_2+\beta_2}{M_2}\right)\right).$$

Note that even if $\mathcal{V}_{11}=\mathcal{V}_{22}$, Γ does not commute with Λ or $\mathcal{V}$ if agents are asymmetric (i.e., when $\alpha_1 \neq \alpha_2$).

A direct calculation implies that the gross indirect utility of any agent from class i is given by

$$U_i \approx \Gamma_i\,\frac{\gamma_i^2}{b_i^2}\alpha_i^2\left(\langle\mathcal{V}_{ii}q_i^i,q_i^i\rangle+2\varepsilon\langle\mathcal{V}_{ij}q_i^i,q_i^{-i}\rangle\right)+2\varepsilon\Gamma_i\frac{\gamma_i^2}{b_i^2}\frac{\gamma_{-i}}{b_{-i}}\frac{M_{-i}\alpha_i\alpha_{-i}}{\alpha_{-i}+\beta_{-i}}\langle\mathcal{V}_{ij}q_i^i,q_{-i}^{-i}\rangle,\ i=1,2\,, \qquad (43)$$

whereas the gross utility gains of the intermediary are given by

$$
\begin{aligned}
&\langle \Gamma \mathbf{Q}, \mathbf{Q} \rangle \approx \xi_{11} \left(b_1^{-1} \frac{M_1 \alpha_1}{\alpha_1 + \beta_1} \right)^2 \langle \mathcal{V}_{11} q_1^1, q_1^1 \rangle \qquad (44) \\
&+ 2\varepsilon \xi_{11} b_1^{-1} \frac{M_1 \alpha_1}{\alpha_1 + \beta_1} \langle q_1^1, \mathcal{V}_{12} \left(b_1^{-1} \frac{M_1 \alpha_1}{\alpha_1 + \beta_1} q_1^2 + \frac{\gamma_1 \gamma_2}{b_1 b_2} \frac{M_2 \alpha_2}{\alpha_2 + \beta_2} q_2^2 \right) \rangle \\
&+ \xi_{22} \left(b_2^{-1} \frac{M_2 \alpha_2}{\alpha_2 + \beta_2} \right)^2 \langle \mathcal{V}_2 q_2^2, q_2^2 \rangle \\
&+ 2\varepsilon \xi_{22} b_2^{-1} \frac{M_2 \alpha_2}{\alpha_2 + \beta_2} \langle q_2^2, \mathcal{V}_{12}^T \left(b_2^{-1} \frac{M_2 \alpha_2}{\alpha_2 + \beta_2} q_2^1 + \frac{\gamma_1 \gamma_2}{b_1 b_2} \frac{M_1 \alpha_1}{\alpha_1 + \beta_1} q_1^1 \right) \rangle \\
&+ 2\varepsilon \xi_{12} b_2^{-1} \frac{M_2 \alpha_2}{\alpha_2 + \beta_2} b_1^{-1} \frac{M_1 \alpha_1}{\alpha_1 + \beta_1} \langle \mathcal{V}_{12} q_2^2, q_1^1 \rangle.
\end{aligned}
$$

To compute the centralized market utility, by Proposition 2.1, price impacts are proportional to $\mathcal{V}$ in this case and are given by $\hat{\beta}_i \mathcal{V},\ i = 1, 2$ for the two classes, and by $\hat{\beta} \mathcal{V}$ for the intermediary. Let

$$
\hat{\Gamma}_i \equiv \frac{0.5\alpha_i + \hat{\beta}_i}{(\alpha_i + \hat{\beta}_i)^2},\ i = 1, 2
$$

and

$$
\hat{\Gamma} \equiv \frac{0.5 + \hat{\beta}}{(1 + \hat{\beta})^2}.
$$

Then, the gross equilibrium indirect utilities for the two classes are given by

$$
\begin{aligned}
\hat{U}_i \;&=\; \frac{\hat{\Gamma}_i}{b^2} \langle \begin{pmatrix} \mathcal{V}_{11} & \varepsilon \mathcal{V}_{12} \\ \varepsilon \mathcal{V}_{12}^T & \mathcal{V}_{22} \end{pmatrix} \left(-\frac{1}{1 + \hat{\beta}} \alpha_i \begin{pmatrix} q_i^1 \\ q_i^2 \end{pmatrix} + \frac{M_{-i}}{\alpha_{-i} + \hat{\beta}_{-i}} \begin{pmatrix} \alpha_{-i} q_{-i}^1 - \alpha_i q_i^1 \\ \alpha_{-i} q_{-i}^2 - \alpha_i q_i^2 \end{pmatrix} \right), \qquad (45) \\
&\left(-\frac{1}{1 + \hat{\beta}} \alpha_i \begin{pmatrix} q_i^1 \\ q_i^2 \end{pmatrix} + \frac{M_{-i}}{\alpha_{-i} + \hat{\beta}_{-i}} \begin{pmatrix} \alpha_{-i} q_{-i}^1 - \alpha_i q_i^1 \\ \alpha_{-i} q_{-i}^2 - \alpha_i q_i^2 \end{pmatrix} \right) \rangle \\
&= \frac{\hat{\Gamma}_i}{b^2} \left(\frac{M_{-i}}{\alpha_{-i} + \hat{\beta}_{-i}} \right)^2 \langle \mathcal{V}_{11} \left(\alpha_{-i} q_{-i}^1 - \kappa_i \alpha_i q_i^1 \right), \left(\alpha_{-i} q_{-i}^1 - \kappa_i \alpha_i q_i^1 \right) \rangle \\
&+ \frac{\hat{\Gamma}_i}{b^2} \left(\frac{M_{-i}}{\alpha_{-i} + \hat{\beta}_{-i}} \right)^2 \langle \mathcal{V}_{22} \left(\alpha_{-i} q_{-i}^2 - \kappa_i \alpha_i q_i^2 \right), \left(\alpha_{-i} q_{-i}^2 - \kappa_i \alpha_i q_i^2 \right) \rangle \\
&+ 2\varepsilon \frac{\hat{\Gamma}_i}{b^2} \left(\frac{M_{-i}}{\alpha_{-i} + \hat{\beta}_{-i}} \right)^2 \langle \mathcal{V}_{12} \left(\alpha_{-i} q_{-i}^2 - \kappa_i \alpha_i q_i^2 \right), \left(\alpha_{-i} q_{-i}^1 - \kappa_i \alpha_i q_i^1 \right) \rangle,
\end{aligned}
$$

where we have defined

$$
\kappa_i \equiv 1 + \frac{\alpha_{-i} + \hat{\beta}_{-i}}{(1 + \hat{\beta}) M_{-i}},\ i = 1, 2.
$$

Assuming $\varepsilon = 0$ and that initial endowments are such that for the agents of class 2 there are no gains from trade in the centralized market. By (45), this happens if, and only if,

$$
\alpha_1 q_1 = \kappa_2 \alpha_2 q_2 \equiv q. \qquad (46)
$$

That is, class one holds the same risk (in proportions) as class two, but a larger exposure (because $\kappa_2 > 1$). In this case, class 2 is always better off in the decentralized market, whereas class 1 is better off if, and only if,

$$\begin{aligned}\Gamma_1 \frac{\gamma_1^2}{b_1^2} \alpha_1^2 \langle \mathcal{V}_{11} q_1^1, q_1^1 \rangle \quad > \quad & \frac{\hat{\Gamma}_1}{b^2} \left(\frac{M_2}{\alpha_2 + \hat{\beta}_2} \right)^2 \langle \mathcal{V}_{11} \left(\alpha_2 q_2^1 - \kappa_1 \alpha_1 q_1^1 \right), \left(\alpha_2 q_2^1 - \kappa_1 \alpha_1 q_1^1 \right) \rangle \\ & + \frac{\hat{\Gamma}_1}{b^2} \left(\frac{M_2}{\alpha_2 + \hat{\beta}_2} \right)^2 \langle \mathcal{V}_{22} \left(\alpha_2 q_2^2 - \kappa_1 \alpha_1 q_1^2 \right), \left(\alpha_2 q_2^2 - \kappa_1 \alpha_1 q_1^2 \right) \rangle.\end{aligned}$$

Substituting from (46), we can rewrite this inequality as

$$\frac{\Gamma_1}{b_1^2} (\gamma_1 \kappa_2)^2 \langle \mathcal{V}_{11} q^1, q^1 \rangle > \frac{\hat{\Gamma}_1}{b^2} K^2 \left(\langle \mathcal{V}_{11} q^1, q^1 \rangle + \langle \mathcal{V}_{22} q^2, q^2 \rangle \right),$$

where $K \equiv (M_2/(\alpha_2 + \hat{\beta}_2))(\kappa_1 \kappa_2 - 1)$. This inequality can only hold if (1) the size of risk in the second market, $\langle \mathcal{V}_{22} q^2, q^2 \rangle$, is sufficiently small and (2) we have $\mathcal{L}_1 > \mathcal{L}_2$ where we have defined

$$\mathcal{L}_1 \equiv \frac{\Gamma_1}{b_1^2} (\gamma_1 \kappa_2)^2, \;\; \mathcal{L}_2 \equiv \frac{\hat{\Gamma}_1}{b^2} K^2. \tag{47}$$

We know that $\beta_1 < \hat{\beta}_1$ and, consequently, $\Gamma_1 < \hat{\Gamma}_1$ always. That is, compensation for the exposure to aggregate risk is always smaller in the decentralized market. However, the vector $\mathbf{Q}$ is different in the centralized and decentralized markets. If $\mathbf{Q}$ is "further away" from q_1 in the decentralized market, this leads to larger gains from trade for agent 1.

We will consider inequality (47) for the case where we expect the gains from trade to be largest; namely, when α_1 is very small and α_2 is very large. Namely, we assume that $\alpha_1 = \alpha_1' \delta, \; \alpha_2 = \alpha_2' \delta^{-1}$ for some very small $\delta > 0$. Then, $\beta_1 \approx \delta \, \beta_1'$, $\beta_2 = \delta^{-1} \, \beta_2'$, where

$$\beta_i' \approx \frac{2 - \alpha_i' b_i' + \sqrt{(\alpha_i' b_i'^2 + 4}}{2 b_i'}, \;\; i = 1, 2\,.$$

and hence

$$\gamma_1 \approx 1 - \frac{\delta}{b_1'}, \;\; \gamma_2 \approx \delta \frac{M_2}{\alpha_2' + \beta_2'}\,.$$

Finally, $\Gamma_1 \approx \delta \frac{0.5\alpha_1' + \beta_1'}{(\alpha_1' + \beta_1'^2 (b_1'^2}$.

In the centralized market, $b \approx b_1$ (indeed, agents from class 3 do not provide any liquidity). Consequently, $\hat{\beta}_1 \approx \hat{\beta}_2 \approx \hat{\beta} \approx \beta_1$ and

$$\kappa_1 \approx 1 - \frac{\alpha_2' \beta_1'}{M_2} + \delta^{-1} \frac{\alpha_2'}{M_2}, \;\; \kappa_2 \approx 1 + \delta \frac{\alpha_1' + \beta_1'}{M_1}$$

and the identity $\gamma_1 = \frac{1}{1+\beta} = \frac{M_1}{\alpha_1+\beta_1+M_1}$ implies that $\frac{\alpha_1'+\beta_1'}{M_1} = b_1'^{-1}$. A direct calculation implies that

$$K \approx 1 - \frac{\beta_1'}{M_2}\delta.$$

A direct (but tedious) calculation implies that the required inequality (47) does hold when δ is sufficiently small. Numerical experiments show that it holds when $\alpha_1 < 0.35$ and $\alpha_2 > 5$.

The intermediary's centralized market utility is approximately given by

$$\begin{aligned} \hat{U} \quad \approx \quad & \frac{\hat{\Gamma}}{b^2}\langle \mathcal{V}_{11}\left(\frac{M_1}{\alpha_1+\hat{\beta}_1}\alpha_1 q_1^1 + \frac{M_2}{\alpha_2+\hat{\beta}_2}\alpha_2 q_2^1\right), \left(\frac{M_1}{\alpha_1+\hat{\beta}_1}\alpha_1 q_1^1 + \frac{M_2}{\alpha_2+\hat{\beta}_2}\alpha_2 q_2^1\right)\rangle \\ + \quad & \langle \mathcal{V}_{22}\left(\frac{M_1}{\alpha_1+\hat{\beta}_1}\alpha_1 q_1^2 + \frac{M_2}{\alpha_2+\hat{\beta}_2}\alpha_2 q_2^2\right), \left(\frac{M_1}{\alpha_1+\hat{\beta}_1}\alpha_1 q_1^2 + \frac{M_2}{\alpha_2+\hat{\beta}_2}\alpha_2 q_2^2\right)\rangle \\ + \quad & 2\varepsilon\langle \mathcal{V}_{12}\left(\frac{M_1}{\alpha_1+\hat{\beta}_1}\alpha_1 q_1^2 + \frac{M_2}{\alpha_2+\hat{\beta}_2}\alpha_2 q_2^2\right), \left(\frac{M_1}{\alpha_1+\hat{\beta}_1}\alpha_1 q_1^1 + \frac{M_2}{\alpha_2+\hat{\beta}_2}\alpha_2 q_2^1\right)\rangle \\ = \quad & \mathcal{L}_3\left(\langle \mathcal{V}_{11}q^1, q^1\rangle + \langle \mathcal{V}_{22}q^2, q^2\rangle + 2\varepsilon\langle \mathcal{V}_{12}q^2, q^1\rangle\right), \end{aligned}$$

where

$$\mathcal{L}_3 = \frac{\hat{\Gamma}}{b^2}\left(\frac{M_1}{\alpha_1+\hat{\beta}_1}\kappa_2 + \frac{M_2}{\alpha_2+\hat{\beta}_2}\right)^2.$$

Thus, for the intermediary to be better off in the decentralized market, we need that

$$\begin{aligned} & \mathcal{L}_3\left(\langle \mathcal{V}_{11}q^1, q^1\rangle + \langle \mathcal{V}_{22}q^2, q^2\rangle + 2\varepsilon\langle \mathcal{V}_{12}q^2, q^1\rangle\right) \\ < \quad & \mathcal{L}_4\langle \mathcal{V}_{11}q^1, q^1\rangle + \mathcal{L}_5\mathcal{V}_{22}\langle q^2, q^2\rangle + 2\varepsilon\mathcal{L}_6\langle \mathcal{V}_{12}q^2, q^1\rangle, \end{aligned} \tag{48}$$

where we have defined

$$\begin{aligned} \mathcal{L}_4 \quad &\equiv \quad \xi_{11}\left(b_1^{-1}\frac{M_1}{\alpha_1+\beta_1}\right)^2\kappa_2^2 \\ \mathcal{L}_5 \quad &\equiv \quad \xi_{22}\left(b_2^{-1}\frac{M_2}{\alpha_2+\beta_2}\right)^2 \\ \mathcal{L}_6 \quad &\equiv \quad \xi_{11}b_1^{-1}\frac{M_1}{\alpha_1+\beta_1}\kappa_2\left(b_1^{-1}\frac{M_1}{\alpha_1+\beta_1}\kappa_2 + \frac{\gamma_1\gamma_2}{b_1b_2}\frac{M_2}{\alpha_2+\beta_2}\right) \\ &\quad +\xi_{22}b_2^{-1}\frac{M_2}{\alpha_2+\beta_2}\left(b_2^{-1}\frac{M_2}{\alpha_2+\beta_2} + \frac{\gamma_1\gamma_2}{b_1b_2}\frac{M_1}{\alpha_1+\beta_1}\kappa_2\right) \\ &\quad +\xi_{12}b_2^{-1}\frac{M_2}{\alpha_2+\beta_2}b_1^{-1}\frac{M_1}{\alpha_1+\beta_1}\kappa_2. \end{aligned}$$

■

Claim. $\mathcal{L}_3 > \mathcal{L}_4$ and $\mathcal{L}_3 > \mathcal{L}_5$.

Proof. Since $\kappa_2 > 1$, we have

$$\mathcal{L}_3 > \frac{\hat{\Gamma}}{b^2}\left(\frac{M_1}{\alpha_1+\hat{\beta}_1} + \frac{M_2}{\alpha_2+\hat{\beta}_2}\right)^2 \; = \; \hat{\Gamma}\frac{(b-(1+\hat{\beta})^{-1})^2}{b^2}$$

and

$$\mathcal{L}_4 = \frac{0.5+\beta}{(1+\beta)^2}\left(\frac{1}{b_1}\frac{M_1}{\alpha_1+\beta_1}\right)^2 = \frac{0.5+\beta}{(1+\beta)^2}\frac{(b_1-(1+\beta)^{-1})^2}{b_1^2}.$$

Since $\hat{\Gamma} > \frac{0.5+\beta}{(1+\beta)^2}$ (compensation for aggregate risk is decreasing in price impact), it suffices to show that

$$(1+\hat{\beta})b > (1+\beta)b_1\,.$$

That is, $2+b+\sqrt{b^2+4} > 2+b_1+\sqrt{b_1^2+4}$. The latter is obvious because $b > b_1$ (the aggregate demand slope is larger in a more liquid market). The claim for $\mathcal{L}_5$ follows by the same arguments. Assuming non-zero correlation. across markets, the inequality $U_1 > \hat{U}_1$ takes the form

$$\mathcal{L}_1\langle \mathcal{V}_{11}q^1, q^1\rangle + 2\varepsilon(\mathcal{L}_1+\mu)\langle \mathcal{V}_{12}q^2, q^1\rangle > \mathcal{L}_2\left(\langle \mathcal{V}_{11}q^1, q^1\rangle + 2\varepsilon\langle \mathcal{V}_{12}q^2, q^1\rangle + \langle \mathcal{V}_{22}q^2, q^2\rangle\right), \tag{49}$$

where

$$\mu \equiv \Gamma_1\frac{\gamma_1^2}{b_1^2}\frac{\gamma_2}{b_2}\frac{M_2}{\alpha_2+\beta_2}\kappa_2.$$

Assuming that $\mathcal{L}_1 > \mathcal{L}_2$ and combining (48) and (49), we get that the dealer-intermediated market dominates the centralized market if, and only if,

$$\frac{(\mathcal{L}_1-\mathcal{L}_2)\langle \mathcal{V}_{11}q^1, q^1\rangle - \mathcal{L}_2\langle \mathcal{V}_{22}q^2, q^2\rangle}{\mathcal{L}_1+\mu-\mathcal{L}_2} > -2\varepsilon\langle \mathcal{V}_{12}q^2, q^1\rangle > \frac{(\mathcal{L}_3-\mathcal{L}_4)\langle \mathcal{V}_{11}q^1, q^1\rangle + (\mathcal{L}_3-\mathcal{L}_5)\langle \mathcal{V}_{22}q^2, q^2\rangle}{\mathcal{L}_3-\mathcal{L}_6}.$$

It follows by a direct (but tedious) calculation from the asymptotic analysis above that, when α_1 is sufficiently small and α_2 is sufficiently large, $\mathcal{L}_1-\mathcal{L}_2 \to 0$, $\mathcal{L}_3-\mathcal{L}_4 \to 0$ and $\mu \to 0$, whereas $\mathcal{L}_6$ converges to a *negative* constant. The negativity of $\mathcal{L}_6$ is the key: By (48), the dealer's utility is affected differently by the covariance term in the centralized versus decentralized market: In the centralized market U is *increasing* in the covariance between the portfolios of the two classes, whereas in the decentralized market it is *decreasing*. Therefore, when the portfolios are sufficiently negatively correlated, the intermediary is better off in the decentralized market. ■

Proof of Lemma 7.1. Let, for brevity $\Gamma = \Gamma_{j\setminus i}$, $\mathcal{V} = \alpha_j\tilde{\mathcal{V}}_j$, $\Lambda = \tilde{\Lambda}_j$, $B = \tilde{B}$. Then,

$$\Gamma = 0.5(\mathcal{V}+\Lambda)^{-1} + 0.5(\mathcal{V}+\Lambda)^{-1}\Lambda(\mathcal{V}+\Lambda)^{-1} = 0.5(B-\Lambda^{-1}) + 0.5(B-\Lambda^{-1})\Lambda(B-\Lambda^{-1})$$

and the claim follows. For Δ, we have

$$\begin{aligned}\Delta &= \Lambda(\Lambda+\mathcal{V})^{-1}\mathcal{V}(\Lambda+\mathcal{V})^{-1}\Lambda = \Lambda(\Lambda+\mathcal{V})^{-1}(\mathcal{V}+\Lambda-\Lambda)(\Lambda+\mathcal{V})^{-1}\Lambda \\ &= \Lambda(\Lambda+\mathcal{V})^{-1}\Lambda - \Lambda(\Lambda+\mathcal{V})^{-1}\Lambda(\Lambda+\mathcal{V})^{-1}\Lambda = \Lambda(B-\Lambda^{-1})\Lambda - \Lambda(B-\Lambda^{-1})\Lambda(B-\Lambda^{-1})\Lambda\end{aligned} \tag{50}$$

and the claim follows by direct calculation. ■

Proof of Proposition 7.3. The proof is based on the following auxiliary "anything goes" result.

Lemma E.3 *Within Example 3, suppose that equilibrium price impact of class 1 is given by $\tilde{\Lambda}_1$.*

Then, any $\hat{\Lambda}_1 \le \Lambda_1$ can be attained as an equilibrium price impact by adding an additional class to the exchange.

Proof. The proof follows directly from the arguments in the proof of Proposition 7.5. Namely, suppose that there are N classes satisfying

$$\tilde{\Lambda}_i^{-1} + (\alpha_i \tilde{\mathcal{V}}_i + \tilde{\Lambda}_i)^{-1} = \tilde{B}^{-1},\ i = 1, \cdots, N.$$

Pick an arbitrary $\hat{\Lambda}_1 \le \Lambda_1$ and define $\hat{B}$ via

$$\hat{\Lambda}_1^{-1} + (\alpha_1 \tilde{\mathcal{V}}_1 + \hat{\Lambda}_1)^{-1} = \hat{B}^{-1}.$$

Then, define $\hat{\Lambda}_i,\ i \ge 2$ via

$$\hat{\Lambda}_i^{-1} + (\alpha_i \tilde{\mathcal{V}}_i + \hat{\Lambda}_i)^{-1} = \hat{B}^{-1},\ i = 2, \cdots, N.$$

The argument in the proof of Proposition 7.5 implies that these price impacts can be sustained in equilibrium if we add an $N+1$-st class with some perceived covariance matrix $\tilde{\mathcal{V}}_{N+1}$, and the proof is complete. ■

Lemma *E.3* implies that, by adding/removing classes to the illiquid exchange, we can achieve arbitrary changes in the price impact. The required claim follows now from Proposition 7.1. ■

Lemma E.4 *Denote*

$$\tilde{\Gamma}_i \equiv (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1} \left(\frac{1}{2} \alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j \right) (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1}$$

and

$$\tilde{\Delta}_j \equiv \frac{1}{2} \tilde{\Lambda} (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1} \alpha_i \tilde{\mathcal{V}}_j (\alpha_i \tilde{\mathcal{V}}_j + \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j.$$

Then, the utility U_j of agent from lass j with initial holdings q_k^0 with $(\mathrm{Id} - \Pi_{\kappa(n)}) q_k^0 = 0$ (i.e., no initial holdings in exchanges $N \setminus K(n)$) is given by

$$U_j(\Lambda_j; q_k^0) = \langle \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle - 2 \langle \alpha_j \tilde{\mathcal{V}}_j \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{q}_k^0 \rangle - \langle \tilde{\Delta}_j \tilde{q}_k^0, \tilde{q}_k^0 \rangle. \tag{51}$$

Proof. Let $\mathcal{V}_j \equiv \mathcal{V}_{N(j)}$. Then,

$$\Gamma_j = 0.5 (\alpha_i \mathcal{V}_j + \Lambda_j)^{-1} + 0.5 (\alpha_j \mathcal{V}_j + \Lambda_j)^{-1} \Lambda_j (\alpha_j \mathcal{V}_j + \Lambda_j)^{-1},$$

and hence,

$$\langle \Gamma_i \mathbf{Q}_{N(i)}, \mathbf{Q}_{N(i)} \rangle = \langle \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{\mathbf{Q}} \rangle,$$

because $(\mathrm{Id} - \Pi_{\kappa(n)}) \mathbf{Q} = 0$ and price impact in exchanges not from $\kappa(n)$ also vanishes.

Furthermore, a direct calculation implies that

$$\alpha_i \mathcal{V}_i \Gamma_i = 0.5(\mathrm{Id} - \Lambda_i(\alpha \mathcal{V}_i + \Lambda_i)^{-1}) + 0.5(\mathrm{Id} - \Lambda_i(\alpha \mathcal{V}_i + \Lambda_i)^{-1})\Lambda_i(\alpha \mathcal{V}_i + \Lambda_i)^{-1},$$

and hence,

$$\langle \alpha_i \mathcal{V}_i \Gamma_i \mathbf{Q}_{N(i)}, q_k^0 \rangle = \langle \alpha_j \tilde{\mathcal{V}}_j \tilde{\Gamma}_j \tilde{\mathbf{Q}}, \tilde{q}_k^0 \rangle.$$

■

Proof of Proposition 7.4. If all matrices commute, we can diagonalize them in a common basis, and the required inequality is equivalent to verifying monotonicity of the scalar function $\delta(\lambda) = 3\lambda^2 b - \lambda^3 b^2$for $\lambda \in (0, 2b^{-1})$. This follows by direct calculation.

Without commutativity, consider a small one dimensional perturbation $\tilde{\Lambda}_{j_2} = \tilde{\Lambda}_{j_1} + \alpha P$ with $P = \langle \cdot, y \rangle y$ for some $y \in \mathbb{R}^{\kappa(n)}$ and a small $\varepsilon > 0$. Then,

$$\begin{aligned}
\langle (\Delta_{j_1} - \Delta_{j_2})q, q \rangle &\approx \varepsilon \langle (3\Lambda BP + 3PB\Lambda - PB\Lambda B\Lambda - \Lambda BPB\Lambda - \Lambda B\Lambda BP)q, q \rangle \\
&= 3\langle q, y \rangle \langle B\Lambda y, q \rangle + 3\langle q, y \rangle \langle B\Lambda q, y \rangle - \langle B\Lambda B\Lambda q, y \rangle \langle q, y \rangle - \langle B\Lambda q, y \rangle \langle \Lambda B y, q \rangle \\
&\quad - \langle q, y \rangle \langle \Lambda B\Lambda B y, q \rangle.
\end{aligned}$$

This is monotone decreasing in α for any $q \perp y$ unless y is an eigenvector of $B\Lambda$, which precisely guarantees that P and $B\Lambda$ commute. ■

Proof of Example 8. For simplicity we consider the case when there are only two classes, 1 and 2, that initially have access to the same liquid exchange and normalize their risk aversion to 1. Then,

$$\Lambda_1 = \Lambda_2 = \frac{1}{M_1 + M_2 - 2} \tilde{\mathcal{V}} \equiv \Lambda(0)\,.$$

Now, suppose that agents of class 1 get access to an additional exchange for an asset that has covariance $\varepsilon \{y_k\} = \varepsilon y \in \mathbb{R}^{K(n)}$ with assets $k \in K(n)$ and whose variance is normalized to 1. Then,

$$\tilde{V}_1 = \tilde{V} - \varepsilon \langle \cdot, y \rangle y \equiv \tilde{V} - \varepsilon P.$$

Substituting conjectured Taylor expansions $\Lambda_i = \Lambda(0) + \varepsilon \Lambda_i^{(1)}$, we have

$$\Lambda_1 = ((M_1 - 1)(\tilde{V}_1 + \Lambda_1)^{-1} + M_2(\tilde{V} + \Lambda_2)^{-1})^{-1}, \ \Lambda_2 = (M_1(\tilde{V}_1 + \Lambda_1)^{-1} + (M_2 - 1)(\tilde{V} + \Lambda_2)^{-1})^{-1}.$$

We get the system

$$\begin{aligned}
\Lambda_1^{(1)} &\approx \Lambda(0)((M_1 - 1)(\Lambda(0) + \tilde{V})^{-1}(\Lambda_1^{(1)} - P)(\Lambda(0) + \tilde{V})^{-1} + M_2(\Lambda(0) + \tilde{V})^{-1}\Lambda_2^{(1)}(\Lambda(0) + \tilde{V})^{-1})\Lambda(0) \\
\Lambda_2^{(1)} &\approx \Lambda(0)(M_1(\Lambda(0) + \tilde{V})^{-1}(\Lambda_1^{(1)} - P)(\Lambda(0) + \tilde{V})^{-1} + (M_2 - 1)(\Lambda(0) + \tilde{V})^{-1}\Lambda_2^{(1)}(\Lambda(0) + \tilde{V})^{-1})\Lambda(0).
\end{aligned}$$

Substituting the expression for $\Lambda(0)$, we get that both $\Lambda_i^{(1)} = -\zeta_i P$ for some $0 < \zeta_1 < \zeta_2$ that only depend on M_1, M_2. Consequently, Λ_1 and Λ_2 differ from each other by a one-dimensional projection and the arguments in (50) apply. ■

Proof of Proposition 7.5. For simplicity, we normalize all risk aversions to 1. Let $j_1 = 1,\ j_2 = 2$. We first show that, for any $\tilde{\mathcal{V}}_1,\ \tilde{\mathcal{V}}_2$ and $\tilde{B}$ there exists a market in which they are realized. To prove this, consider a market with three classes and let us show that we can pick $\tilde{\mathcal{V}}_3$ accordingly. First, equation $\tilde{\Lambda}_i^{-1} + (\tilde{\mathcal{V}}_i + \tilde{\Lambda}_i)^{-1} = \tilde{B}$ implies (by Lemma C.2) that

$$\tilde{\Lambda}_i \ = \ \tilde{\mathcal{V}}_i^{1/2} f(\tilde{\mathcal{V}}_i^{1/2} \tilde{B} \tilde{\mathcal{V}}_i^{1/2}) \tilde{\mathcal{V}}_i^{1/2}$$

and

$$(\tilde{\Lambda}_i + \tilde{\mathcal{V}}_i)^{-1} = \tilde{\mathcal{V}}_i^{-1/2} g(\tilde{\mathcal{V}}_i^{1/2} \tilde{B} \tilde{\mathcal{V}}_i^{1/2}) \tilde{\mathcal{V}}_i^{-1/2}.$$

Denote

$$A \ \equiv (\tilde{\mathcal{V}}_1 + \tilde{\Lambda}_1)^{-1} + (\tilde{\mathcal{V}}_2 + \tilde{\Lambda}_2)^{-1}.$$

Then, $\tilde{\Lambda}_3$ satisfies

$$\tilde{\Lambda}_3 = (A + (M_3 - 1)(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1})^{-1} = (\tilde{B} - (\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1})^{-1}$$

and therefore, to complete the proof, it suffices to show that there exist positive definite matrices $\tilde{\Lambda}_3,\ \tilde{\mathcal{V}}_3$ satisfying

$$\tilde{\Lambda}_3^{-1} - (M_3 - 1)(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = A,\ \tilde{\Lambda}_3^{-1} + (\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = \tilde{B}.$$

Solving this system, we get

$$(\tilde{\mathcal{V}}_3 + \tilde{\Lambda}_3)^{-1} = M_3^{-1}(\tilde{B} - A),\ \tilde{\Lambda}_3^{-1} = M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1})\,.$$

Since $A + (M_3 - 1)\tilde{B}^{-1} > \tilde{B}^{-1} - A$ whenever $M_3 > 1$, the matrix

$$\tilde{\mathcal{V}}_3 = (M_3^{-1}(\tilde{B} - A))^{-1} - (M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1}))^{-1}$$

is positive definite. Finally, $\tilde{B} > 2\tilde{\mathcal{V}}_1^{-1} > \tilde{\mathcal{V}}_1^{-1} + \tilde{\mathcal{V}}_2^{-1} > A$, completing the proof of existence.

Lemma E.5 *None of the functions* $f,\ f_1,\ g$ *is matrix monotone.*

Proof. By the Löwner Theorem (Donoghue (1974)), it suffices to show that none of these functions can be analytically continued to the whole upper half-plane. This follows directly from the fact that $2i$ is a branching point for all these functions. ■

By Lemma, $f_1 = \frac{2-a+\sqrt{a^2+4}}{2}$ is not matrix monotone on any interval and, consequently, for any positive definite matrix X_1 of sufficiently high dimension, there exists a matrix $X_2 \geq X_1$ such that $f_1(X_2) \not\leq f_1(X_1)$. Let $X_1 = \tilde{B}^{-1/2}\tilde{\mathcal{V}}_1\tilde{B}^{-1/2}$. Then, define $\tilde{\mathcal{V}}_2 \equiv \tilde{B}^{1/2} X_2 \tilde{B}^{1/2}$.

Now, by Lemma *C.2*,

$$\Lambda_1 = \tilde{B}^{1/2} f_1(X_1) \tilde{B}^{1/2} \not\geq \tilde{B}^{1/2} f_1(X_2) \tilde{B}^{1/2} = \Lambda_2,$$

and the proof is complete. ■

Proof of Example 9. Suppose that agent j_1 participates in a greater number of liquid exchanges than agent j_2 so that his perceived covariance matrix $\tilde{\mathcal{V}}_{j_1}$ satisfies $\tilde{\mathcal{V}}_{j_1} \leq \tilde{\mathcal{V}}_{j_2}$. Pick $\tilde{\mathcal{V}}_{j_1}$, $\tilde{\mathcal{V}}_{j_2}$ and $\tilde{B}$ satisfying the conditions of Proposition 7.5. Then, there exists a vector π such that

$$\langle \Lambda_{j_1}\pi, \pi\rangle < \langle \Lambda_{j_2}\pi, \pi\rangle.$$

Now, pick a distribution of initial holdings so that $\tilde{\mathbf{Q}} = \tilde{B}^{-1}\pi$. Denote by U_i the utility of agent $j_i, i = 1, 2$. Then, by Lemma 7.1,

$$U_1+\langle \tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}}\rangle = \langle 0.5\tilde{B}\tilde{\Lambda}_{j_1}\tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}}\rangle = 0.5\langle \Lambda_{j_1}\pi, \pi\rangle < 0.5\langle \Lambda_{j_2}\pi, \pi\rangle = \langle 0.5\tilde{B}\tilde{\Lambda}_{j_1}\tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}}\rangle = U_2+\langle \tilde{B}\tilde{\mathbf{Q}}, \tilde{\mathbf{Q}}\rangle.$$

■