A Closer Look at the Aumann-Serrano and Foster-Hart Measures of Riskiness

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Abstract

Hart (2011) argues that the Aumann and Serrano (2008) and Foster and Hart (2009) measures of riskiness have an objective and universal appeal, assuming expected utility preferences. We show that mean-riskiness decision-making criteria using either measure violate expected utility. Alongside much experimental evidence, this motivates exploring universality beyond expected utility. We demonstrate that riskiness measures satisfying Hart's other behavioral requirements do not generally exist when non-expected utility maximizers are additionally incorporated into the argument. We identify other attributes of the Aumann-Serrano and Foster-Hart measures that raise questions concerning their usefulness in a decision making, risk management, or risk assessment setting.

Keywords: Riskiness; Risk Aversion; Index of Riskiness; Gamble; Risky asset.

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1 Introduction

In two influential papers, Aumann and Serrano (2008) and Foster and Hart (2009) introduce measures of riskiness that appear to have many desirable attributes. A gamble's riskiness strictly increases if it is subjected to a mean-preserving spread and strictly decreases with an increase in the mean. Riskiness is subadditive, meaning that the riskiness of a portfolio is less than the sum over the riskiness of each portfolio component. Riskiness is positively homogeneous in that scaling every outcome of a gamble by a constant scales the riskiness by the same proportion. The most attractive of the attributes of the proposed measures, not claimed by any other measures in the literature, is their "objectivity" as perhaps best expounded in Hart (2011). The measure of Aumann and Serrano (2008), denoted R^{AS}, applies whenever a risky profit opportunity is deemed unacceptable regardless of wealth to anyone with what Hart (2011) implicitly defines as unobjectionable preferences. The measure of Foster and Hart (2009), denoted R^{FH}, applies whenever for some level of wealth everyone with unobjectionable preferences deem the opportunity unacceptable. These attributes — universal in the sense that they apply uniformly either across wealth levels or across preferences — lend a sense of objectivity, if not normative weight, to R^{AS} and R^{FH}.

In this paper, we take issue with, primarily, the claim of objectivity of R^{AS} and R^{FH}. Specifically, we question Hart's (2011) definition of unobjectionable preferences which admits only those satisfying the Independence Axiom (IA) of expected utility. We demonstrate that any differentiable ranking of gambles using the mean and either R^{FH} or R^{AS} results in preferences that violate expected utility. Thus attempts to replace mean-variance analysis (or similar approaches) using "mean-riskiness" undermine the very assumptions that endow these riskiness measures with a sense of objectivity. In addition, cumulative evidence dating back to the 1950's reveals systematic inadequacies in the empirical validity of IA, and has contributed to the development of a good number of alternative models with better descriptive power that satisfy the basic requirements of transitivity, continuity, and monotonicity. These considerations call for a reexamination of Hart's (2011) analysis and the role played by IA.

To summarize our main finding, when we relax reliance on IA in Hart's (2011) definition of unobjectionable preferences we find that R^{AS} and R^{FH} are no longer universal. In particular, a mild departure from IA results in a negative result: There are no measures featuring the standard attributes mentioned above that apply uniformly across preferences or wealth levels. In other words, the results in Hart (2011) are, in some sense, knife-edge. Our findings pose a challenge to advocating a normative approach to risk measurement and/or management using R^{AS} and R^{FH}. If a significant proportion of the population eschews IA in practice, including those that would employ R^{AS} or R^{FH} in “mean-riskiness” analysis, then such advocacy may not be convincing.

Beyond the “universality” critique outlined above, we also identify several features of R^{AS} and R^{FH} that potentially render them unsuitable for the purposes of risk management. Specifically, these measures do not distinguish between prospects that can only result in insignificant gains and losses versus those that exhibit arbitrarily large gains and losses. Moreover, a profit opportunity involving a high likelihood of substantial loss can be assessed as being as risky as one whose likelihood of substantial loss is tiny. These issues arise because R^{AS} and R^{FH} confound mean and dispersion. We prove that one cannot disentangle the two effects while retaining the standard requirements of riskiness measures mentioned earlier. Thus it may not be appropriate to rely on these measures to determine risk management policies such as capital requirements.

Finally, the practical use of R^{AS} and R^{FH} is constrained to instances in which one can clearly identify a risk-free choice alternative that, in turn, can be identified with the decision maker’s “wealth”. We argue that at times it may not be possible to arrive at a meaningful assessment of the decision maker’s wealth. Yet, the need for measuring and managing risk may be more pressing in such circumstances.

The derivation of the Aumann and Serrano (2008) and Foster and Hart (2009) measures of riskiness is elegant and economically motivated. We hope to place the use of these measures in an appropriate context that appears not to have been addressed in the literature. Notwithstanding the eloquent arguments put forth in Hart (2011) on behalf of R^{AS} and R^{FH}, it would seem prudent to avoid giving them decisive weight in a decision making setting without a more complete understanding of their limitations. In that regard, R^{AS} and R^{FH} are similar to other measure of riskiness currently in use.

2 Framework

In deriving their risk measures, Aumann and Serrano (2008), Foster and Hart (2009) and Hart (2011) consider all“gambles” in the set $\mathcal{G}$ defined as follows.

Definition 1. *$\tilde{g} \in \mathcal{G}$ if and only if $\tilde{g}$ is a real-valued random variable such that*

i. $0 < \text{Prob}[\tilde{g} < 0]$

ii. $\text{E}[\tilde{g}] > 0$

iii. The probability distribution of $\tilde{g}$ has finite support *where* Prob[$\cdot$] *denotes the probability of an event and* E[$\cdot$] *denotes the expected value operation.*

Properties *i* and *ii* are binding in the sense that the Aumann and Serrano (2008) and Foster and Hart (2009) measures are ill-defined for distributions violating *i* and *ii*. While seemingly restrictive, property *i* is well-suited to situations in which there is a forward market price for any payoff distribution. For instance, consider a portfolio whose payoff distribution in one year, $\tilde{S}$, is a log-normal distribution (and therefore strictly positive). If the current portfolio value is S_0, then in choosing to accept or reject the stock portfolio one is essentially choosing between the foregone "sure" forward value of $S_0 \times (1 + r_f)$ (where r_f is the one-year risk-free rate) and the risky outcome of $\tilde{S}$. The gamble inherent in opting for the portfolio is the relative profit from the investment, $\tilde{g} = \tilde{S} - S_0 \times (1 + r_f)$, which in the absence of arbitrage will exhibit the property $0 < \text{Prob}[\tilde{g} < 0] < 1$. Note that the cost, or present value, of $\tilde{g}$ is zero because the cost of the stock is S_0 and the cost of the risk-free bond that delivers $S_0 \times (1 + r_f)$ in one year is also S_0. Thus one can interpret property *i* as restricting attention to zero present value profit opportunities. Taken in that spirit, the addition of property *ii* is tantamount to restricting attention to investment profits with a strictly positive risk-premium. Although we will return to pointing out the limitations imposed by properties *i-ii*, for now we shall maintain our focus on the domain defined by $\mathcal{G}$.[1] Inspired by the market example, we will employ the terms "gamble" and "profit" interchangeably when referring to $\tilde{g} \in \mathcal{G}$.

Let $\mathcal{D}$ be the closure of the set of all finite-outcome and non-negative real-valued random variables.[2] Let $\succcurlyeq_{\text{FSD}}$ (resp. $\succcurlyeq_{\text{SSD}}$) refer to the ordering of prospects in $\mathcal{D}$ via first-degree (resp. second degree) stochastic dominance. A binary relation, $\succcurlyeq$ over $\mathcal{D}$ is said to be *FSD monotonic* whenever $\tilde{x} \succcurlyeq_{\text{FSD}} \tilde{y} \Rightarrow \tilde{x} \succcurlyeq \tilde{y}$. We similarly define what it means for $\succcurlyeq$ to be *SSD monotonic.* When referring to preference relations over $\mathcal{D}$, we will restrict attention to the set $\mathcal{U}$ of binary relations over $\mathcal{D}$ that satisfy the following:

Basic Preference Assumptions $\succcurlyeq \in \mathcal{U}$ iff $\succcurlyeq$ is a weak, continuous, FSD- and SSD-monotonic ordering over $\mathcal{D}$.[3]

Denote by F_g the cumulative distribution function associated with any $\tilde{g} \in \mathcal{G}$. For any investment profits $\tilde{g}, \tilde{g}' \in \mathcal{G}$ and $\alpha \in [0, 1]$, denote by $\alpha\tilde{g} \oplus (1 - \alpha)\tilde{g}'$ the random variable with probability distribution $\alpha F_g + (1 - \alpha)F_{g'}$. The gamble $\alpha\tilde{g} \oplus (1 - \alpha)\tilde{g}'$ is the probabilistic α-mixture generated

[1] Schulze (2014) and Hellmann and Riedel (2013) have pointed out issues that arise when attempting to extend the Aumann and Serrano (2008) and Foster and Hart (2009) to continuous distributions. While meriting attention, especially in applications, our focus differs.

[2] Following Hart (2011), we restrict attention to non-negative payoffs.

[3] A weak ordering is complete and transitive. By continuity, we mean that all upper and lower contour sets of $\succcurlyeq$ are closed in the topology of weak convergence on $\mathcal{D}$.

via an α-coin flip (where the probability of "heads" is α) in which "heads" signifies that the outcomes will be determined by $\tilde{g}$, and "tails" signifies that the outcomes will be determined by $\tilde{g}'$. A similar definition applies to random variables in $\mathcal{D}$. The Independence Axiom (IA) of von Neumann and Morgenstern can be stated as follows:

IA (Independence Axiom). *For any* $\tilde{x}, \tilde{y} \in \mathcal{D}$,

$$\tilde{x} \succcurlyeq \tilde{y} \;\Leftrightarrow\; \forall \alpha \in [0,1] \text{ and } \tilde{z} \in \mathcal{D}, \alpha\tilde{x} \oplus (1-\alpha)\tilde{z} \succcurlyeq \alpha\tilde{y} \oplus (1-\alpha)\tilde{z}.$$

The basic preference assumptions above together with IA imply that $\succcurlyeq$ has an expected utility representation with a weakly concave and nondecreasing utility function (See, for example, Herstein and Milnor, 1953).

2.1 The Aumann and Serrano (2008) and Foster and Hart (2009) measures of riskiness

Consider a decision maker facing the problem of accepting a prospect versus its forward market value. In a sufficiently complete market this is canonical and one can pose the problem as a choice between the forward value of wealth, w — i.e., the future amount achieved if current wealth is invested in a risk-free asset — versus $w + \tilde{g}$, where $\tilde{g}$ can be viewed as the (stochastic) profit from the prospect as constructed relative to its forward value. Hart (2011) defines the following two notions of dominance relative to a set of preferences $\hat{\mathcal{U}}$:

Definition 2 (Wealth-Uniform Dominance). $\tilde{g} \succcurlyeq_{WU} \tilde{h}$ *iff for any* $\succcurlyeq_i \in \hat{\mathcal{U}}$,

$$w \succcurlyeq_i \tilde{g} + w \;\; \forall w > 0 \;\;\Rightarrow\;\; w \succcurlyeq_i \tilde{h} + w \;\; \forall w > 0.$$

Definition 3 (Utility-Uniform Dominance). $\tilde{g} \succcurlyeq_{UU} \tilde{h}$ *iff for any* $w > 0$

$$w \succcurlyeq_i \tilde{g} + w \;\; \forall \succcurlyeq_i \in \hat{\mathcal{U}} \;\;\Rightarrow\;\; w \succcurlyeq_i \tilde{h} + w \;\; \forall \succcurlyeq_i \in \hat{\mathcal{U}}.$$

Taken together, the intuition for these dominance criteria could be described as follows: For a "less risky than" ordering to be economically meaningful it has to reflect the consensus of a multitude of rational decision makers. Wealth-uniform dominance applies to preferences expressing uniform distaste for a profit prospect regardless of wealth. Utility-uniform dominance applies when, fixing wealth, the distaste applies across preferences. Uniform rejection for any positive wealth level also depends on the nature of preferences considered in $\hat{\mathcal{U}}$ for the simple reason that many preference relations do not exhibit wealth-uniform rejection of *any* profit in $\mathcal{G}$.[4]

[4]This is generally true of all expected utility functions with decreasing absolute risk aversion tending towards zero as wealth tends to infinity.

Uniform rejection across preferences is restrictive and sensitive to the preferences considered in $\hat{\mathcal{U}}$. Hart (2011) argues that the set $\hat{\mathcal{U}}$ should coincide with $\mathcal{U}_H$, defined as follows.

Definition 4. *Each continuous monotonic weak ordering in $\mathcal{U}_H$ satisfies*

1. $\succcurlyeq \in \mathcal{U}$.
2. *Acceptance increases with wealth: For any $\tilde{g} \in \mathcal{G}$, $\tilde{g} + w \succcurlyeq_i w$ implies that $\tilde{g} + w' \succcurlyeq_i w'$ whenever $w' > w$.*
3. *Acceptance decreases with relative wealth: For any $\tilde{g} \in \mathcal{G}$, $\tilde{g} + w \succcurlyeq_i w$ implies that $\lambda(\tilde{g} + w) \succcurlyeq_i \lambda w$ whenever $\lambda \in (0, 1)$.*
4. *Rejection at some wealth: For any $\tilde{g} \in \mathcal{G}$, if $\tilde{g} \neq 0$ then $\exists w > 0$ such that $w \succcurlyeq_i \tilde{g} + w$*
5. *$\succcurlyeq_i$ satisfies the Independence Axiom (IA) of von Neumann and Morgenstern*

The main result in Hart (2011) is the following theorem:

Theorem 1. *Set $\hat{\mathcal{U}} = \mathcal{U}_H$ from Definition 4. Then $\tilde{g} \succcurlyeq_{WU} h \ \Leftrightarrow\ R^{AS}(\tilde{g}) \leq R^{AS}(\tilde{h})$ and $\tilde{g} \succcurlyeq_{UU} h \ \Leftrightarrow\ R^{FH}(\tilde{g}) \leq R^{FH}(\tilde{h})$, where $R^{AS}(\tilde{g})$ and $R^{FH}(\tilde{g})$ are defined by*

$$E\Big[\exp\big(-\frac{\tilde{g}}{R^{AS}(\tilde{g})}\big)\Big] = 1, \tag{1}$$

and

$$E\Big[\ln\big(1 + \frac{\tilde{g}}{R^{FH}(\tilde{g})}\big)\Big] = 0. \tag{2}$$

The potential significance of this result is its scope. Indeed, Hart (2011) advocates that the measures R^{AS} and R^{FH} be deemed objective in the sense that they reflect a riskiness ordering that is independent of individual preferences. This lends R^{AS} and R^{FH} an objective and universal appeal, suggesting that they could be used to rank the riskiness of prospects by individuals as well as institutions. Suppose that $R^{\mathrm{AS}}(\tilde{g}) \leq R^{\mathrm{AS}}(\tilde{h})$ and $R^{\mathrm{FH}}(\tilde{g}) \leq R^{\mathrm{FH}}(\tilde{h})$. Then one may be tempted to assert normatively that under a broad range of circumstances where $\tilde{g}$ is forgone so should be $\tilde{h}$. This interpretation, however, is inappropriate. It is more appropriate to draw implication from $R^{\mathrm{AS}}(\tilde{g}) \leq R^{\mathrm{AS}}(\tilde{h})$ at the individual level: If the investment opportunity $\tilde{g}$ is deemed unattractive to any individual regardless of available wealth then that same individual should also avoid $\tilde{h}$. By contrast, $R^{\mathrm{FH}}(\tilde{g}) \leq R^{\mathrm{FH}}(\tilde{h})$ should be interpreted conditional on available means: If everyone with available wealth of w finds $\tilde{g}$ unattractive, then anyone with wealth w

should also avoid $\tilde{h}$.[5] The key to both the derivation of the representations and to understanding their scope of interpretation is in specifying the sort of preferences characterizing the set of individuals comprising "everyone". This is contained in Definition 4.

3 Quibbles

Our critique of the Aumann and Serrano (2008) and Foster and Hart (2009) measures fall into two categories. First, we raise concerns about the "universality" of $\mathcal{U}_H$, which is intimately related both to the representations given by R^{AS} and R^{FH} as well as the interpretation (and operationalization) of these measures. While one may take issue with any of the criteria in Definition 4, we are content to admit any or all of them save for the last. By now, there are decades worth of accumulated data suggesting that individual choice behavior does not generally satisfy IA. Moreover, considerable amount of work has been invested in demonstrating that violations of IA are not "irrational" in any meaningful sense. Moreover, such preferences are now routinely used in micro- and macroeconomic applications.

Our second set of concerns arises from certain behaviors of R^{AS} and R^{FH} with respect to "small" profits. The source of these behaviors has to do with the way that R^{AS} and R^{FH} depend on the mean of a gamble in a non-separable fashion. In turn, this non-separability may make it difficult to deploy R^{AS} and R^{FH} in a risk-management setting.

3.1 Universality

The primary motive for measuring risk is to assist in decision making. Correspondingly, knowing that $\tilde{g} \succcurlyeq_{UU} \tilde{h}$ and/or $\tilde{g} \succcurlyeq_{WU} \tilde{h}$ would be useful if, along with other information, it may help an individual or an institution (private, public, or regulatory) rank the desirability of $w + \tilde{g}$ versus $w + \tilde{h}$. For instance, Foster and Hart (2009) suggest using the generalized Sharpe ratio, $\frac{E[\tilde{g}]}{R^{\text{FH}}(\tilde{g})}$, to rank the desirability of profit opportunities.[6] In employing $\succcurlyeq_{WU}$ or $\succcurlyeq_{UU}$ in a decision making criterion, by way of R^{AS} and R^{FH}, one needs to be wary of a consistency conundrum. Specifically, the R^{AS} and R^{FH} rankings of riskiness are based on the assumption that ranking is done on behalf of individuals with preferences in $\mathcal{U}_H$. It would therefore make sense to insist that a

[5] While it may be straight forward to check whether a certain individual will reject a profit opportunity at all wealth levels, it may seem more daunting to check, for a fixed wealth w, whether $w \succcurlyeq w + \tilde{g}$ for *all* $\succcurlyeq \in \hat{\mathcal{U}}$. Because the logarithm is the globally least risk-averse utility function in $\mathcal{U}_H$, it follows that $\tilde{g}$ will be (weakly) rejected by all $\succcurlyeq \in \mathcal{U}_H$ whenever $w \leq R^{\text{FH}}(\tilde{g})$. It is in this sense that R^{FH} can be operationalized.

[6] Aumann and Serrano (2008) touch on this point as well.

function used in trading off between, say, mean and R^{FH} also satisfies IA.[7]

The next result demonstrates that there is no expected utility function that delivers the same ranking as a general differentiable function of $E[\tilde{g}]$ and $R(\tilde{g})$:

Proposition 1. *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$ is non-decreasing in its first argument, strictly decreasing in its second argument, and everywhere differentiable. Then $f\big(E[\tilde{g}], R^{AS}(\tilde{g})\big)$ and $f\big(E[\tilde{g}], R^{FH}(\tilde{g})\big)$ are not expected utility representations.*

Proof. First, assume that $f(\mu, R)$ is strictly increasing in μ over some interval. Suppose $f(\mu, R) = f(\mu', R')$ with $\mu < \mu'$ and $R < R'$. Let $\tilde{g} = e^x(e^{y\tilde{e}} - 1)$ where $x, y \in \mathbb{R}$; the binary random variable $\tilde{e}$ pays $\sqrt{\frac{1-p}{p}}$ with probability p and pays $-\sqrt{\frac{p}{1-p}}$ otherwise; $\tilde{e}$ has zero mean and unit variance. It is straight forward to show that $R^{\text{FH}}(\tilde{g}) = e^x$, and $E[\tilde{g}] = e^x(E[e^{y\tilde{e}}] - 1)$. Fixing $p \in (0, 1)$, $E[e^{y\tilde{e}}] - 1$ maps $y \in \mathbb{R}$ onto $\mathbb{R}_+$. Thus for any $p \in (0, 1)$, one can find some $\tilde{g} \in \mathcal{G}$ so that $E[\tilde{g}] = \mu$ and $R^{\text{FH}}(\tilde{g}) = R$, and there are therefore an infinite number of distinct binary distributions (differentiated by the skewness of $\tilde{e}$) that deliver the same μ and R. One can likewise parameterize a binary distribution for $\tilde{g}'$ with $E[\tilde{g}'] = \mu'$ and $R^{\text{FH}}(\tilde{g}') = R'$.

Consider a probabilistic ϵ-mixture of any $\tilde{g}$ and $\tilde{g}'$ parameterized as above. Then for any $\epsilon \in [0, 1]$, $\mu_\epsilon = \mu + \epsilon(\mu' - \mu)$ and R_ϵ solves

$$0 = \epsilon E\Big[\ln\big(1 + \frac{\tilde{g}'}{R_\epsilon}\big)\Big] + (1 - \epsilon)E\Big[\ln\big(1 + \frac{\tilde{g}}{R_\epsilon}\big)\Big].$$

The derivative of R_ϵ with respect to ϵ and evaluated at $\epsilon = 0$ is equal to $RE\big[\ln\big(1 + \frac{\tilde{g}'}{R}\big)\big]/E\big[1 - e^{-y\tilde{e}}\big]$ (after substituting $e^x(e^{y\tilde{e}} - 1)$ for $\tilde{g}$). If $f\big(E[\tilde{g}, R^{\text{FH}}(\tilde{g})\big)$ is an expected utility representation, then $f(\mu, R) = f(\mu_\epsilon, R_\epsilon)$, and using the differentiability of $f(\cdot, \cdot)$ one can expand $f(\mu_\epsilon, R_\epsilon)$ to first order in ϵ and set the result equal to $f(\mu, R)$ to obtain,

$$\frac{(\mu - \mu')f_1(\mu, R)}{Rf_2(\mu, R)} = \frac{E\big[\ln\big(1 + \frac{\tilde{g}'}{R}\big)\big]}{E\big[1 - e^{-y\tilde{e}}\big]}. \tag{3}$$

The left side of (3) depends only on R, μ, and μ', and thus so must the right side. Moreover, this must be true for *any* $\tilde{g}$ and $\tilde{g}'$ satisfying the criteria stated at the beginning of the proof. To see that this is impossible, consider the limit where $p \equiv \text{Prob}[\tilde{e} > 0] \to 0$ versus the limit where $p \to 1$. In the first instance, $y \sim \sqrt{p} \ln \frac{\mu}{pR}$ and $E[e^{-y\tilde{e}}] \to 1$. In the second instance, $y \sim \frac{\ln 1 + \frac{\mu}{R}}{\sqrt{1-p}}$ and $E[e^{-y\tilde{e}}] \to \infty$. Thus, $\frac{(\mu-\mu')f_1(\mu,R)}{Rf_2(\mu,R)}$ is not well defined because the denominator on the right in (3) can vary independently of the choice of μ, μ', R and R'.

[7]Note that this problem does not plague the standard Sharpe ratio because it is the output of an optimizing problem (i.e., mean-variance optimization) and is therefore necessarily consistent with the underlying preferences assumptions.

To repeat a similar analysis for R^{AS}, consider $\tilde{g} = \mu + \sigma\tilde{e}$ where $\tilde{e}$ is the binary random variable defined above. It is straight forward to show that for any μ and p, setting $R = R^{\mathrm{AS}}(\tilde{g})$ is solved by some $\sigma \in \mathbb{R}_+$. In particular, $\lim_{p\to 0} \sigma \to \frac{\mu}{\sqrt{p}}$ and $\lim_{p\to\infty} \sigma \to R\sqrt{1-p}\ln\big(\frac{e^{\frac{\mu}{R}}-1}{1-p}\big)$. Using arguments similar to the previous case, one deduces that if $f\big(E[\tilde{g}], R^{\mathrm{FH}}(\tilde{g})\big)$ is an expected utility representation, then

$$\frac{(\mu-\mu')f_1(\mu,R)}{Rf_2(\mu,R)} = \frac{E\big[1-e^{-\frac{\tilde{g}'}{R}}\big]}{E\big[\frac{\tilde{g}}{R}e^{-\frac{\tilde{g}}{R}}\big]}.$$

The expectation in the denominator approaches 0 as $p \to 0$ and approaches $-\infty$ as $p \to 1$. Thus, here too, $\frac{(\mu-\mu')f_1(\mu,R)}{Rf_2(\mu,R)}$ is not well defined.

It remains to establish the result for the case in which $f(\mu, R) = f(R)$ is constant with respect to μ. Because $f(R)$ is monotonically equivalent to $-R$, we can restrict attention to demonstrating that R^{AS} and R^{FH} themselves violate expected utility. This is a consequence of Proposition 3.ii in this paper. The latter implies that, in the three-outcome simplex with zero as the middle outcome, both R^{AS} and R^{FH} exhibit indifference sets (curves) that are constant rays emanating from the origin. This is in violation of expected utility which requires that equivalent sets are parallel lines in the simplex. ∎

Proposition 1 together with the overwhelming evidence that expected utility does not adequately reflect how individuals trade off risk against reward make a compelling case that Definition 4 should be weakened to allow for non-expected utility preferences. The following corollary to Proposition 1 suggests a direction for relaxing part 5 of Definition 4 by expanding $\hat{\mathcal{U}}$ to include preferences satisfying the Betweenness Axiom of Chew (1983, 1989) and Dekel (1986):

BA (Betweenness Axiom)**.** *For any $\tilde{x}, \tilde{y} \in \mathcal{D}$,*

$$\tilde{x} \sim \tilde{y} \;\Leftrightarrow\; \forall\alpha \in [0,1], \alpha\tilde{x} \oplus (1-\alpha)\tilde{y} \sim \tilde{x}.$$

Corollary to Proposition 1. *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}_+ \Rightarrow \mathbb{R}$ is non-decreasing in its first argument, strictly decreasing in its second argument, and everywhere differentiable. Then $f\big(E[\tilde{g}], R^{AS}(\tilde{g})\big)$ and $f\big(E[\tilde{g}], R^{FH}(\tilde{g})\big)$ satisfy the Betweenness Axiom if and only if $f(\cdot,\cdot)$ is constant in its first argument.*

Proof. If $f(\cdot,\cdot)$ is not constant in its first argument, the proof proceeds exactly as in Proposition 1. It remains to show that R^{AS} and R^{FH} satisfy the Betweenness Axiom. Suppose that $R^{\mathrm{AS}}(\tilde{g}) = R^{\mathrm{AS}}(\tilde{h}) = R$, and consider any $\alpha \in (0,1)$. Then

$$E[e^{-\frac{\alpha\tilde{g}\oplus(1-\alpha)\tilde{h}}{R}}] = \alpha E[e^{-\frac{\tilde{g}}{R}}] + (1-\alpha)E[e^{-\frac{\tilde{h}}{R}}] = \alpha + (1-\alpha) = 1.$$

Thus $R^{\mathrm{AS}}(\alpha\tilde{g} \oplus (1-\alpha)\tilde{h}) = R$ meaning that R^{AS} satisfies the Betweenness Axiom. A similar argument establishes the result for R^{FH} ∎

If a decision criterion using the Aumann and Serrano (2008) or Foster and Hart (2009) measures cannot satisfy IA but may satisfy BA, then it seems sensible to allow for preferences satisfying Betweenness to be included as part of the defining set of permissible decision criteria, $\hat{\mathcal{U}}$. Unfortunately, doing so unravels the program. Substituting BA for IA in Definition 4 of admissible preferences forces $\succcurlyeq_{WU}$ to be incomplete, while $\succcurlyeq_{UU}$ violates strict FSD and SSD monotonicity. Thus, moving only "slightly away" from expected utility, by allowing for flat though non-parallel indifference surfaces in the probability simplex, renders $\succcurlyeq_{WU}$ and $\succcurlyeq_{UU}$ inconsistent with the orderings of riskiness represented by R^{AS} and R^{FH}. To see how this result comes about consider the BA-satisfying preferences defined by the following certainty equivalent function, $\mathrm{CE}[\tilde{x}]$ (itself defined implicitly):

$$0 = E\Big[\xi \ln\Big(\frac{\tilde{x}}{\mathrm{CE}}\Big) + (1+\eta\,\mathbf{1}_{\mathrm{CE}>\tilde{x}})\frac{\Big(\frac{\tilde{x}}{\mathrm{CE}}\Big)^{1-\gamma} - 1}{1-\gamma}\Big], \tag{4}$$

where $\eta, \gamma, \xi \geq 0$, with $\xi > 0$ if $\gamma < 1$, and $\mathbf{1}_{v>z}$ is an indicator function taking the value of 1 if $v > z$ and zero otherwise. The certainty equivalent above exhibits all of the properties defining $\mathcal{U}_H$ save for the last (it can violate IA but always satisfies BA).[8] When $\gamma \geq 1$, $\xi = 0$ and $\eta \geq 0$, $\mathrm{CE}[\tilde{x}]$ belongs to the class of Disappointment Aversion preferences axiomatized by Gul (1991) and nests expected utility with constant relative risk aversion greater than or equal to 1. It is also easy to find parameters for which (4) exhibits the ordering commonly displayed in the classic Allais (1953) Paradox (in violation of IA). Thus, from all appearances there should be no major objection to inclusion of the preferences described by $\mathrm{CE}[\tilde{x}]$. This said, inclusion of the preferences in (4) also makes $\succcurlyeq_{UU}$ and $\succcurlyeq_{WU}$ incompatible with R^{AS} and R^{FH}. We first illustrate this with examples before providing a general argument in Proposition 2.

Example 1. Assume $w = 600$. Let a binary profit prospect that pays a with probability p and $-b$ otherwise be denoted as $(-b, 1-p; a, p)$. Consider the binary profit prospects $\tilde{g} = (-599.9, \frac{1}{2}; 2400, \frac{1}{2})$ and $\tilde{h} = (-100, \frac{1}{2}; 111, \frac{1}{2})$. We calculate that $R^{\mathrm{FH}}(\tilde{g}) = 799.8$ and $R^{\mathrm{FH}}(\tilde{h}) = 1009.1$. Consider the preferences $\succcurlyeq$ parameterized by $\gamma = \frac{1}{2}$, $\eta = 0$, and $\xi = \frac{1}{10}$ in (4). One can check that $\mathrm{CE}[w + \tilde{g}] = 551.4$, and thus $\tilde{g}$ is soundly rejected. If R^{FH} represents $\succcurlyeq_{UU}$ for a set of preferences $\hat{\mathcal{U}}$ that includes $\mathcal{U}_H$ and $\succcurlyeq$, then it should be the case that $w \succcurlyeq \tilde{h}$. However, this is not so given that $\mathrm{CE}[w + \tilde{h}] = 600.4$. ∎

We observe that $\succcurlyeq$ in Example 1 is Fréchet differentiable (i.e., it can always be locally approximated by an expected utility functional). As Machina (1982) points out, preferences that

[8] We establish these claims in the proof to Proposition 2.

are Fréchet differentiable retain much of the tools of economic analysis derived using expected utility without requiring that preferences satisfy IA (thereby allowing for a greater range of attitudes towards risk). Definition 4 of $\mathcal{U}_H$, because of its insistence on IA, assigns a special role to one utility function: the logarithm. It is the globally least risk-averse function that satisfies the other conditions and therefore serves as a bulwark for all the other preferences in $\mathcal{U}_H$ — if $\tilde{g}$ is rejected by expected log-utility at w, then it will be rejected by all other functions in $\mathcal{U}_H$. When IA is relaxed, as in the case of the preferences in (4), expected log-utility loses its special status. Example 1 specifies preferences that exhibit FSD and SSD monotonicity, continuity (even smoothness), decreasing absolute risk aversion, and increasing relative risk aversion. Moreover, $\succcurlyeq$ also rejects any non-trivial profit opportunity at some level of wealth (because $\xi > 0$). However, it is also globally *less* risk averse than expected log-utility and thus the rejection thresholds specified by expected log-utility cannot apply to it.

If the preferences in (4) are added to $\mathcal{U}_H$, a natural question to raise at this point is whether an alternative preference relation takes the place of expected log-utility in determining threshold wealth levels (and thus $\succcurlyeq_{UU}$). While the answer is "yes", the implications are disappointing. The reason is that the globally least risk averse preference relation in (4) is arbitrarily close to one that is risk-neutral. For such preferences, rejection of a positive-mean profit opportunity will only take place if wealth is arbitrarily close to the magnitude of the highest possible loss. In other words, the only possible representation for $\succcurlyeq_{UU}$ becomes $R^{UU}(\tilde{g}) = |\min \tilde{g}|$, where the min is understood to be taken over all possible outcomes of $\tilde{g}$. This representation is not strictly FSD or SSD monotonic and is clearly inadequate for most purposes of risk measurement or management.

Turning our attention to R^{AS}, we provide a different example.

Example 2. For the profit opportunity, $\tilde{g} = (-100, \frac{1}{2}; 110, \frac{1}{2})$, we calculate that $R^{\text{AS}}(\tilde{g}) = 1100.8$. Likewise, for the profit opportunity, $\tilde{h} = (-1000, \frac{1}{2}; 110000, \frac{1}{2})$, we calculate that $R^{\text{AS}}(\tilde{h}) = 1442.7$ so that $\tilde{g} \succcurlyeq_{WU} \tilde{h}$ according to the Aumann and Serrano (2008) index. This means that any investor in $\mathcal{U}_H$ that rejects $\tilde{g}$ regardless of wealth level, will also do the same with $\tilde{h}$. In addition, all expected utility preferences in $\mathcal{U}_H$ with absolute risk aversion greater than $\frac{1}{1100.8}$ (resp. $\frac{1}{1442.7}$) will reject $\tilde{g}$ (resp. $\tilde{h}$).

Consider supplementing $\mathcal{U}_H$ with $\succcurlyeq$ in (4) parameterized by $\gamma = \frac{1}{2}$, $\eta = 1$, and $\xi = 1$ and let $\text{CE}[w + \tilde{g}] = w + c_g$. For $w \gg 100$, one can expand (4) to first order in $\frac{1}{w}$ which yields $c_g \to -16$. Thus $w \succcurlyeq w + \tilde{g}$ for sufficiently high w. Because acceptance increases with wealth for this preference specification (see the proof of Proposition 2), $\tilde{g}$ is always rejected by $\succcurlyeq$. On the other hand, when wealth is $w = 1100$, we calculate that $\text{CE}[w + \tilde{h}] = 7613.7$ meaning that $w + \tilde{h}$ is overwhelmingly preferred to w. Thus, if $\mathcal{U}_H$ is supplemented with $\succcurlyeq$, then it is no longer the case that $\tilde{g} \succcurlyeq_{WU} \tilde{h}$. Moreover, because some preferences in $\mathcal{U}_H$ always reject $\tilde{h}$ and never reject $\tilde{g}$ (for instance, any expected utility preferences with absolute risk aversion between $\frac{1}{1442.7}$ and $\frac{1}{1100.8}$), it cannot be the case that $\tilde{h} \succcurlyeq_{WU} \tilde{g}$. In other words, supplementing $\mathcal{U}_H$ with $\succcurlyeq$ renders $\succcurlyeq_{WU}$ incomplete. ∎

To obtain a better understanding of Example 2, we first note that a "counter-example" to the representation of $\succcurlyeq_{WU}$ by R^{AS} can only arise from a preference ordering that rejects a profit opportunity at all wealth levels. Rabin (2000) essentially demonstrates that this can only happen with an expected utility representation if large outcomes are exponentially discounted relative to low outcomes. This feature of expected utility is the driving force in deriving R^{AS} as a representation of $\succcurlyeq_{WU}$. With the preferences in (4), rejection of $\tilde{g} \in \mathcal{G}$ at all wealth levels is not possible unless $\eta > 0$, which in turn implies deviation from "second-order risk aversion" at all wealth levels — something ruled out by expected utility (see, for example, the discussion in Segal and Spivak, 1990). Deviations from second-order risk aversion can be instrumental in addressing the Rabin (2000) critique of expected utility as well as the equity premium puzzle (e.g., Epstein and Zin, 1990). Employing $\eta > 0$ delivers a finite downside "penalty" to the certainty equivalent regardless of wealth level. This ensures that some profit opportunities will always be rejected while large outcomes are not exponentially discounted relative to low outcomes. Thus, allowing for the possibility of deviations from second-order risk aversion voids the equivalence between uniform rejection and exponential discounting that exists under expected utility, and breaks the connection between $\succcurlyeq_{WU}$ and the least risk averse expected exponential-utility maximizer who rejects $\tilde{g}$.

With these examples in mind, we can now state the more general result.

Proposition 2. *Let $\mathcal{U}_H^*$ be the set of preferences satisfying properties 1-4 in Definition 4 and the Betweenness Axiom, and set $\hat{\mathcal{U}} = \mathcal{U}_H^*$. Then the orderings induced by R^{AS} and R^{FH} are not consistent with $\succcurlyeq_{WU}$ and $\succcurlyeq_{UU}$, respectively. Moreover, $\succcurlyeq_{WU}$ is incomplete and $\succcurlyeq_{UU}$ is represented by $R^{UU}(\tilde{g}) = |\min \tilde{g}|$, where the* min *is understood to be taken over all possible outcomes of $\tilde{g}$.*

Proof. We show that the preferences in (4) satisfy properties 1-4 in Definition 4. The rest of the proof follows from Examples 1 and 2 and the associated discussion. Strict FSD monotonicity follows from the fact that $\mathrm{CE}[\tilde{x}]$ is a solution to the implicit equation $0 = E[\phi(\tilde{x}, \mathrm{CE})]$ where ϕ is strictly increasing in its first argument, strictly decreasing in its second argument, and $\phi(x, x) = 0$. Under the parametric constraints for ξ, γ and η, $\phi(x, c)$ is strictly concave in x and strict SSD monotonicity follows (see Chew and Mao, 1995), establishing property 1 in Definition 4.

Next, suppose that $\text{CE}[w+\tilde{g}] \geq w$. Then the implication that $E[\phi(\tilde{x}, w)] \geq 0$ can be written out as

$$E\Big[\xi \ln\Big(\frac{w+\tilde{g}}{w}\Big) + (1+\eta\,\mathbf{1}_{0>\tilde{g}})\frac{\left(\frac{w+\tilde{g}}{w}\right)^{1-\gamma}-1}{1-\gamma}\Big] \geq 0.$$

Consider

$$F(\lambda) \equiv E\Big[\xi \ln\Big(1+\lambda\tilde{g}\Big) + (1+\eta\,\mathbf{1}_{0>\tilde{g}})\frac{\Big(1+\lambda\tilde{g}\Big)^{1-\gamma}-1}{1-\gamma}\Big].$$

Over the interval $\lambda \in [0, \frac{1}{w}]$, the function $F(\lambda)$ is continuous and concave; moreover, $F(0) = 0$ and $F(\frac{1}{w}) \geq 0$. Thus, $F(\lambda) \geq 0$ everywhere in $[0, \frac{1}{w}]$. In particular, this means that $\text{CE}[w'+\tilde{g}] > w'$ for any $w' \geq w$ and that $\text{CE}[w+\tau\tilde{g}] > w$ for any $\tau \in [0, 1]$. This establishes properties 2 in Definition 4. Property 3 follows from the fact that the certainty equivalent in (4) exhibits constant relative risk aversion (it scales with the prospect): $\text{CE}[\lambda(w+\tilde{g})] = \lambda\text{CE}[(w+\tilde{g})]$. Property 4 follows trivially from the fact that $\text{Prob}(\tilde{g} < 0) > 0$ and that, under the parameter restrictions, $\lim_{x\to 0}\phi(x, c) \to \infty$. ∎

Proposition 2 demonstrates that the risk orderings corresponding to R^{AS} and R^{FH} are not robust to relaxing IA to BA. The latter is still restrictive in that it requires indifference surfaces in the probability simplex to be linear. While such a restriction may not be fully descriptive of observed choice behavior (see, for instance, Camerer and Ho, 1994), there are many alternatives to expected utility that allow nonlinear indifference surfaces. These include the rank-dependent utility (RDU) model of Quiggin (1982) and Yaari (1987), which are themselves special cases of the *implicit rank-linear utility* models introduced in Chew and Epstein (1989), and the Quadratic Utility model of Chew, Epstein, and Segal (1991).[9] All of these models nest expected utility as a special case. When $\hat{\mathcal{U}}$ is expanded to include any of these classes of models, $\succcurlyeq_{WU}$ is no longer complete and therefore no longer represented by R^{AS}.[10] Only when $\hat{\mathcal{U}}$ is restricted to include "pure" RDU models can $\succcurlyeq_{UU}$ still be represented by R^{FH}.[11] For the other models mentioned, this is no longer the case.[12] The representation results for R^{AS} and R^{FH} fail for an even larger class of

[9]Cumulative Prospect Theory (CPT) of Tversky and Kahneman (1992) also poses a challenge to R^{AS} and R^{FH}. We do not include it in our analysis because it is not clear how to apply property 4 of Definition 4 to CPT models.

[10]This is because all of the mentioned models allow for deviations from second-order risk aversion. For instance, they all include the RDU model, $U(\tilde{g}) = \int u(x)d[1-\big(1-F_g(x)\big)^2]$ which can be easily calibrated to reject $\tilde{g}$ from Example 2 at all wealth levels, and yet accept $\tilde{h}$ at some wealth level.

[11]To ensure global risk aversion, the probability transformation function in RDU must be concave (see Chew, Karni, and Safra, 1987). From this, it is straight forward to show that the only rank-dependent models consistent with properties 1-4 in Definition 4 are also at least as risk averse as expected log-utility, and thus R^{FH} still provides the rejection threshold.

[12]To prove this for the more general implicit rank-linear utility beyond the Betweenness class, one can adopt the implicit model in (4) using a rank-ordered expectation instead. The Quadratic Utility model has a representation of

well-behaved though as yet unaxiomatized models that nest expected utility, thus it appears that Theorem 1 is perhaps better viewed as a knife-edge result rather than reflective of universal tastes.[13]

3.2 Scale Conundrums

Consider the position of a regulator who wishes to limit the risk undertaken by banks and other institutions that are deemed "too big to fail". A typical risk-management approach might specify that the institution has to maintain a nominal amount of safe and liquid capital commensurate with the riskiness of profit opportunity (e.g., a loan or a zero-cost derivative position). Traditionally, such policies have been based on measures such as the standard deviation, or sometimes the value at risk, of $\tilde{g}$. Here, we argue that using R^{AS} and R^{FH} to specify risk-based capital instead may yield less satisfactory results. Suppose that riskiness was measured only using R^{AS} or R^{FH} and thus capital requirements were set using only a monotonic transformation of one of these measures. Then the following proposition raises concerns.

Proposition 3.

i. Given any $\epsilon \in (0, \infty)$, let Δ_ϵ be the set of all finite-outcome gambles with strictly positive mean, non-zero probability of a negative outcome, and whose support is a strict subset of $(-\epsilon, \epsilon)$. Then $R^{AS}(\Delta_\epsilon) = R^{FH}(\Delta_\epsilon) = (0, \infty)$.

ii. Let δ_0 correspond to the sure profit of zero. Then for any $\tilde{g} \in \mathcal{G}$ and $\alpha \in [1, 0)$, $R^{AS}(\tilde{g}) = R^{AS}(\alpha\tilde{g} \oplus (1-\alpha)\delta_0)$ and $R^{FH}(\tilde{g}) = R^{FH}(\alpha\tilde{g} \oplus (1-\alpha)\delta_0)$.

Proof. To prove the first part, consider any binary gamble in Δ_ϵ that is symmetric about its mean, μ. Denoting the standard deviation as σ, for the low outcome to be negative it must be that $\sigma > \mu$. It is straight forward to show that $R^{\text{FH}} = \frac{\sigma^2 - \mu^2}{2\mu}$ and $R^{\text{AS}} = \frac{\sigma}{x}$ where x solves

the form $U(\tilde{g}) = \int\int \phi(x, y) dF_g(x) dF_g(y)$. Using $\phi(x, y) = \ln(\frac{x+y}{2})$, this model satisfies Properties 1-4 of Definition 4 but is globally less risk averse than expected log-utility (which is necessary for R^{FH} to represent $\hat{\mathcal{U}}$). We also note that $\phi(x, y) = \min\big(u(x), u(y)\big)$ leads to $U(\tilde{g}) = \int u(x) d[1 - \big(1 - F_g(x)\big)^2]$, implying as noted in the previous footnote, that $\succcurlyeq_{WU}$ is incomplete.

[13] A simple example is the following: Let $\text{CE}_u[w+\tilde{g}]$ correspond to the certainty equivalent of $w+\tilde{g}$ from an expected utility model with utility function u. Then for any $\beta \in (0, 1)$, the certainty equivalent $\text{CE}_{u_1}[w+\tilde{g}]^\beta \text{CE}_{u_2}[w+\tilde{g}]^{1-\beta}$ will violate IA unless u_1 and u_2 are affine equivalent (expected utility forms a strict subset of this class of models). Setting $u_1(x) = \ln x$ and $u_2(x) = \sqrt{x}$ leads to a certainty equivalent that is compatible with properties 1-4 in Definition 4 and less risk-averse than expected log-utility (inconsistent with R^{FH}). Setting, instead, $u_2(x) = -e^{-ax}$ and, say, $\beta = \frac{1}{2}$ can lead to inconsistency with R^{AS}. If $\hat{\mathcal{U}}$ contains all preferences in this class satisfying properties 1-4 in Definition 4, it is straight forward to show that a result identical to Proposition 2 obtains.

$\ln\big(\cosh(x)\big) = \frac{\mu}{\sigma}x$. In both cases, as $\frac{\mu}{\sigma}$ tends to zero, R tends to infinity; likewise, as $\frac{\mu}{\sigma}$ tends to one, R tends to zero.

To prove part *ii*, consider

$$E\Big[\exp\big(-\frac{\alpha\tilde{g}\oplus(1-\alpha)\delta_0}{R^{\mathrm{AS}}(\tilde{g})}\big)\Big] = \alpha E\Big[\exp\big(-\frac{\tilde{g}}{R^{\mathrm{AS}}(\tilde{g})}\big)\Big] + (1-\alpha)E\Big[\exp\big(\frac{0}{R^{\mathrm{AS}}(\tilde{g})}\big)\Big] = \alpha + (1-\alpha) = 1,$$

where the last equality follows from the fact that $R^{\mathrm{AS}}(\tilde{g})$ solves $E\Big[\exp\big(\frac{\tilde{g}}{R^{\mathrm{AS}}(\tilde{g})}\big)\Big] = 1$. The calculation demonstrates that $R^{\mathrm{AS}}(\tilde{g})$ solves $E\Big[\exp\big(\frac{\alpha\tilde{g}\oplus(1-\alpha)\delta_0}{R^{\mathrm{AS}}(\tilde{g})}\big)\Big] = 1$. Thus $R^{\mathrm{AS}}(\tilde{g}) = R^{\mathrm{AS}}(\alpha\tilde{g}\oplus(1-\alpha)\delta_0)$. An analogous argument, also documented in Foster and Hart (2009), establishes the result for $R^{\mathrm{FH}}(\tilde{g})$. ∎

Proposition 3.i establishes that within the set of (arbitrarily) small gambles R^{AS} and R^{FH} take on every conceivable value. The riskiness measures R^{AS} and R^{FH} do not convey the scale of the outcomes in the sense that a report of $R^{\mathrm{AS}}(\tilde{g})$ says nothing about whether differences in potential outcomes of $\tilde{g}$ can be sizable. Related to this, Proposition 3.ii establishes that R^{AS} and R^{FH} do not convey the scale of the *likelihood* of non-zero profits. The bottom line is that these measures necessitate a potentially unintuitive or descriptively inaccurate link between the ranking of small or low-probability profits and those of profit opportunities with moderate or large variance. In the context of risk-based capital regulation, Proposition 3.i suggests that a risk involving arbitrarily small gains and losses will require as much capital as one that has arbitrarily large gains and losses . Likewise, according to Proposition 3.ii, a risk involving only a tiny likelihood of substantial loss will require as much capital as one with a high likelihood of substantial loss.[14]

The reason for the result in Proposition 3, partly illustrated in the proof, is that R^{AS} and R^{FH} diverge as $E[\tilde{g}]$ tends to zero. Even though riskiness intuitively declines as either the magnitude of possible outcomes shrinks or as the likelihood of non-zero profit shrinks, the mean of the gamble simultaneously shrinks, potentially allowing R^{AS} and R^{FH} to stay constant or even increase — all depending on the rate at which the mean decreases. Every measure of riskiness that is FSD- and SSD-monotonic necessarily confounds "risk" and "reward". What is special about R^{AS} and R^{FH} is their divergent behavior as $E[\tilde{g}] \to 0$. In that sense, they may be better viewed as measures of a risk's performance (similar to the inverse Sharpe Ratio) rather than an appropriate dollar-denominated risk premium.[15] One may hope to fix this by considering an

[14] In the case of R^{FH}, Foster and Hart (2009) recognize property ii of Proposition 3 and term it "dilution". They rationalize the invariance of R^{FH} to dilution by noting that an infinitely repeated sequence of a gamble leads to a non-zero probability of bankruptcy if and only if its diluted counterpart does as well. We comment on this alternative rationale for measuring risk in Section 4.

[15] Both Aumann and Serrano (2008) and Foster and Hart (2009) argue that measures of riskiness should be FSD-

alternative risk measure of the form $f(E[\tilde{g}], R(\tilde{g}))$ where f is increasing in both its arguments, homogeneous of degree one, finite as $E[\tilde{g}] \to 0$. One potential example is $\sqrt{E[\tilde{g}]R(\tilde{g})}$, which can be shown to be finite whenever the outcomes of $\tilde{g}$ are bounded and scales proportionately with $\tilde{g}$. In particular, for $E[\tilde{g}]$ tending to zero while holding the variance constant, it can be shown that this measure converges to the standard deviation of $\tilde{g}$. However, $\sqrt{E[\tilde{g}]R^{\mathrm{AS}}(\tilde{g})}$ and $\sqrt{E[\tilde{g}]R^{\mathrm{FH}}(\tilde{g})}$ are not monotonic in $E[\tilde{g}]$, thus undermining one of the attractive advantages of these measure over variance (and other coherent measures).[16] This problem is pervasive as is demonstrated in the following result.

Proposition 4. *Consider any $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}_+$ that is increasing in both arguments, is homogeneous of degree one, and such that for any constant $a > 0$, $\lim_{x \to 0} f(x, \frac{a}{x}) < \infty$. Then for some $\tilde{g} \in \mathcal{G}$ and either $R(\tilde{g}) = R^{AS}(\tilde{g})$ or $R(\tilde{g}) = R^{FH}(\tilde{g})$, $f(E[\tilde{g}], R(\tilde{g}))$ is increasing in $E[\tilde{g}]$.*

Proof. By the homogeneity of f, define $f(\mu, R) = \mu f(1, \frac{R}{\mu}) \equiv \mu\varphi(\frac{R}{\mu})$. The limit condition becomes $\lim_{\mu \to 0} \mu\varphi(\frac{a}{\mu^2}) < \infty$, which means that $\lim_{x \to \infty} \varphi(x) \sim b\sqrt{x} + o(\sqrt{x})$, where b is a constant.[17] Consider now that $R = R(\mu)$. Taking the derivative,

$$\frac{d}{d\mu}\,\mu\varphi\Big(\frac{R(\mu)}{\mu}\Big) = \varphi + \Big(\frac{d}{d\mu}\,R(\mu) - \frac{R}{\mu}\Big)\varphi'. \tag{5}$$

For the adjusted measure to be decreasing in μ, it must be that $\varphi \leq \big(-\frac{d}{d\mu}\,R(\mu) + \frac{R}{\mu}\big)\varphi'$. Write $\tilde{g} = \mu + \tilde{\varepsilon}$ for some zero-mean random variable $\tilde{\varepsilon}$. For the Foster and Hart (2009) measure, we calculate that

$$\frac{d}{d\mu}\,R^{\mathrm{FH}}(\tilde{g}) = \frac{E\Big[\frac{R}{R+\tilde{g}}\Big]}{E\Big[\frac{\tilde{g}}{R+\tilde{g}}\Big]}.$$

For $\tilde{g} = e^x(e^{y\tilde{e}} - 1)$ considered in Proposition 1, with $R^{\mathrm{FH}}(\tilde{g}) = e^x$, the derivative of $R^{\mathrm{FH}}(\tilde{g})$ reduces to

$$\frac{d}{d\mu}\,R^{\mathrm{FH}}(\tilde{g}) = \frac{E[e^{-y\tilde{e}}]}{1 - E[e^{-y\tilde{e}}]}.$$

In the case of the binary $\tilde{e}$ discussed in the proof of Proposition 1, recall that we are free to choose $E[e^{-y\tilde{e}}] \in (1, \infty)$, thus for any μ and R fixed, we can choose $\frac{d}{d\mu}\,R^{\mathrm{FH}}(\tilde{g}) \in (-\infty, -1)$. Using

monotonic. We do not take issue with that statement. The dollar "risk-premium", generated by subtracting from w the certainty equivalent of $w + \tilde{g}$ using any one of the models in $\mathcal{U}$, can also be viewed as a measure of riskiness. It is FSD- and SSD-monotonic in $\mathcal{G}$, and does not exhibit the scale conundrums of R^{AS} and R^{FH}. The main shortcoming of the dollar risk-premium just described is that it is not"universal". The arguments in Section 3.1 suggest that the universality advantages of R^{AS} and R^{FH} are limited.

[16] Consider, for example, the binary gamble $\tilde{g}$ that with probability $p > 0$ yields $\mu + \sigma\sqrt{\frac{1-p}{p}}$ and otherwise yields $\mu - \sigma\sqrt{\frac{p}{1-p}}$, where $\sigma, \mu > 0$. For such a gamble to be admissible, it must be that $\mu < \sigma\sqrt{\frac{p}{1-p}}$. It is easy to check that for $p = 0.75$ and $\sigma = 1$ the measures $\sqrt{\mu R^{\mathrm{AS}}(\tilde{g})}$ and $\sqrt{\mu R^{\mathrm{AS}}(\tilde{g})}$ have inverted-U shapes for $\mu \in [0.4, 0.9]$.

[17] $o(\sqrt{x})$ denotes a term with the property that $\lim_{x \to \infty} x^{-\frac{1}{2}} o(\sqrt{x}) \to 0$.

this result in Equation (5), a necessary condition for $f(E[\tilde{g}], R^{\text{FH}}(\tilde{g}))$ to decrease with μ is that $\varphi \leq \left(1 + \frac{R}{\mu}\right)\varphi'$. Setting $x = \frac{R}{\mu}$, the asymptotic condition for φ then imply $b\sqrt{x} + o(\sqrt{x}) \leq (1+x)\left(\frac{b}{2\sqrt{x}} + o(\frac{1}{\sqrt{x}})\right)$, which is violated for x sufficiently large.

For the Aumann and Serrano (2008) measure, we calculate that

$$\frac{d}{d\mu}\, R^{\text{AS}}(\tilde{g}) = \frac{1}{E\left[\frac{\tilde{g}}{R}e^{-\frac{\tilde{g}}{R}}\right]}.$$

For $\tilde{g} = \mu + \sigma\tilde{e}$, as considered in Proposition 1, the denominator can taken on any value in $(-\infty, 0)$, meaning that for any fixed R and μ we can find a corresponding $\tilde{g}$ such that $\frac{d}{d\mu}\, R^{\text{AS}}$ takes any desired value in $(-\infty, 0)$. Using this result in Equation (5), a necessary condition for $f(E[\tilde{g}], R^{\text{AS}}(\tilde{g}))$ to decrease with μ is that $\varphi \leq \frac{R}{\mu}\varphi'$. This time the asymptotic condition for φ imply $b\sqrt{x} + o(\sqrt{x}) \leq x\left(\frac{b}{2\sqrt{x}} + o(\frac{1}{\sqrt{x})})\right)$, which is also violated for x sufficiently large. ∎

Returning once more to regulating risk-based capital, suppose that the capital requirements were determined using some function of R^{AS} and R^{FH} and the mean of the profit opportunity. Proposition 4 tells us that such capital requirements would feature the perverse implication that, at least in some cases, a positive shift in profit across all possible outcomes will lead to *higher* capital requirements. One way to avoid this would be to include additional information (other than the mean, R^{AS} and R^{FH}) about the distribution of profits — in other words, employ additional risk measures in the exercise. In summary, Propositions 3 and 4 raise serious questions about the use of R^{AS} and R^{FH} as tools in risk measurement and management.

In contrast with R^{AS} and R^{FH}, which are not separable in the mean of the gamble evaluated, conventional measures of riskiness like variance, value-at-risk, and/or coherent measures á la Artzner, Delbaen, Eber, and Heath (1999) convey, or can be adjusted to reflect, a sense of the magnitude of risk independent of the mean. For such mean-adjusted measures, large variance gambles engender greater risks because they allow for a greater likelihood of large differences in outcomes. Propositions 3 indicates that these statements are not true of R^{AS} and R^{FH}, while Proposition 4 implies that any attempt to mean-adjust R^{AS} and R^{FH} would generally result in measures that are not FSD-monotonic.

3.3 Incomplete markets

The measures of riskiness R^{AS} and R^{FH} apply only to "gambles" that have positive mean and a non-zero probability of a negative outcome. In Section 2 we argue that this is suitable for a setting in which any risky position can be liquidated and the proceeds invested in some risk-free asset. In many cases, reality presents a sufficiently good approximation to this situation that there is no reason to quibble with the approach taken in Aumann and Serrano (2008), Foster and Hart (2009), and Hart (2011). There are, however, important settings in which this approximation materially fails. Difficulties arise, for instance, if one attempts to evaluate R^{AS} or R^{FH} for a prospect that is non-tradeable or highly illiquid. Consider any one of the various financial institutions that were heavily invested in complicated structured products during the Winter of 2008-2009. A market price might not have existed even if it might have been possible to assign a probability distribution to the value of the institution's portfolio. At a time when risk measurement was most needed and while one could still calculate the variance or value-at-risk of such a portfolio, one might not have been be able to calculate its Aumann and Serrano (2008) or Foster and Hart (2009) measures.

Another example of a situation that is perhaps more pervasive is that of risk assessment at the non-institutional level. A small enterprise (or family) may have need for risk assessment that cuts across preferences, underscoring the attractiveness of measures that are "universal". Suppose that such an entity was capable of calculating the distribution of possible future outcomes for each of several financial and real investment strategies.[18] While standard utility theory, or standard approaches to the measurement of risk, would at this point produce a ranking of options this would not be the case if one attempted to calculate R^{AS} or R^{FH}. Leaving aside the arguments put forth in Section 3.1, there remains the thorny issue of evaluating the "wealth" of the enterprise, which itself can and typically does carry significant uncertainty (e.g., real estate valuation, the value of human capital, etc.). In summary, the additional requirement of wealth as a reference point for assessing R^{AS} and R^{FH} is far from innocuous and limits their consistent application across a satisfactory range of situations.[19]

4 Further remarks

Foster and Hart (2009) provide an additional rationale for the use of their measure, different from that expounded in Hart (2011). Specifically, $R^{\mathrm{FH}}(\cdot)$ can be viewed as the unique "simple rule" solution to the following problem. Suppose a decision maker with initial wealth w_1 faces a sequence of independent profit opportunities, $(\tilde{g}_1, \tilde{g}_2, \ldots)$, such that $\tilde{g}_t \in \mathcal{G}$ for all t. Suppose,

[18] This is a necessary assumption for any decision making approach under risk.

[19] The Cumulative Prospect Theory of Tversky and Kahneman (1992) also features a reference point that is crucial in determining the merits of various decisions. An important difference is that Prospect Theory is motivated by psychological biases observed in experiments and generally limits itself to contexts where a status quo is obvious, without making claims to being normative or broadly operationalizable.

further, that this decision maker seeks to find a simple rule that guarantees solvency and is of the following form: Accept any opportunity from the sequence whenever $R^{\mathrm{FH}}(\tilde{g}_t) \leq w_t$. This, too, is an elegant formulation justifying the usage of $R^{\mathrm{FH}}(\cdot)$ in a particular and limited context for inter-temporal investment. That said, this formulation is not justified by more general considerations of inter-temporal utility maximization with solvency constraints. To see this, let each of the $\tilde{g}_t$'s be an independently distributed binary random variable paying 110 with probability $\frac{1}{2}$ and -100 otherwise. We calculate that $R^{\mathrm{FH}}(\tilde{g}_t) = 1100$. This means that if $w_1 = 1070$ then, when restricted to simple rules, the entire sequence will be rejected. Consider, however, the "less simple rule" for which $\tilde{g}_1$ is accepted at $w_1 \geq 1070$, $\tilde{g}_2$ is accepted only if the outcome of $\tilde{g}_1$ is high, and $\tilde{g}_t$ is otherwise rejected for all $t > 2$. Clearly this alternative strategy is solvent. Moreover, a straight-forward calculation demonstrates that an expected log-utility maximizer would receive a higher expected utility by accepting this strategy than by rejecting the entire sequence and keeping $w_1 = 1070$ (as directed by the simple rule based on R^{FH}). Thus, the restriction to simple rules is not optimal even when considered by an objective function that rejects any finite probability of insolvency. In that sense, R^{FH} can only be associated with solvent strategies that are generally suboptimal.

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